The next generation techniques for anisotropic flow analyses

arXiv:2101.05619, arXiv:2106.05760

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Technical University of Munich
“Offshell-2021”, 08/07/2021

European Research Council
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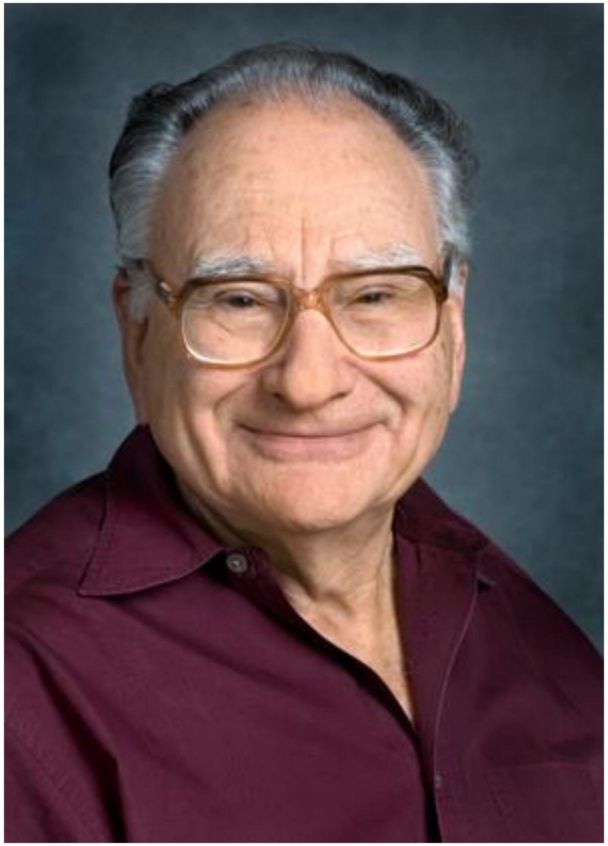
Tribute to Art Poskanzer (1931-2021)



Group photo from the workshop
'Initial State Fluctuations and Final State Correlations',
held at ECT* in Trento in July, 2012

Outline

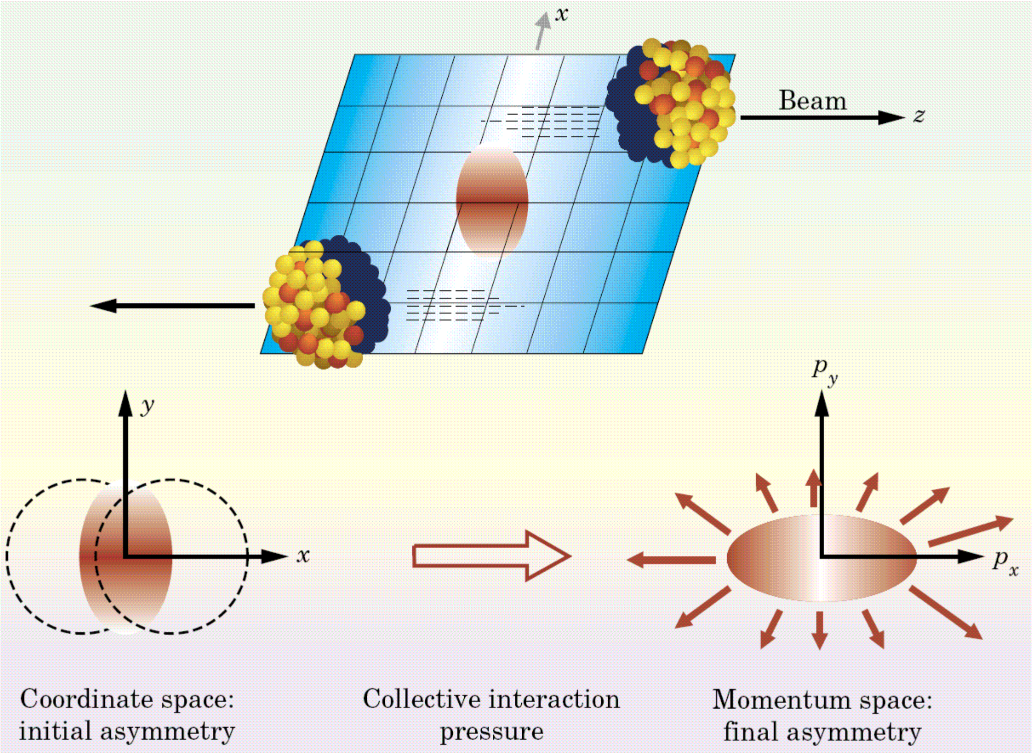
- Introduction
  - Anisotropic flow
  - Multiparticle correlations and cumulants
- 'Multivariate cumulants in flow analyses: The Next Generation'

