A new strategy for the determination of α_s

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Based on:

- Mattia Dalla Brida, Roman Höllwieser, Francesco Knechtli, Tomasz Korzec, Alberto Ramos, Rainer Sommer. (ALPHA) *Non-perturbative renormalization by decoupling*. [arXiv: 1912.06001]

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ALPHA Collaboration

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Where do we stand [L. Del Debbio talk]

- α_s is a key quantity for LHC physics
- Lattice QCD offers the most precise determinations ($\delta\alpha_s \approx 0.7\%$)
  - Several "large volume" methods ($q\bar{q}$, moments HQ correlators, ...)
  - Finite size scaling **[Lüscher, Wolff '91]**: dedicated approach
- Computing Λ is a hard multiscale problem (corrections $\alpha^k(Q) \sim \log^k(Q/\Lambda)$)
- Precision is limited by
  - "large volume": Perturbative uncertainties
  - finite size scaling: Statistical uncertainties
- Several "hot" topics in LQCD are irrelevant for α_s

$$\delta\alpha_s \approx 0.4\% \Longleftrightarrow \delta\Lambda^{(3)} \approx 2.5\%$$

QED, charm quark effects, ... well below our target precision.

New dedicated approach

Heavy quarks as a tool in non-perturbative renormalization

- The "hard" multiscale problem (i.e. Λ/μ_{had}) is solved in **pure gauge!**
- Matching condition: Pure gauge and world with heavy quarks are similar

Important in remote talks

Specially important for fellows:

- If something is not clear, **stop me**
- If you get lost at some point, **stop me**
- Ask as many questions as you want (we have time!)

MASSLESS RENORMALIZATION SCHEMES

Computation of observables

$$O(Q) \overset{\alpha\to 0}{\sim} \sum_n c_n(Q/\mu)\alpha^n_{\overline{\rm MS}}(\mu)$$

- Coupling $\alpha_{\overline{\rm MS}}(\mu)$ defined by: counterterms only include the divergences
- Divergences do not depend on values of quark masses
- $\overline{\rm MS}$ is an example of a massless scheme.
- In LQCD we also prefer massless schemes: Conditions imposed at zero mass
  - Schrödinger Functional (SF)
  - (S)RI/MOM
  - …

Massless renormalization schemes: Tremendous advantages

- Renormalization group functions are mass independent

$$\mu \frac{\mathrm{d}\bar{g}^2(\mu)}{\mathrm{d}\mu} = \beta(\bar{g}, \cancel{m})\,.$$

- RGI invariants that characterize the running (i.e $\Lambda, M, B_K, \ldots$) **only** exists in massless schemes

$$\Lambda_s = \mu \left[b_0 \bar{g}_s^2(\mu)\right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}_s^2(\mu)}} \exp\left\{-\int_0^{\bar{g}_s(\mu)} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right]\right\}$$

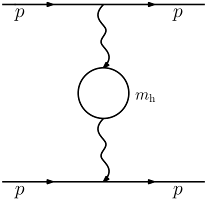
- Precision: high loop computations available in perturbation theory

$$\beta_{\overline{\mathrm{MS}}}(\bar{g}) \overset{\bar{g}\to 0}{\sim} -\bar{g}^3(b_0 + b_1\bar{g}^2 + b_2^{\overline{\mathrm{MS}}}\bar{g}^4 + b_3^{\overline{\mathrm{MS}}}\bar{g}^6 + b_4^{\overline{\mathrm{MS}}}\bar{g}^8 + \text{unknown})$$

  Always universal but universal only in massless schemes

- In LQCD: easier to define the chiral point ($m_q = 0$) than the physical point ($m_q =??$)

Decoupling of heavy quarks: Perturbation theory



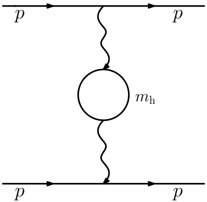
Quark-Quark scattering with N_l light and one heavy

$$T = \frac{\alpha_{\overline{\mathrm{MS}}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\mathrm{MS}}}(\mu)}{\pi^2}\left\{T_1(p,m) + \frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2} + \mathrm{c}\right\} + \mathcal{O}(\alpha^3)$$

Five Stages of understanding: (I) Denial

- If I choose $\mu \approx m_h(\mu)$ the $T_1(p,m)$ gets large...
- The computation has to be wrong, because this heavy quark cannot break perturbation theory

Decoupling of heavy quarks: Perturbation theory

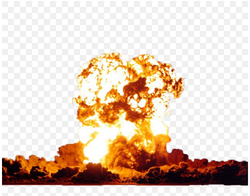


Quark-Quark scattering with N_l light and one heavy

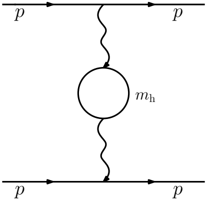
$$T = \frac{\alpha_{\overline{\rm MS}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi^2}\left\{T_1(p,m) + \frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2} + \mathrm{c}\right\} + \mathcal{O}(\alpha^3)$$

Five Stages of understanding: (II) Anger

- So the existence of a quark with $m_h \sim 2000$ TeV is breaking perturbation theory at scale $p \approx \mu \approx 20$ GeV.
- Nonsense!!!!!!
- Nothing works!!!!!



Decoupling of heavy quarks: Perturbation theory



Quark-Quark scattering with N_l light and one heavy

$$T = \frac{\alpha_{\overline{\rm MS}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi^2}\left\{ T_1(p,m) + \frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2} + {\rm c}\right\} + \mathcal{O}(\alpha^3)$$

Five Stages of understanding: (III) Bargaining

ALICE: Look, If I only could say that

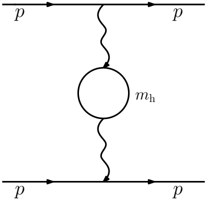
$$\frac{\alpha'(\mu)}{\pi} = \frac{\alpha_{\overline{\rm MS}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi^2}\frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2}$$

Then everything would make sense:

$$T = \frac{\alpha'(\mu)}{\pi} + \frac{\alpha'^2(\mu)}{\pi^2}\left[T_1(p,m) + {\rm c}\right] + \mathcal{O}(\alpha^3)$$

But then the coupling would depend on $m_{\rm h}$!

