TOR VERGATA
UNIVERSITY OF ROME
School of Mathematics, Physical and Natural Sciences

nazario tantalo
nazario.tantalo@roma2.infn.it

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Finite volume effects in QCD+QED by using relativistic EFTs

Lucini et al. JHEP02 (2016) 076, arXiv:1612.00199

$$\frac{m_P(L)-m_P}{m_P} = \frac{e^2}{4\pi}\left\{\frac{Q_P^2\xi(1)}{2m_P L} + \frac{Q_P^2\xi(2)}{\pi(m_P L)^2} - \frac{1}{4\pi m_P L^4}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell(2\ell)!\,\mathcal{T}_\ell\,\xi(2+2\ell)}{\ell!\,L^{2(\ell-1)}}\right\} + O(L^{-\infty})$$

the aim of this lecture is to explain how to obtain this formula, the FVE on the mass of a stable hadron in QCD+QED$_C$, by using an approach based on relativistic effective theories

before starting the journey, however, let's analyze the structure of the formula:

- the first two terms are *universal*: depend only on the mass and on the charge of the particle and coincide with those of a point–like hadron
- a part from the universal $1/L$ term, there are no inverse odd powers of L
- structure dependent terms start to contribute at $O(L^{-4})$
- the $\xi(n)$ are geometric objects that depend on the spatial boundary conditions

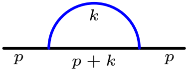
$$\xi(s) = \sum_{\boldsymbol{n}\neq \boldsymbol{0}} \frac{e^{i\pi(n_1+n_2+n_3)}}{|\boldsymbol{n}|^s}$$

- the structure dependent coefficients are related to the derivatives of the forward Compton scattering amplitude

$$\mathcal{T}_\ell = \left.\frac{d^\ell}{d(\boldsymbol{k}^2)^\ell} T_{\mu\mu}(i|\boldsymbol{k}|, \boldsymbol{k})\right|_{|\boldsymbol{k}|=0}$$

- let's consider a simple example: the self–energy of a particle of mass m_0 interacting with a particle of mass m_π

$$\Delta\Sigma(p, L) = \left\{ \frac{1}{L^3} \sum_{\boldsymbol{k}} - \int \frac{d^3 k}{(2\pi)^3} \right\} \int \frac{dk_0}{2\pi} \frac{1}{\{(p+k)^2 + m_0^2\}\,\{k^2 + m_\pi^2\}}$$



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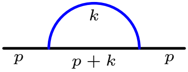

- by introducing a Feynman parameter we can understand the singularity structure of the denominator

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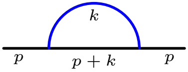

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- now, by calling $s=-p^2$, we have

$$M^2(x) = sx^2 - x(s-m_0^2) + (1-x)m_\pi^2$$

$$= s\left\{ \left[x - \frac{s+m_\pi^2-m_0^2}{2s}\right]^2 + \frac{\left[m_0^2-(s-m_\pi^2)^2\right]\left[(s+m_\pi^2)^2-m_0^2\right]}{4s^2} \right\} \geq M^2_{min}$$

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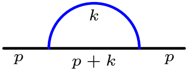

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- the important point to be noticed is that this object is always positive as long as

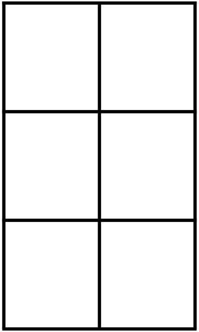
$$m_0 \leq \sqrt{s} < m_0 + m_\pi \;, \qquad M^2_{min} > 0$$

- a very useful trick to understand the phenomenology of FVE is Poisson's summation formula

$$\left\{ \frac{1}{L^3} \sum_{\boldsymbol{k}} - \int \frac{d^3k}{(2\pi)^3} \right\} f(\boldsymbol{k}) =$$

$$= \sum_{\boldsymbol{n} \neq \boldsymbol{0}} e^{i\boldsymbol{\theta}\cdot\boldsymbol{n} - \epsilon \boldsymbol{n}^2} \int \frac{d^3k}{(2\pi)^3} e^{iL\boldsymbol{n}\cdot\boldsymbol{k}} f(\boldsymbol{k}) = \sum_{\boldsymbol{n} \neq \boldsymbol{0}} e^{i\boldsymbol{\theta}\cdot\boldsymbol{n} - \epsilon \boldsymbol{n}^2} \tilde{f}(\boldsymbol{n}L)$$