  A. Bilandzic, M. Lesch, C. Mordasini, F. Taghavi, arXiv:2101.05619
- 'Event-by-event cumulants of azimuthal angles'

  A. Bilandzic, arXiv:2106.05760, prepared for **'Offshell-2021'**

Anisotropic flow phenomenon

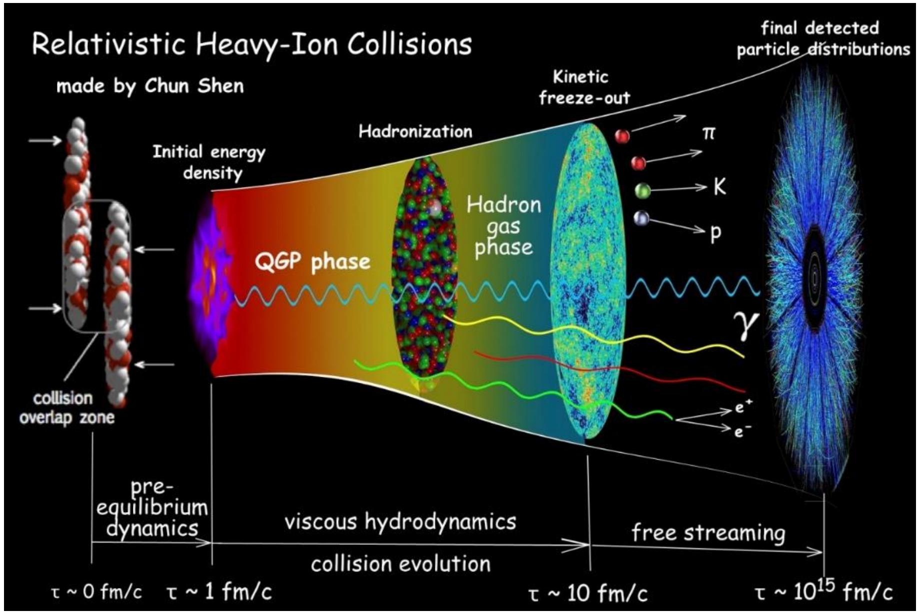
- Transfer of anisotropy from the initial coordinate space into the final momentum space via the thermalized medium:



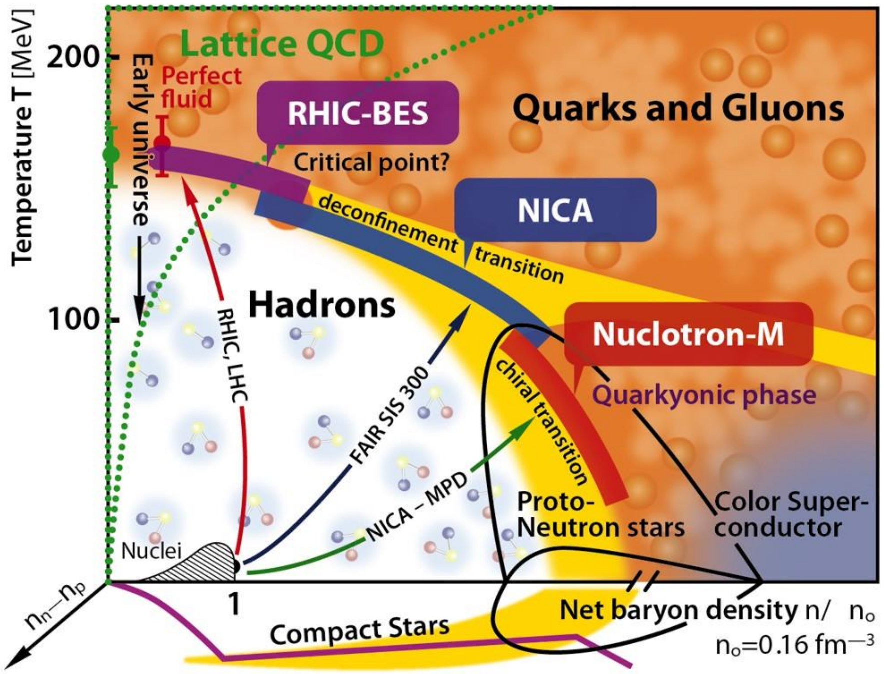
- J.Y. Ollitrault, Phys. Rev. D **46** (1992) 229