Decoupling of heavy quarks: Perturbation theory



Quark-Quark scattering with N_l light and one heavy

$$T = \frac{\alpha_{\overline{\mathrm{MS}}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\mathrm{MS}}}(\mu)}{\pi^2}\left\{T_1(p,m) + \frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2} + \mathrm{c}\right\} + \mathcal{O}(\alpha^3)$$

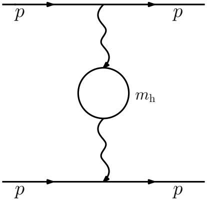
Five Stages of understanding: (IV) The right question

BOB: And this coupling of yours...

$$\frac{\alpha'(\mu)}{\pi} = \frac{\alpha_{\overline{\mathrm{MS}}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\mathrm{MS}}}(\mu)}{\pi^2}\frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2}$$

How would it run?

Decoupling of heavy quarks: Perturbation theory



Quark-Quark scattering with N_l light and one heavy

$$T = \frac{\alpha_{\overline{\rm MS}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi^2}\left\{ T_1(p,m) + \frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2} + \mathrm{c}\right\} + \mathcal{O}(\alpha^3)$$

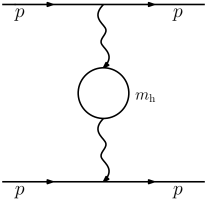
Five Stages of understanding: (V) All fits nicely

$$\frac{\alpha'(\mu)}{\pi} = \frac{\alpha_{\overline{\rm MS}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi^2}\frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2}$$

And determine

$$\beta' = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\alpha'(\mu) = \left(\mu^2\frac{\partial}{\partial\mu^2} + \beta\frac{\partial}{\partial\alpha} + \gamma\frac{\partial}{\partial m_h}\right)\left[\alpha'_{\overline{\rm MS}}(\mu) + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi}\frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2}\right]$$

Decoupling of heavy quarks: Perturbation theory



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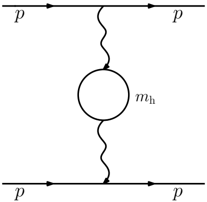
And determine

$$\beta' = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\alpha'(\mu) \overset{\alpha'\to 0}{\sim} \frac{\alpha'^2(\mu)}{\pi}\left(\beta - \frac{1}{6}\right) + \mathcal{O}(\alpha^3)$$

Decoupling of heavy quarks: Perturbation theory



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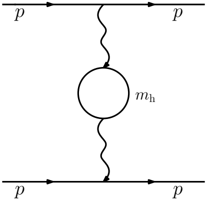
And determine

$$\beta' = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\alpha'(\mu) \overset{\alpha'\to 0}{\sim} -\frac{\alpha'^2(\mu)}{\pi}\left(\frac{11}{4} - \frac{1}{6}N_{\rm f} + \frac{1}{6}\right) + \mathcal{O}(\alpha^3)$$

Decoupling of heavy quarks: Perturbation theory



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Five Stages of understanding: (V) All fits nicely

$$\frac{\alpha'(\mu)}{\pi} = \frac{\alpha_{\overline{\rm MS}}(\mu)}{\pi} + \frac{\alpha^2_{\overline{\rm MS}}(\mu)}{\pi^2}\frac{1}{6}\log\frac{m_h^2(\mu)}{\mu^2}$$

And determine

$$\beta' = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\alpha'(\mu) \overset{\alpha'\to 0}{\sim} -\frac{\alpha'^2(\mu)}{\pi}\left[\frac{11}{4} - \frac{1}{6}(N_f - 1)\right] + \mathcal{O}(\alpha^3)$$

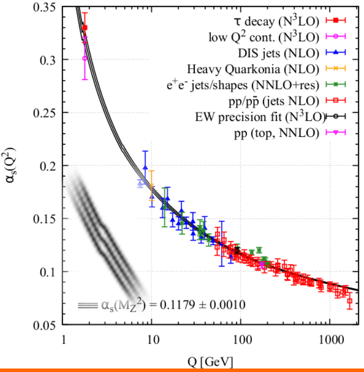
$\alpha'(\mu)$ is the running coupling with $N_l = N_f - 1$ flavors!

Decoupling of heavy quarks in massless schemes

Matching between theories

- At energy scales Q just forget about all quarks with $m > Q$
- "Nice" perturbative expressions if you only use **active** quarks
- Matching between effective theory (with **active quarks**) and fundamental theory (with **active** and heavy quarks)

$$\alpha_{\overline{\mathrm{MS}}}^{(N_\mathrm{f}-1)}(\mu) = \alpha_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}(\mu) \times \left\{1 + a_1(m_h/\mu)\alpha_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}(\mu) + \dots\right\}$$



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$$\alpha_{\overline{\rm MS}}^{(N_{\rm f}-1)}(\mu) = \alpha_{\overline{\rm MS}}^{(N_{\rm f})}(\mu) \times \left\{1 + a_1(m_h/\mu)\alpha_{\overline{\rm MS}}^{(N_{\rm f})}(\mu) + \dots\right\}$$

Abuse of language: A single $\alpha_{\overline{\rm MS}}(\mu)$ that "jumps" at quark thresholds

- $\alpha_{\overline{\rm MS}}(4\,{\rm GeV})$: This is the four flavor coupling
- $\alpha_{\overline{\rm MS}}(10\,{\rm GeV})$: This is the five flavor coupling
- $\alpha_{\overline{\rm MS}}(M_Z)$: This is the five flavor coupling

Caveats

Power corrections are neglected (more later)

Decoupling of heavy quarks in massless schemes

$$\Lambda_{\overline{\rm MS}}^{(N_{\rm f})} \xrightarrow{P(M/\Lambda)} \Lambda_{\overline{\rm MS}}^{(N_{\rm f}')}$$