$$\boldsymbol{k} = \frac{2\pi\boldsymbol{n} + \boldsymbol{\theta}}{L}$$



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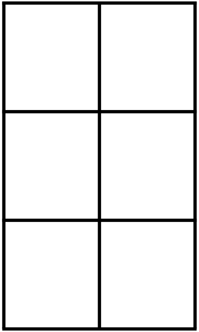

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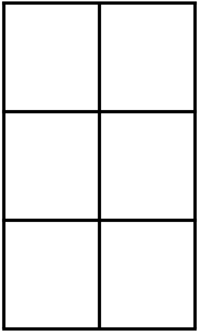

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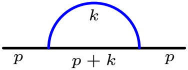
- for later use we notice that

$$\left\{ \frac{1}{L^3} \sum_{\boldsymbol{k}\neq\boldsymbol{k}_\star} - \int \frac{d^3k}{(2\pi)^3} \right\} f(\boldsymbol{k}) = -\frac{f(\boldsymbol{k}_\star)}{L^3} + \sum_{\boldsymbol{n}\neq\boldsymbol{0}} e^{i\boldsymbol{\theta}\cdot\boldsymbol{n} - \epsilon \boldsymbol{n}^2} \tilde{f}(\boldsymbol{n}L)$$

- in our simple exercise, if

$$m_P \leq \sqrt{s} < m_P + m_\pi \ , \qquad m_\pi \neq 0$$



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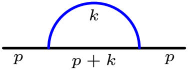

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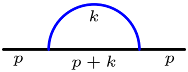

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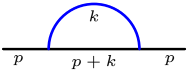

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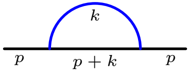

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$$\leq \frac{1}{16\pi^2} \sum_{\boldsymbol{n}\neq\boldsymbol{0}} \int_0^\infty \frac{du}{u} e^{-\frac{M_{min}|\boldsymbol{n}|L}{2}\left(u+\frac{1}{u}\right)} = O\left(e^{-LM_{min}}\right)$$

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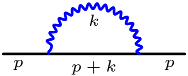

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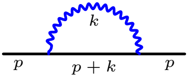

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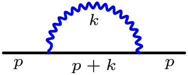

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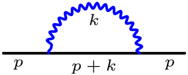

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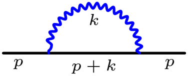

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- in the massless case, with the external particle on–shell $p^2 = -m_0^2$, the power law FVE come from the *infrared* $k^2 = 0$ *singularity*

$$\Delta\Sigma(p, L) = \frac{1}{L^3} \sum_{\boldsymbol{k}} \int \frac{dk_0}{2\pi} \frac{1}{\{2p \cdot k + k^2\}\, k^2}$$

$$= \sum_{\boldsymbol{q}} \int \frac{dq_0}{2\pi} \frac{1}{\{2Lp \cdot q + q^2\}\, q^2} \sim \frac{1}{2L} \left(\sum_{\boldsymbol{q}} \int \frac{dq_0}{2\pi} \frac{1}{p \cdot q\, q^2} \right) ,$$

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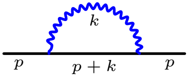

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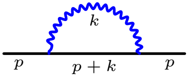

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- in general we have

$$\frac{1}{L^3} \sum_{\boldsymbol{k}} \int \frac{dk_0}{2\pi} \frac{1}{k^\alpha} = O(L^{\alpha-4})$$

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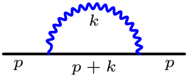
$$= \sum_{\boldsymbol{q}} \int \frac{dq_0}{2\pi} \frac{1}{\{2Lp \cdot q + q^2\}\, q^2} \sim \frac{1}{2L} \left(\sum_{\boldsymbol{q}} \int \frac{dq_0}{2\pi} \frac{1}{p \cdot q\, q^2} \right) ,$$



$$k = \frac{q}{L}$$

- in general we have

$$\frac{1}{L^3} \sum_{\boldsymbol{k}} \int \frac{dk_0}{2\pi} \frac{1}{k^\alpha} = O(L^{\alpha-4})$$