Quark-Gluon Plasma (QGP)

Heavy-ion evolution



Phase diagram of QCD



- In the QGP stage quarks are deconfined
- Anisotropic flow is a sensitive probe of QGP properties (e.g. of its shear viscosity)

Fourier series

- We use Fourier series to describe anisotropic emission of particles in the plane transverse to the beam direction after each heavy-ion collision:

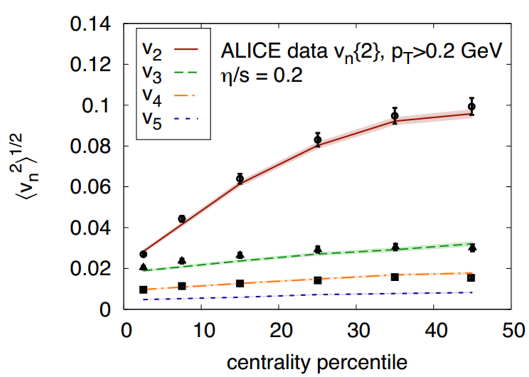
$$f(\varphi) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{\infty} v_n \cos[n(\varphi - \Psi_n)]\right]$$

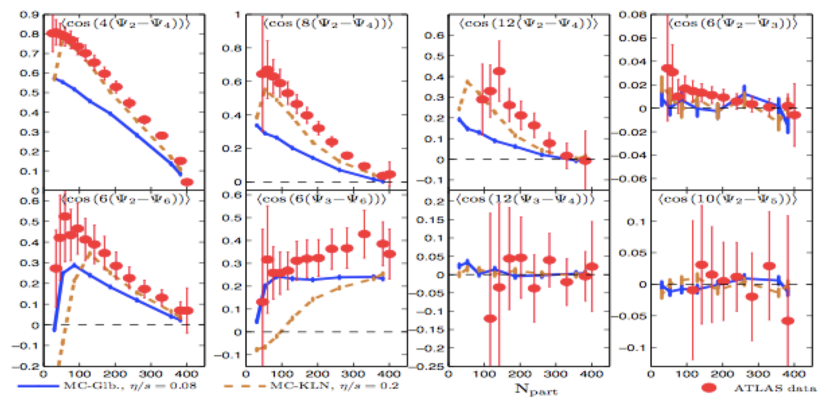
- $\boldsymbol{v_n}$: flow amplitudes
- $\boldsymbol{\Psi_n}$: symmetry planes
- Anisotropic flow is quantified with $\boldsymbol{v_n}$ and $\boldsymbol{\Psi_n}$
  - $\boldsymbol{v_1}$ is directed flow
  - $\boldsymbol{v_2}$ is elliptic flow
  - $\boldsymbol{v_3}$ is triangular flow
  - $\boldsymbol{v_4}$ is quadrangular flow, etc.