Relation between Λ parameters

If you happen to know $\Lambda_{\overline{\rm MS}}^{(6)}$, then

1. Determine $\alpha_{\overline{\rm MS}}^{(6)}(\mu) = \bar{g}_{\overline{\rm MS}}^2(\mu)/(4\pi)$ at some scale $\mu \approx m_{\rm t}$

$$\frac{\Lambda_{\overline{\rm MS}}^{(6)}}{\mu} = \left[b_0 \bar{g}_{\overline{\rm MS}}^2(\mu)\right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}_{\overline{\rm MS}}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}_{\overline{\rm MS}}(\mu)} dx \left[\frac{1}{\beta_{\overline{\rm MS}}^{(6)}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right]\right\}$$

2. Match across the top threshold (4 loops known!)

$$\frac{\bar{g}'^2(\mu)}{4\pi} = \alpha_{\overline{\rm MS}}^{(5)}(\mu) = \alpha_{\overline{\rm MS}}^{(6)}(\mu) \times \left\{1 + a_1(m_{\rm t}/\mu)\alpha_{\overline{\rm MS}}^{(6)}(\mu) + \dots\right\}$$

3. Determine the Λ parameter of the 5 flavor theory

$$\frac{\Lambda_{\overline{\rm MS}}^{(5)}}{\mu} = \left[b_0 \bar{g}_{\overline{\rm MS}}'^2(\mu)\right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}_{\overline{\rm MS}}'^2(\mu)}} \exp\left\{-\int_0^{\bar{g}'_{\overline{\rm MS}}(\mu)} dx \left[\frac{1}{\beta_{\overline{\rm MS}}^{(5)}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right]\right\}$$

Decoupling of heavy quarks in massless schemes

$$\frac{\Lambda_{\overline{\rm MS}}^{(N_f)}}{\mu} = \left[b_0 \bar{g}^2_{\overline{\rm MS}}(\mu)\right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}^2_{\overline{\rm MS}}(\mu)}} \exp\left\{-\int_0^{\bar{g}_{\overline{\rm MS}}(\mu)} dx \left[\frac{1}{\beta^{(N_f)}_{\overline{\rm MS}}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right]\right\}$$

Some numerical examples

- Start with $\Lambda_{\overline{\rm MS}}^{(6)} \approx 91.1$ MeV
- Determine $\alpha_{\overline{\rm MS}}^{(6)}(m_t) \Longrightarrow \alpha_{\overline{\rm MS}}^{(5)}(m_t)$
- Get $\Lambda_{\overline{\rm MS}}^{(5)} \approx 215$ MeV
- Determine $\alpha_{\overline{\rm MS}}^{(5)}(m_b) \Longrightarrow \alpha_{\overline{\rm MS}}^{(4)}(m_b)$
- Get $\Lambda_{\overline{\rm MS}}^{(4)} \approx 298$ MeV
- Determine $\alpha_{\overline{\rm MS}}^{(4)}(m_c) \Longrightarrow \alpha_{\overline{\rm MS}}^{(3)}(m_c)$
- Get $\Lambda_{\overline{\rm MS}}^{(3)} \approx 312$ MeV
- We cannot get $\Lambda_{\overline{\rm MS}}^{(2)}$: No valid perturbative matching at $\mu \approx m_s < \Lambda$

Perturbative uncertainties **ridiculously small** in this game!

Checkpoint

- Massless schemes are needed for precision
- One should use perturbative expressions with only the number of **active** quarks
- Matching between theories

$$\alpha_{\overline{\mathrm{MS}}}^{(3)} \to \alpha_{\overline{\mathrm{MS}}}^{(4)} \to \alpha_{\overline{\mathrm{MS}}}^{(5)} \to \alpha_{\overline{\mathrm{MS}}}^{(6)} .$$

- Matching between theories **also requires to write everything in terms of quark masses in the effective theory**

$$m_{\overline{\mathrm{MS}}}^{(N_{\mathrm{f}}-1)}(\mu) = m_{\overline{\mathrm{MS}}}^{(N_{\mathrm{f}})}(\mu) \times \left\{1 + b_2(m_h/\mu)\left[\alpha_{\overline{\mathrm{MS}}}^{(N_{\mathrm{f}})}(\mu)\right]^2 + \ldots\right\}$$

Decoupling of heavy quarks: Non-perturbatively

- Large coefficients in PT **is a problem of PT**
- In Lattice QCD we can use as many (heavy) flavors as we want
- But simulating heavy quarks is challenging:
  - m_{h} is large
  - am_{h} **has to be** small

  Requires large computational resources!

Large PT coefficients

Multi-scale problem

Large computational cost

Decoupling of heavy quarks

- In gluonic quantities (i.e. $r_0, t_0, w_0, \sigma, \dots$) heavy quarks make small effects.

$$\frac{r_0}{\sqrt{t_0}}\Big|_{N_\mathrm{f}=2+1+1} = \frac{r_0}{\sqrt{t_0}}\Big|_{N_\mathrm{f}=2+1} + \mathcal{O}\left(\frac{1}{m_\mathrm{c}^2}\right)$$

$M/\Lambda \rightarrow$	M_c/Λ		0
R	$1/M$-scaled	$1/M^2$-scaled	
$\sqrt{t_0}/w_0$	0.34(5)%	0.16(2)%	5.4%
$\sqrt{t_c/t_0}$	0.28(3)%	0.13(1)%	3.2%
r_1/r_0	0.45(13)%	0.21(6)%	$\approx$ 4.0%
$r_0/\sqrt{t_0}$	0.05(28)%	0.02(12)%	3.0%

Table: Comparison of $N_\mathrm{f}=0$ and $N_\mathrm{f}=2$ with two heavy quarks **[ALPHA Phys.Rev.Lett. 114]**