- by starting from $\alpha = 4$ one has infrared divergences: these do not appear in the case of the corrections to the masses but the infrared logs plague the extraction of S–matrix elements at intermediate stages of the calculations

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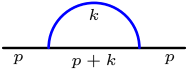

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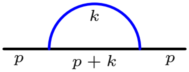

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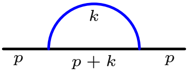

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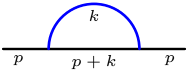

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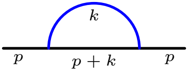

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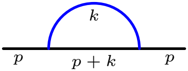

what we are going to learn:

- if we are interested in physical processes in which P (stable because of flavour) is the *only hadron* that appears in the external states, provided that we renormalize the parameters of the theory, we can write down effective field theories in which the degrees of freedom associated with other hadrons do not appear explicitly
- let's try to better understand this statement by starting, again, with a very simple example. . .

- let's consider a two–point function in QCD where the interpolating operator $P(x)$ has, for example, the quantum numbers of a flavoured pseudoscalar meson, e.g. $P(x) = \bar{b}(x)\gamma_5 u(x)$

$$C_{PP}(p) = \int d^4x \, e^{-ip\cdot x} \, T\langle 0|P(x)\, P^\dagger(0)|0\rangle$$

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- by performing a spectral–analysis of this object we have

$$C_{PP}(p) = \sum_n \frac{|\langle 0|P(0)|n(\boldsymbol{p})\rangle|^2}{2E_n(\boldsymbol{p})} \int dt\, e^{-itp_0}\, \left\{\theta(t)e^{-tE_n(\boldsymbol{p})} + \theta(-t)e^{tE_n(\boldsymbol{p})}\right\}$$

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$$= \sum_n \frac{|\langle 0|P(0)|n(\boldsymbol{p})\rangle|^2}{p_0^2 + \boldsymbol{p}^2 + [E_n^2(\boldsymbol{p}) - \boldsymbol{p}^2]}$$

$$= \frac{|\langle 0|P(0)|P(\mathbf{0})\rangle|^2}{p^2 + m_P^2} \left\{1 + \frac{p^2 + m_P^2}{|\langle 0|P(0)|P(\mathbf{0})\rangle|^2} \sum_{n\neq P} \frac{|\langle 0|P(0)|n(\mathbf{0})\rangle|^2}{p^2 + M_n^2}\right\}$$

where in the last line we have already assumed that we are in the infinite volume (the sum has to be read as the integral of the spectral density and M_n is the rest mass of the multi–particle state)

- if we now normalize the correlator,

$$\Delta^{-1}(p) = \left\{ \frac{C_{PP}(p)}{|\langle 0|P(0)|P(\mathbf{0})\rangle|^2} \right\}^{-1} \quad = \quad Z^{-1}(p^2)(p^2 + m_P^2)$$

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- we have that the function $Z^{-1}(p^2)$ is analytic if $-p^2 < (m_P + 2m_\pi)^2$ and can be Taylor–expanded in this region

$$Z^{-1}(p^2) = 1 + \sum_{n=1}^{\infty} \frac{z_n}{n!} (p^2 + m_P^2)^n \ , \qquad z_n = \left. \frac{d^n Z^{-1}(p^2)}{dp^{2n}} \right|_{p^2 = -m_P^2} = O\left(\frac{1}{(2m_\pi)^{2n}} \right)$$

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- it is now trivial to write down an effective theory that reproduces the correlator in the region $-p^2 < (m_P + 2m_\pi)^2$,

$$\mathcal{L}_{kin}(x) = \phi_P^\star(x) \left\{ -\partial^2 + m_P^2 + \sum_{n=1}^{\infty} \frac{z_n}{n!} (-\partial^2 + m_P^2)^{n+1} \right\} \phi_P(x)$$

- let's have a closer look at

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- at the single particle pole, the only one we are interested in, all the other terms are irrelevant... by using the leading order equations of motions we can write a lagrangian that has the very same physical content

$$\mathcal{L}_{kin}(x) \mapsto \mathcal{L}_{pt} = \phi^\star_P(x)\left\{-\partial^2 + m_P^2\right\}\phi_P(x)$$

  this is the lagrangian of a free point–like particle

- if we now turn–on QED, in the region $-p^2 < (m_P + 2m_\pi)^2$ we can still write an effective theory for a single hadron but, in this case, we have to include explicitly the degrees of freedom associated with the photons

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- the structure dependent terms are contained in $\mathcal{L}_{str}(x)$: since we are working at $O(e^2)$ we shall exclude operators that contain more than two powers of A_μ, this simplifies our task, but what kind of operators can go into $\mathcal{L}_{str}(x)$?