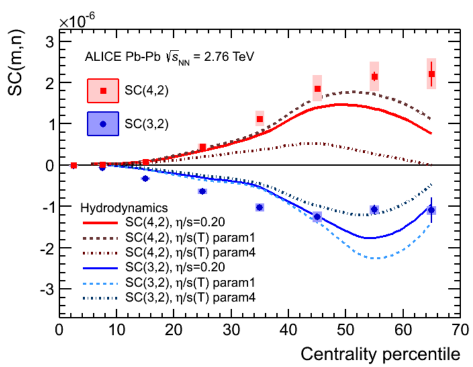
S. Voloshin and Y. Zhang, Z.Phys. C70 (1996) 665-672

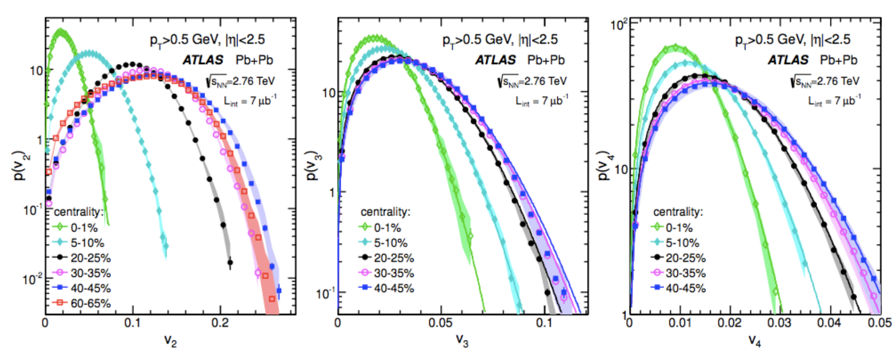
Flow observables

- Individual flow harmonics: $v_1, v_2, v_3, v_4, \ldots$
- Correlations between harmonics: $\langle v_m^2 v_n^2 \rangle - \langle v_m^2 \rangle \langle v_n^2 \rangle$
- Symmetry plane correlations: $\langle \cos[mn(\Psi_m - \Psi_n)] \rangle$
- Probability density function: $\mathbf{P}(v_n)$
- …



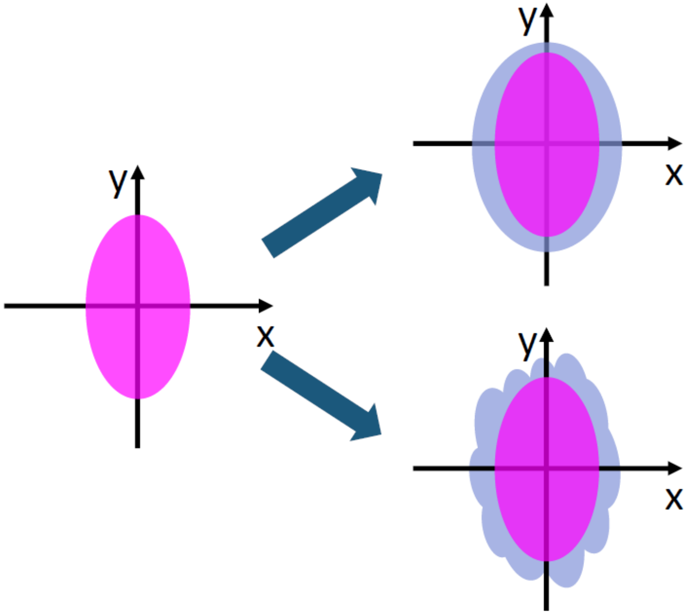






Shear vs. bulk viscosities

- Can we separate the effects of shear (η) and bulk (ξ) viscosities?