Quantity	$N_\mathrm{f}=2$	$N_\mathrm{f}=0$	sea effects [%]
m_V/m_P	1.05405(60)	1.05274(46)	0.124(71)
m_S/m_P	1.258(14)	1.224(20)	2.7(1.9)
m_T/m_P	1.271(38)	1.321(33)	3.9(4.1)

Table: Charmed mesons in $N_\mathrm{f}=0$ and $N_\mathrm{f}=2$ with two heavy quarks **[S. Cali et al. Eur.Phys.J.C 79 (2019) 7, 607]**

Decoupling of heavy quarks

- In gluonic quantities (i.e. $r_0, t_0, w_0, \sigma, \ldots$) heavy quarks make small effects.

$$\left.\frac{r_0}{\sqrt{t_0}}\right|_{N_\mathrm{f}=2+1+1} = \left.\frac{r_0}{\sqrt{t_0}}\right|_{N_\mathrm{f}=2+1} + \mathcal{O}\left(\frac{1}{m_\mathrm{c}^2}\right)$$

	$M/\Lambda \rightarrow$	M_c/Λ		0
R		1/M-scaled	1/M^2-scaled	

- We can certainly ignore the bottom/top quarks when computing $M_p, f_k, K \rightarrow \pi \ldots$
- Probably/maybe even *charm* can be ignored?
- Challenge for $M_p, f_k, K \rightarrow \pi \ldots$: QED, $m_u - m_d$, ... **[A. Patella, N. Tantalo talks]**

Quantity	$N_\mathrm{f} = 2$	$N_\mathrm{f} = 0$	sea effects [%]
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Decoupling of heavy quarks

Λ parameters **[ALPHA Nucl.Phys.B 943 (2019) 114612]**

- Related by perturbation theory (4 loops)

$$\frac{\Lambda^{(4)}_{\overline{\mathrm{MS}}}}{\Lambda^{(3)}_{\overline{\mathrm{MS}}}} = \frac{1}{P^{(k)}(M/\Lambda)} + \mathcal{O}(\alpha^4(m_{\mathrm{c}})) + \mathcal{O}\left(\frac{1}{m_{\mathrm{c}}^2}\right)$$

- Uncertainties are small (i.e. 1%) and very small (i.e. 0.4%).
- Remember: current uncertainty in $\Lambda^{(4)}_{\overline{\mathrm{MS}}}$ is 4%

Checkpoint

- Quark loops are very suppressed: c, b, t effects are sub-percent
- Decoupling in massless schemes allows to determine

$$\Lambda^{(3)} \Longleftrightarrow \Lambda^{(4)} \Longleftrightarrow \Lambda^{(5)} \Longleftrightarrow \Lambda^{(6)}$$

with uncertainties much smaller than our current target precision

3M: A universe with three heavy degenerate quarks $(M \gg \Lambda)$

Alice uses fundamental theory

$$S_{\text{fund}}[A_\mu, \psi, \bar{\psi}] = \int \mathrm{d}^4x \left\{ -\frac{1}{2g^2} \mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu}) + \sum_{i=1}^{3} \bar{\psi}_i(\gamma_\mu D_\mu + M)\psi_i \right\}$$

Bob uses effective theory

$$S_{\text{eff}}[A_\mu] = -\frac{1}{2g_{\text{eff}}^2} \int \mathrm{d}^4x \,\{\mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu})\} + \frac{1}{M^2} \sum_k \omega_k \int \mathrm{d}^4x \,\mathcal{L}_k^{(6)} + \ldots$$

3M: A universe with three heavy degenerate quarks $(M \gg \Lambda)$

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Bob uses effective theory

$$S_{\text{eff}}[A_\mu] = -\frac{1}{2g_{\text{eff}}^2} \int \mathrm{d}^4x \,\{\mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right)\} + \cancel{\frac{1}{M^2} \sum_k \omega_k \int \mathrm{d}^4x\, \mathcal{L}_k^{(6)}} + \ldots$$

Decoupling

- Dimensionless "low energy quantities" $\sqrt{t_0}/r_0, w_0/\sqrt{8t_0}, r_0/w_0, \ldots$ from effective theory

$$\frac{\mu_1^{\text{fund}}(M)}{\mu_2^{\text{fund}}(M)} = \frac{\mu_1^{\text{eff}}}{\mu_2^{\text{eff}}} + \mathcal{O}\left(\frac{\mu^2}{M^2}\right)$$

Renormalization in 3M: Alice determines the strong coupling

$$\frac{\Lambda}{\mu} = \left[b_0\bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0^2 x}\right]\right\}.$$

- Determine non-perturbatively the β-function in the fundamental ($N_f = 3$) theory, mass-less scheme.
- Integral up to $\bar{g}^{(3)}(\mu_{\rm ref})$ (in a mass-less scheme!) gives:

$$\frac{\Lambda^{(3)}}{\mu_{\rm ref}}$$

- Only needs to compute a dimensionless ratio

$$\mu_{\rm ref}\sqrt{8t_0(M)}$$

- Result

$$\sqrt{8t_0(M)}\Lambda^{(3)} = \frac{\Lambda^{(3)}}{\mu_{\rm ref}} \times \mu_{\rm ref}\sqrt{8t_0(M)}$$

Renormalization in 3M: Bob determines the strong coupling

$$\frac{\Lambda}{\mu} = \left[b_0 \bar{g}^2(\mu)\right]^{-\frac{b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}^2(\mu)}} \exp\left\{-\int_0^{\bar{g}(\mu)} \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}.$$

- Determine non-perturbatively the β-function in the effective ($N_f = 0$) theory.
- Integral up to $\bar{g}^{(0)}(\mu'_{\mathrm{ref}})$ gives:

$$\frac{\Lambda^{(0)}}{\mu'_{\mathrm{ref}}}$$

- Determine the dimensionless ratio

$$\mu'_{\mathrm{ref}}\sqrt{8t_0}$$

- Match across quark threshold to convert to $\Lambda^{(3)}$ (using perturbation theory)

$$\Lambda^{(3)}\sqrt{8t_0} = \Lambda^{(0)}\sqrt{8t_0} \times \frac{1}{P(\Lambda/M)}.$$

- Matching factor $P(\Lambda/M)$ [ALPHA 1809.03383]:
  - Known in perturbation theory up to three-loops. Power series in $\alpha(m^\star)$
  - "Good" PT: corrections very small even at $m_c^\star$.