- among the possible gauge–invariant operators we have for example

$$O_D(x) = \phi^\star_P(x) D_\mu D_\nu D_\mu D_\nu \phi_P(x)$$

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- we can write

$$O_D(x) = F_{\mu\nu} \phi_P^\star(x) D_\mu D_\nu \phi_P(x) + \phi_P^\star(x) D_\nu D^2 D_\nu \phi_P(x)$$

$$= \frac{3}{2} F_{\mu\nu}^2 \phi_P^\star(x)\phi_P(x) + \partial_\mu F_{\mu\nu} \phi_P^\star(x) D_\nu \phi_P(x) + \phi_P^\star(x) D^4 \phi_P(x)$$

- by using now the equations of motions

$$\left\{-D^2 + m_0^2\right\}\phi_P(x) = 0\ , \qquad \partial_\mu F_{\mu\nu}(x) = -ie\left\{2\phi_P^\star D_\nu \phi_P - \partial_\nu\left(\phi_P^\star \phi_P\right)\right\}(x)$$

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- O_D becomes to a linear combinations of the following operators

$$O_1(x) = \phi_P^\star \phi_P(x)\, F_{\mu\nu}^2\ ,$$

$$O_A(x) = \partial_\mu F_{\mu\nu}(x)\partial_\nu\left\{\phi_P^\star \phi_P\right\}(x) = \partial_\nu\left\{\phi_P^\star \phi_P \partial_\mu F_{\mu\nu}\right\}(x)\ ,$$

$$O_B(x) = \left\{\partial_\mu F_{\mu\nu}\right\}^2(x) = \partial_\nu\left\{F_{\nu\rho}\partial_\mu F_{\mu\rho}\right\}^2(x)$$

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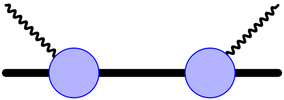
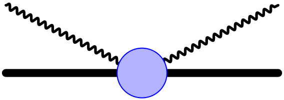
- the operators O_A and O_B do not contribute to the action (total derivatives) and we are left with O_1

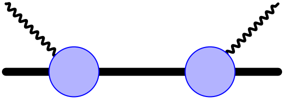
- by using similar manipulations, it is easy to convince ourselves that a basis for structure dependent operators that contribute at $O(e^2)$ is given by

$$O_1^n(x) = \phi_P^\star \phi_P(x) \partial^{2n} \left\{F_{\mu\nu}\right\}^2 ,$$

$$O_2^n(x) = \phi_P^\star \phi_P(x) \partial^{2n} \left\{\partial_\mu F_{\mu\nu}\right\}^2 , \qquad n = 0, 1, \cdots$$







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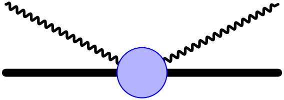

- therefore

$$\mathcal{L}_{str}(x) = \phi_P^\star \sum_{n=0}^{\infty} \left\{ c_1^n \left[\partial^{2n} F_{\mu\nu} F_{\mu\nu} \right] + c_2^n \left[\partial^{2n} \partial_\mu F_{\mu\alpha} \partial_\nu F_{\nu\alpha} \right] \right\} \phi_P(x)$$

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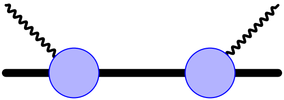

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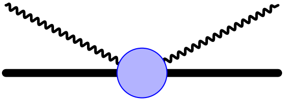