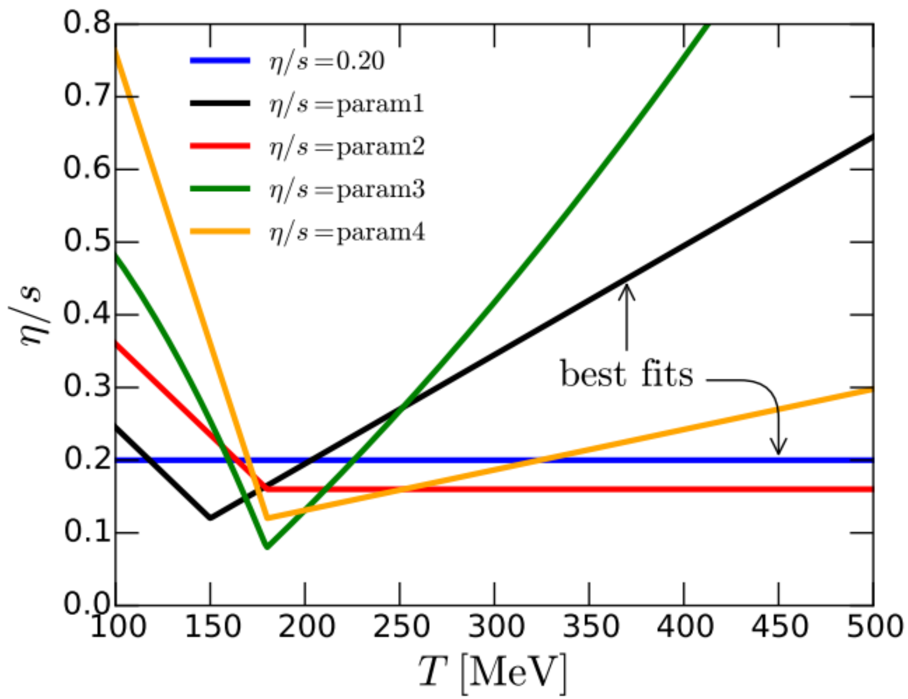
Credits: C. Mordasini

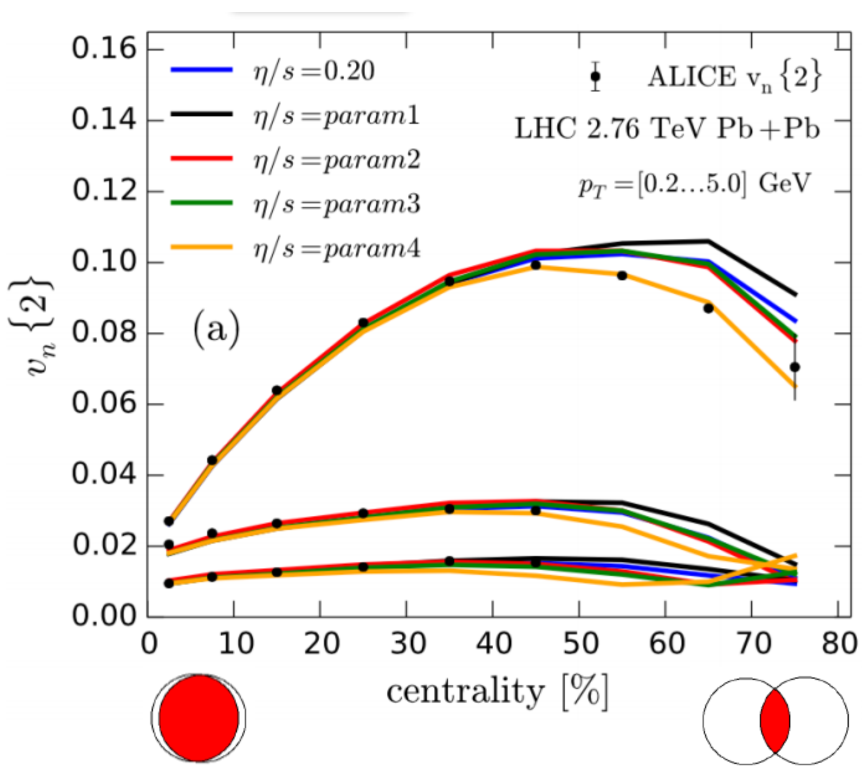
- Isotropic fluctuations
  - Neighbouring layers move at equal velocities
  - Generally preserves the ellipse shape
  - Main sensitivity to ξ/s
- Shape fluctuations
  - Neighbouring layers move at different velocities
  - Sensitivity to η/s

- Can we develop new observables with potential to separate these different sources of fluctuations?