Relation between Alice and Bob computation

Relation between Alice and Bob results:

$$\Lambda^{(3)}\sqrt{8t_0(M)} = \Lambda^{(0)}\sqrt{8t_0} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{1}{t_0 M^2}\right)$$

Bob is telling us that $\Lambda^{(3)}$ can be computed from $\Lambda^{(0)}$

$$\Lambda^{(3)} = \lim_{M\to\infty} \frac{1}{\sqrt{8t_0(M)}} \times \Lambda^{(0)}\sqrt{8t_0} \times \frac{1}{P(\Lambda/M)}$$

We need

- Running in pure gauge: $\Lambda^{(0)}\sqrt{8t_0}$
- A scale in a world with degenerate massive quarks: $\sqrt{8t_0(M)}$ in fm/MeV.

Lattice QCD can simulate *unphysical* worlds

$$\sqrt{8t_0(M)} = \sqrt{8t_0^{\text{phys}}} \times \frac{\sqrt{8t_0(M)}}{\sqrt{8t_0^{\text{phys}}}}$$

with $\sqrt{8t_0^{\text{phys}}} = 0.415(4)(2)$ fm from **[Bruno et al. '17]**

Non-perturbative renormalization by decoupling

Master relation

$$\frac{\Lambda^{(N_\mathrm{f})}}{\mu_\mathrm{dec}(M)} = \frac{\Lambda^{(0)}}{\mu_\mathrm{dec}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{\mu_\mathrm{dec}^2}{M^2}\right)$$

With a proper limit:

$$\Lambda^{(N_\mathrm{f})} = \lim_{M\to\infty} \mu_\mathrm{dec}(M) \times \frac{\Lambda^{(0)}}{\mu_\mathrm{dec}} \times \frac{1}{P(\Lambda/M)}$$

Where

- Pure gauge running: $\Lambda^{(0)}/\mu_\mathrm{dec}$
- A scale with N_f massive degenerate quarks: $\mu_\mathrm{dec}(M)$

NOTE: this is not completely trivial

$$\frac{\mu_\mathrm{dec}(M)}{\mu_\mathrm{dec}^\mathrm{phys}} = \lim_{a\to 0} \frac{a\mu_\mathrm{dec}(M)}{a\mu_\mathrm{dec}^\mathrm{phys}}$$

is a difficult extrapolation (M wants to be large, aM wants to be small).

Our setup: Choices optimized to be able to simulate heavy quarks

$$\Lambda^{(3)} = \mu_{\rm dec}(M) \times \frac{\Lambda^{(0)}}{\mu_{\rm dec}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{\mu_{\rm dec}}{M}\right) + \mathcal{O}\left(\frac{\mu_{\rm dec}^2}{M^2}\right)$$

- Work in finite volume schemes with Schrödinger Functional boundary conditions: $T \times L^3$ with Dirichlet bcs. in time. ($\mu \sim 1/L$): "Only" two scales.
- Use Gradient Flow couplings

$$\bar{g}^2(\mu) = \mathcal{N}^{-1}(c, a/L)\, t^2 \langle E(t)\rangle\Big|_{\mu^{-1}=\sqrt{8t}=cL}.$$

- Fix $\bar{g}^2(\mu_{\rm dec})|\Big|_{N_{\rm f}=3, M=0, T=L} = 3.95$. This defines $\mu_{\rm dec} \sim 800$ MeV
- Small volume $\Longrightarrow$ We can simulate heavy quarks (i.e. $a \sim 30\ \text{GeV}^{-1}$)
- Matching condition ($\{N_{\rm f} = 3, M\} \leftrightarrow \{N_{\rm f} = 0\}$) between massive scheme and effective theory

$$\bar{g}^2(\mu_{\rm dec}(M))\Big|_{N_{\rm f}=3, M, T=2L} = \bar{g}^2(\mu_{\rm dec})\Big|_{N_{\rm f}=0, T=2L}.$$

  Matching: QCD in a finite volume!
- Convenient variable: $z = M/\mu_{\rm dec}$

Our setup: Choices optimized to be able to simulate heavy quarks

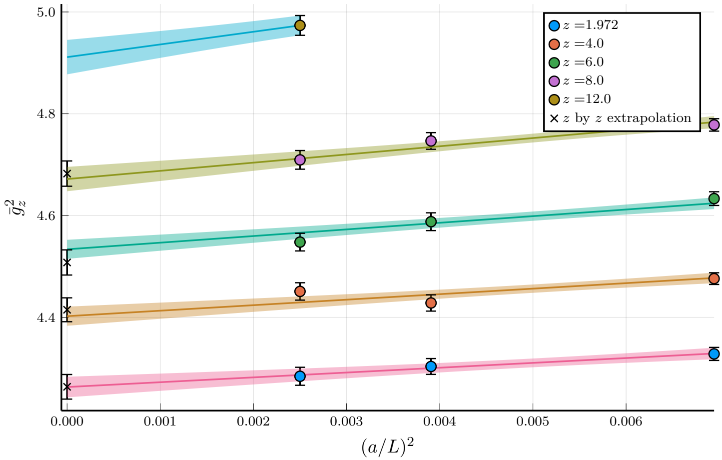
$$\Lambda^{(3)} = \mu_{\rm dec}(M) \times \frac{\Lambda^{(0)}}{\mu_{\rm dec}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{\mu_{\rm dec}}{M}\right) + \mathcal{O}\left(\frac{\mu_{\rm dec}^2}{M^2}\right)$$

We only need to fill in a table!