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$$\mathcal{L}_{str}(x) = \phi_P^\star \sum_{n=0}^{\infty} \left\{ c_1^n \left[\partial^{2n} F_{\mu\nu} F_{\mu\nu}\right] + c_2^n \left[\partial^{2n} \partial_\mu F_{\mu\alpha} \partial_\nu F_{\nu\alpha}\right] \right\} \phi_P(x)$$

- there is much more physics here, the coefficients $c_{1,2}^n$ can be fixed by matching the (derivatives of the) Compton scattering amplitude

$$c_{1,2}^n \quad \leftarrow \quad \mathcal{T}_\ell = \frac{d^n}{d(\boldsymbol{k}^2)^n} T_{\mu\mu}(i|\boldsymbol{k}|, \boldsymbol{k}) \Big|_{|\boldsymbol{k}|=0}$$

Lucini et al. JHEP02 (2016) 076, arXiv:1612.00199

$$\frac{m_P(L)-m_P}{m_P}=\frac{e^2}{4\pi}\left\{\frac{Q_P^2\xi(1)}{2m_PL}+\frac{Q_P^2\xi(2)}{\pi(m_PL)^2}-\frac{1}{4\pi m_PL^4}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell(2\ell)!\,\mathcal{T}_\ell\,\xi(2+2\ell)}{\ell!\,L^{2(\ell-1)}}\right\}+O(L^{-\infty})$$

the aim of this lecture is to explain how to obtain this formula, the FVE on the mass of a stable hadron in QCD+QED$_C$, by using an approach based on relativistic effective theories

before starting the journey, however, let's analyze the structure of the formula:

- the first two terms are *universal*: depend only on the mass and on the charge of the particle and coincide with those of a pointlike hadron
- a part from the universal $1/L$ term, there are no inverse odd powers of L
- structure dependent terms start to contribute at $O(L^{-4})$

- the $\xi(n)$ are geometric objects that depend on the spatial boundary conditions

$$\xi(s)=\sum_{\boldsymbol{n}\neq\boldsymbol{0}}\frac{e^{i\pi(n_1+n_2+n_3)}}{|\boldsymbol{n}|^s}$$

- the structure dependent coefficients are related to the derivatives of the forward Compton scattering amplitude

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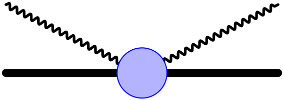
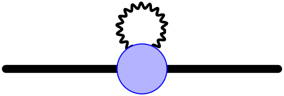


- let's analyze now the contributions to $m_P(L)$ coming from the operators

$$O_1^n(x) = \phi_P^\star \phi_P(x) \partial^{2n} \left\{F_{\mu\nu}\right\}^2 \ ,$$

$$O_2^n(x) = \phi_P^\star \phi_P(x) \partial^{2n} \left\{\partial_\mu F_{\mu\nu}\right\}^2 \ , \qquad n = 0, 1, \cdots$$



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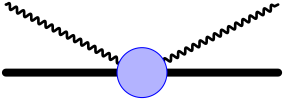

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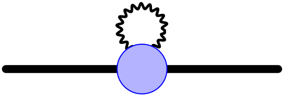
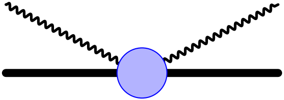


- these are tadpole contributions to the self–energy that, since the operators $\partial^{2n} F_{\mu\nu} F_{\mu\nu}$ and $\partial^{2n} \partial_\mu F_{\mu\alpha} \partial_\nu F_{\nu\alpha}$ have at least two derivatives of the photon field, are such that

$$\Delta m_P^2(L) \propto \frac{1}{L^3} \sum_{\boldsymbol{k}} \int \frac{dk^0}{2\pi} \frac{c_1^n O(k^{2n+2}) + c_2^n O(k^{2n+4})}{k^2} = c_1^n O\left(\frac{1}{L^{2n+4}}\right) + c_2^n O\left(\frac{1}{L^{2n+6}}\right)$$



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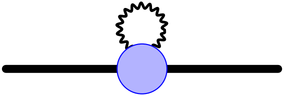

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- remember the power–counting

$$\frac{1}{L^3} \sum_{\boldsymbol{k}} \int \frac{dk_0}{2\pi} \frac{1}{k^\alpha} = O(L^{\alpha-4})$$

many details are missing and have to be filled–in, but I hope it is clear at this level that:

- FVE on the mass of a stable hadron in QCD+QED can be analyzed by using the effective field theory lagrangian

$$\mathcal{L}(x) = \phi_P^\star(x)\left\{-D^2 + m_P^2\right\}\phi_P(x) + \frac{1}{4}F_{\mu\nu}^2(x)$$

$$+\,\phi_P^\star \sum_{n=0}^{\infty}\left\{c_1^n\,\left[\partial^{2n}F_{\mu\nu}F_{\mu\nu}\right] + c_2^n\,\left[\partial^{2n}\partial_\mu F_{\mu\alpha}\partial_\nu F_{\nu\alpha}\right]\right\}\phi_P(x)$$

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- with the result

$$\frac{\Delta m_P(L)}{m_P} = \frac{e^2}{4\pi}\left\{\frac{Q_P^2\xi(1)}{2m_P L} + \frac{Q_P^2\xi(2)}{\pi(m_P L)^2} - \frac{1}{4\pi m_P L^4}\sum_{\ell=1}^{\infty}\frac{(-1)^\ell(2\ell)!\,\mathcal{T}_\ell\,\xi(2+2\ell)}{\ell!\,L^{2(\ell-1)}}\right\} + O(L^{-\infty})$$

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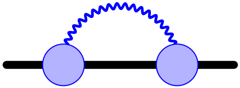

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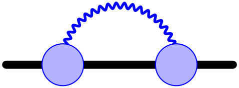
$$\tilde{A}_\mu(t, \boldsymbol{k} = \boldsymbol{0}) = \int_L d^3x A_\mu(t, \boldsymbol{x}) = 0$$



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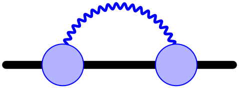

- this is a non–local constraint that generates $1/L^3$ FVE, remember

$$\left\{ \frac{1}{L^3} \sum_{\boldsymbol{k}\neq\boldsymbol{0}} - \int \frac{d^3k}{(2\pi)^3} \right\} f(\boldsymbol{k}) = -\frac{f(\boldsymbol{0})}{L^3} + \sum_{\boldsymbol{n}\neq\boldsymbol{0}} \tilde{f}(\boldsymbol{n}L)$$

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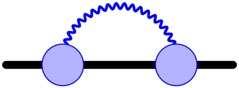

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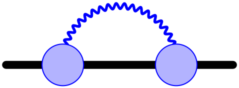

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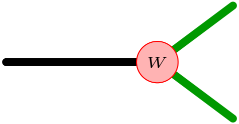
- all the (multi)particle states that can propagate between the two electromagnetic currents contribute to this term and cannot be neglected in an effective field theory description if one is interested in L^{-3} FVE (see also **Fodor et al. Phys.Lett.B 755 (2016)**)

RM123+SOTON, PRD 91 (2015), PRD 95 (2017), arXiv:1612.00199

- along the same lines discussed in the case of the masses one can also understand FVE in leptonic decays $P^- \mapsto \ell\bar{\nu}(\gamma)$



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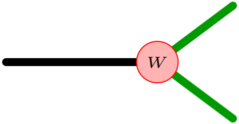

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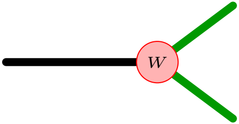

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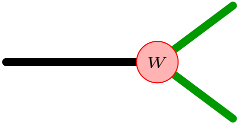

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  - structure–dependent terms can be classified with the techniques of the previous slides, e.g.

$$\mathcal{O}_V(x) = c_V\, \epsilon^{\mu\nu\rho\sigma} D_\mu \phi_P\, F_{\nu\rho}\, \bar{\ell}\gamma_\sigma \nu , \qquad F_{\nu\rho} = \partial_\nu A_\rho - \partial_\rho A_\nu , \qquad \text{sub–leading in } \frac{E_\gamma}{m_\pi}$$

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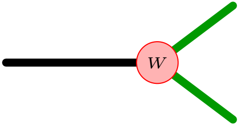

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  - it comes out that in the expansion around vanishing photon energies both the leading (infrared divergent) and the next-to-leading terms are universal: this implies that $O(L^{-1})$ finite volume effects are universal