‘Classical’ flow observables

- Insensitivity to temperature dependence of η/s





H. Niemi, K. J. Eskola, R. Paatelainen, Phys. Rev. C 93, 024907 (2016)

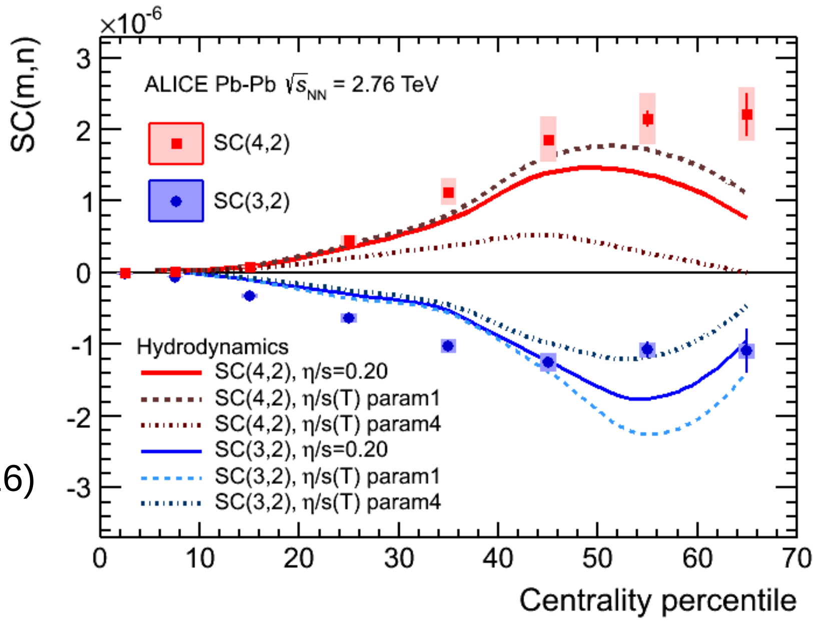
Symmetric Cumulants SC(*m*,*n*)

- How to quantify experimentally the correlation between two different flow amplitudes?
  - Symmetric Cumulants (Section IVC in Phys. Rev. C **89** (2014) no.6, 064904)

$$
\begin{aligned}
\langle\langle\cos(m\varphi_1+n\varphi_2-m\varphi_3-n\varphi_4)\rangle\rangle_c &= \langle\langle\cos(m\varphi_1+n\varphi_2-m\varphi_3-n\varphi_4)\rangle\rangle \\
&\quad -\langle\langle\cos[m(\varphi_1-\varphi_2)]\rangle\rangle\,\langle\langle\cos[n(\varphi_1-\varphi_2)]\rangle\rangle \\
&= \left\langle v_m^2 v_n^2\right\rangle - \left\langle v_m^2\right\rangle\left\langle v_n^2\right\rangle
\end{aligned}
$$

- SC observables are sensitive to differential η/s(T) parametrizations
- Individual flow amplitudes are dominated by averages ⟨η/s(T)⟩
- Independent constraints both on initial conditions and QGP properties

ALICE Collaboration, Phys. Rev. Lett. 117, 182301 (2016)



Multiparticle correlations and cumulants

Multiparticle azimuthal correlations

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- The most general result, which relates multiparticle azimuthal correlators and flow degrees of freedom:

$$\langle \cos[n_1\varphi_1 + \cdots + n_k\varphi_k)] \rangle = v_{n_1} \cdots v_{n_k} \cos[n_1\Psi_{n_1} + \cdots + n_k\Psi_{n_k})]$$

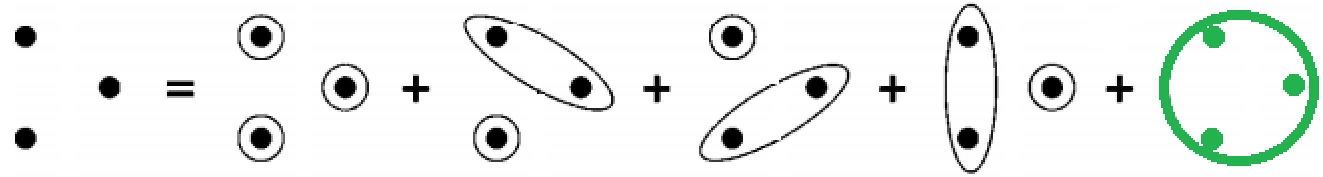
R. S. Bhalerao, M. Luzum and J.-Y. Ollitrault, Phys. Rev. C **84** 034910 (2011)