M [GeV]	$\mu_{\rm dec}(M)$ [GeV]	$\bar{g}^2(\mu_{\rm dec}(M))\big\rvert_{N_{\rm f}=3,M,T=2L}$	$\Lambda^{(0)}/\mu_{\rm dec}$	$1/P(\Lambda/M)$	$\Lambda^{(3)}$
-	-	-	-	-	-
-	-	-	-	-	-
-	-	-	-	-	-
-	-	-	-	-	-
...	...	...	...	...	...

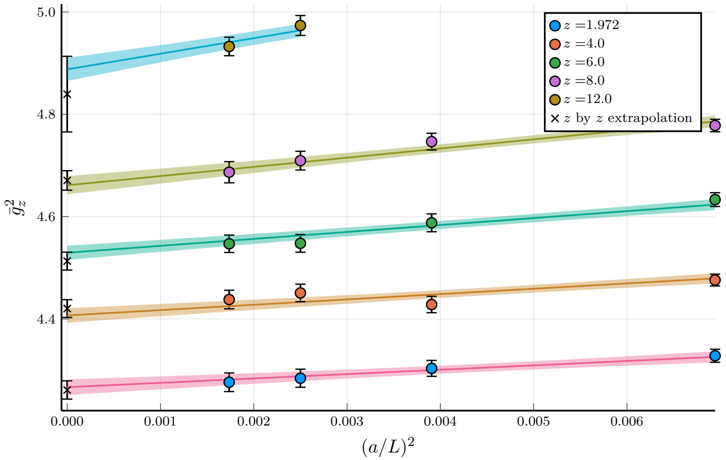
- Difficult continuum extrapolations to determine $\bar{g}^2(\mu_{\rm dec}(M))\big\rvert_{N_{\rm f}=3,M,T=2L}$
- Use combined Heavy-Quark / Symanzik effective theories: Global fit

Continuum extrapolations



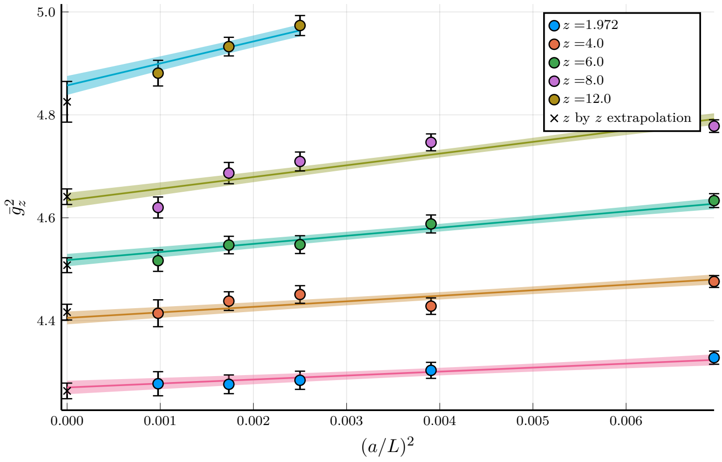
Continuum extrapolations with $L/a = 12, 16, 20$

Continuum extrapolations



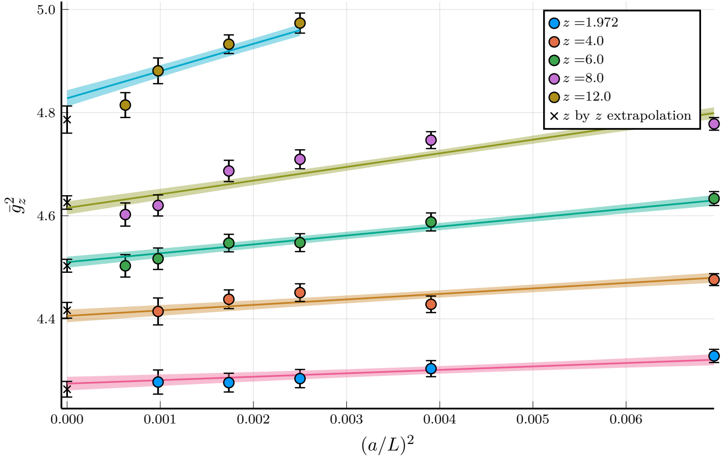
Continuum extrapolations with $L/a = 12, 16, 20, 24$

Continuum extrapolations



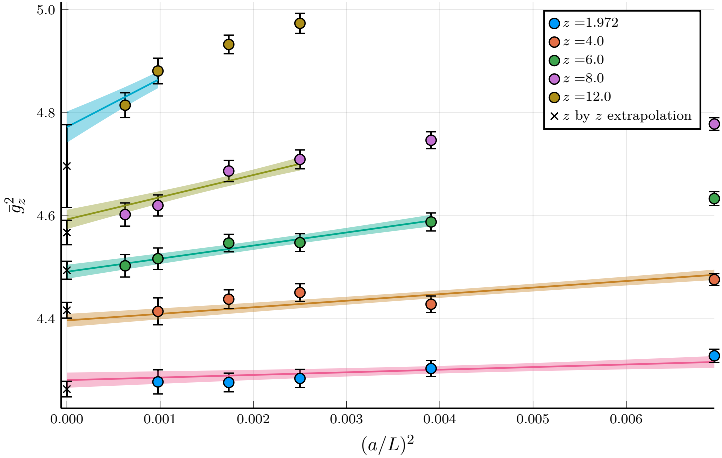
Continuum extrapolations with $L/a = 12, 16, 20, 24, 32$

Continuum extrapolations



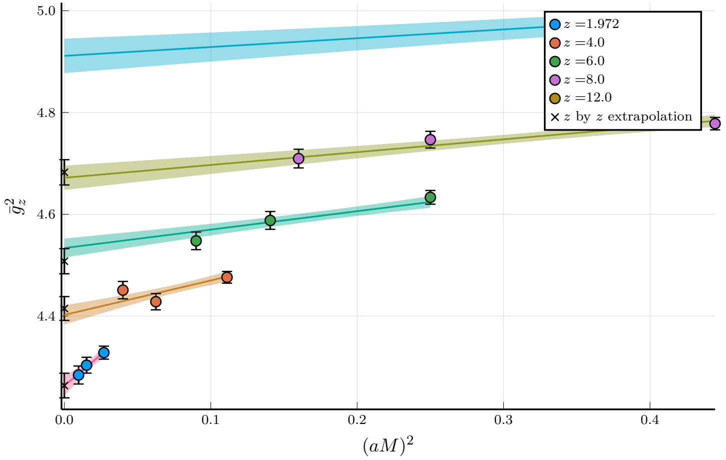
Continuum extrapolations with $L/a = 12, 16, 20, 24, 32, 40$

Continuum extrapolations



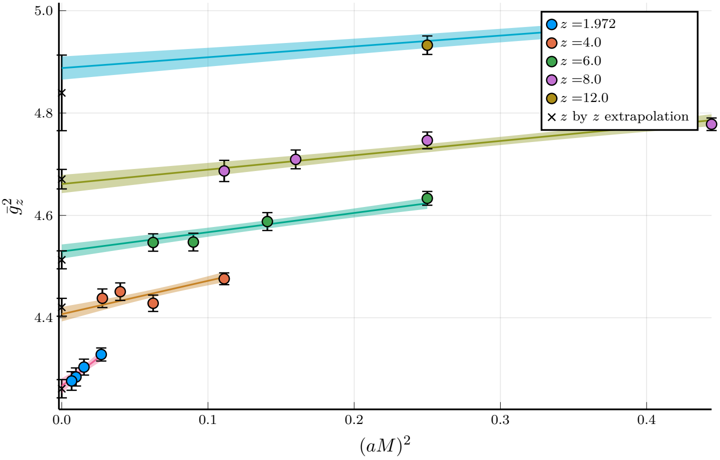
Continuum extrapolations with $L/a = 12, 16, 20, 24, 32, 40$. Use only $aM < 0.4$

Continuum extrapolations: We also extrapolate $aM \to 0$



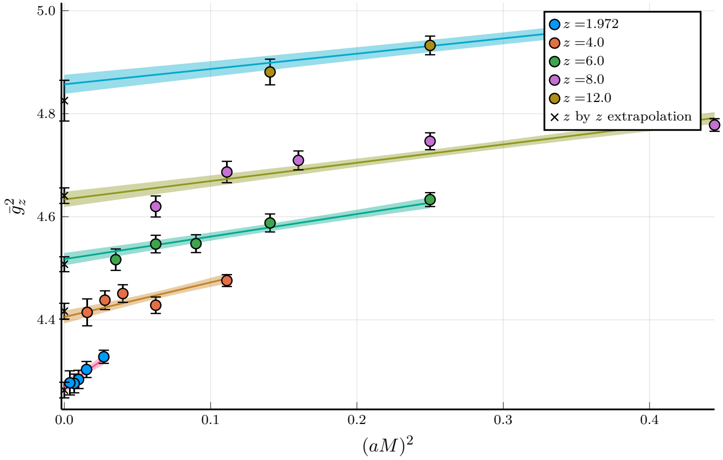
- Continuum extrapolations with $L/a = 12, 16, 20$.
- Heavy quark description broken at $z = 2$

Continuum extrapolations: We also extrapolate $aM \to 0$



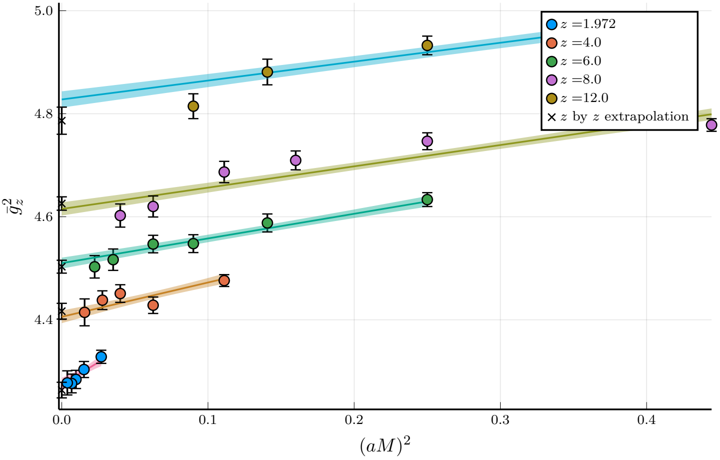
- Continuum extrapolations with $L/a = 12, 16, 20, 24$.
- Heavy quark description broken at $z = 2$

Continuum extrapolations: We also extrapolate $aM \to 0$



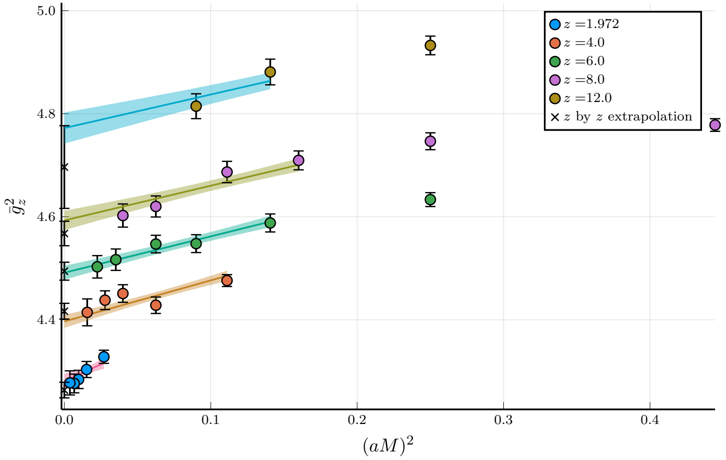
- Continuum extrapolations with $L/a = 12, 16, 20, 24, 32$.
- Heavy quark description broken at $z = 2$

Continuum extrapolations: We also extrapolate $aM \to 0$



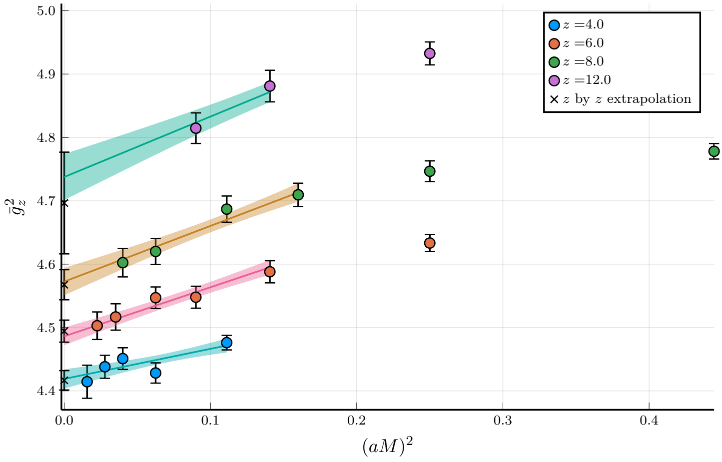
- Continuum extrapolations with $L/a = 12, 16, 20, 24, 32, 40$.
- Heavy quark description broken at $z = 2$

Continuum extrapolations: We also extrapolate $aM \to 0$



- Continuum extrapolations with $L/a = 12, 16, 20, 24, 32, 40$.
- Heavy quark description broken at $z = 2$
- Use only $aM < 0.4$.

Continuum extrapolations: We also extrapolate $aM \to 0$



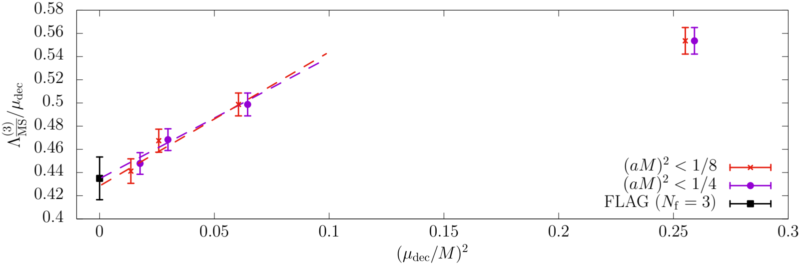
- Continuum extrapolations with $L/a = 12, 16, 20, 24, 32, 40$.
- Heavy quark description broken at $z = 2$: Cut $z > 2$
- Use only $aM < 0.4$.

Table can be filled (PRELIMINARY)

M [GeV]	$\mu_{\mathrm{dec}}(M)$ [GeV]	$\bar{g}^2(\mu_{\mathrm{dec}}(M))\Big\rvert_{N_{\mathrm{f}}=3,M,T=2L}$	$\Lambda^{(0)}/\mu_{\mathrm{dec}}$	$\frac{1}{P(\Lambda/M)}$	$\Lambda^{(3)}$ [MeV]
1.6	0.789(15)	4.559(39)	0.689(11)	0.7662(44)	416(11)
3.2	0.789(15)	4.421(16)	0.725(11)	0.6693(37)	382.7(96)
4.7	0.789(15)	4.466(37)	0.741(12)	0.6198(34)	362.0(92)
6.3	0.789(15)	4.507(60)	0.757(13)	0.5871(32)	350.3(92)

Determination of $\Lambda^{(3)}$ from decoupling

ALPHA [Phys.Lett.B 807 (2020) 135571]



Tremendous advantage

- The hard multi-scale problem (i.e. Λ/μ) can be solved in pure gauge
- Compatible with both "large volume" approaches and "finite size scaling"

Checkpoint

- Continuum extrapolations with heavy quarks are difficult!

Conclusions: Non-perturbative renormalization by decoupling

- Lattice QCD can determine scales at un-physical values of the parameters

$$\Lambda^{(N_\mathrm{f})} = \lim_{M\to\infty} \mu_\mathrm{dec}(M) \times \frac{\Lambda^{(0)}}{\mu_\mathrm{dec}} \times \frac{1}{P(\Lambda/M)}$$

  with
  - $\mu_\mathrm{dec}(M)$: Scale with N_f heavy quarks ($M \gg \Lambda$)
  - $\Lambda^{(0)}/\mu_\mathrm{dec}$: Computed non-perturbatively in pure gauge
  - $P(\Lambda/M)$: Perturbative relation between fundamental and effective theories
- Method is generic
  - Similar expressions for other RGI invariants: $M, \hat{B}_K, \hat{B}_B, \ldots$
  - Valid in finite or infinite volume renormalization schemes
  - If you can, just compute $\sqrt{8t_0}, w_0$ with 3-4 quarks as heavy as possible!
- Still working (larger L/a) but It works!!
  - Finite volume setup: Small PT corrections $\mathcal{O}(\alpha^3(m^\star))$, window problem ameliorated
  - $\mu_\mathrm{dec}(M) = 789(15)$ MeV. Applied with $M = 1.6, \ldots, 6.3$ GeV
  - Non-perturbative running in pure gauge from $\mu = 789$ MeV to $\mu = \infty$
  - $\Lambda^{(3)}$ in agreement with current knowledge
- Probably best approach to reduce error in α_s substantially
  - Running done in pure gauge!
  - Better precision in pure gauge: Small lattice spacing, efficient algorithms.
  - Switch massive $\leftrightarrow$ massless schemes little effect in total error.
  - $\lim_{M\to\infty}$ can be controlled

Conclusions: Precision physics with lattice QCD

- Electromagnetic corrections
- Charm effects (i.e. $N_{\mathrm{f}} = 2+1+1$ vs. $N_{\mathrm{f}} = 2+1$)
- Statistical precision in difficult observables.
- Connecting hadronic physics with EW scale without assumptions on low scale physics (i.e. perturbation theory).
- Continuum extrapolations with heavy quarks are **difficult**: Easy to get wrong results unless access to very fine lattice spacings
  - Ameliorated with finite volume setup
  - Is this under control in current simulations?q

$$aM_{\mathrm{c}} < 0.4 \Longrightarrow a < 4.5\,\mathrm{GeV}^{-1}$$