- A lot of non-trivial and independent flow observables for different choices of harmonics $\boldsymbol{n_i}$
  - Examples: 2- and 4-particle azimuthal correlations

$$\begin{aligned}\langle \cos[n(\varphi_1 - \varphi_2)] \rangle &= v_n^2 \\ \langle \cos[n(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4)] \rangle &= v_n^4\end{aligned}$$

Multiparticle cumulants

- Consider the following diagram representation of the most general decomposition of 3-particle correlation:



- The very last term, which cannot be decomposed further, is by definition 3-particle cumulant
  - Cumulant term exists for any number of particles, it is always unique, and it isolates the genuine collective contribution
- Introduced in flow analyses by Ollitrault *et al*

N. Borghini, P. M. Dinh, and J.-Y. Ollitrault, Phys. Rev. C **63**, 054906 (2001)
N. Borghini, P. M. Dinh, and J.-Y. Ollitrault, Phys. Rev. C **64**, 054901 (2001)

3-particle cumulants in general

- Working recursively from higher to lower orders, we eventually have 3-particle cumulant expressed in terms of measured 3-, 2-, and 1-particle averages

$$
\begin{aligned}
\langle X_1 X_2 X_3 \rangle_c &= \langle X_1 X_2 X_3 \rangle \\
&- \langle X_1 X_2 \rangle \langle X_3 \rangle - \langle X_1 X_3 \rangle \langle X_2 \rangle - \langle X_2 X_3 \rangle \langle X_1 \rangle \\
&+ 2 \langle X_1 \rangle \langle X_2 \rangle \langle X_3 \rangle
\end{aligned}
$$

- General result, true for any choice of stochastic observables
- In the same way, cumulants can be expressed in terms of measurable averages for any number of observables
  - The number of terms grows rapidly with the number of observables

Multivariate cumulants in flow analyses: The Next Generation

A. Bilandzic, M. Lesch, C. Mordasini, F. Taghavi, https://arxiv.org/abs/2101.05619

Fundamental properies of cumulants

- We reviewed everything from scratch and supported proofs for:
  - Statistical independence
  - Reduction
  - Semi-invariance
  - Homogeneity
  - Multilinearity
  - Additivity
  - …

For all technical details, see Section II and Appendix A in arXiv:2101.05619

- The main strategy in the paper is divided into two steps:
  - Confront all existing observables in the field named cumulants with these fundamental properties
  - For the ones which fail to satisfy them, provide the alternative definitions which do satisfy all fundamental properties of cumulants

AB, M. Lesch, C. Mordasini, F. Taghavi, arXiv:2101.05619

Main conclusions

- **The main conclusion #1:** One cannot perform cumulant expansion in one set of stochastic observables, then in the resulting expression perform the transformation to some new set of observables, and then claim that the cumulant properties are preserved in the new set of observables
  - After such transformation, the fundamental properties of cumulants are lost in general

- **The main conclusion #2:** The formal properties of cumulants are valid only if there are no underlying symmetries due to which some terms in the cumulant expansion would vanish identically
  - Due to symmetries, $\left\langle e^{in\varphi_i}\right\rangle = 0$, $\left\langle e^{in(\varphi_i+\varphi_j)}\right\rangle = 0$, etc., all vanish
  - There are no obvious symmetries for $\left\langle v_k^2\right\rangle$, $\left\langle v_k^2 v_l^2\right\rangle$, etc., to vanish

AB, M. Lesch, C. Mordasini, F. Taghavi, arXiv:2101.05619

Choice of fundamental observable

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- Cumulants as used in flow analyses in the last ~20 years:
  - Cumulant expansion is performed on azimuthal angles
  - Azimuthal correlators which are not isotropic are dropped
  - The final result is merely re-expressed in terms of flow degrees of freedom v_n and Ψ_n via analytic relation

$$\langle\cos[n_1\varphi_1+\cdots+n_k\varphi_k)]\rangle = v_{n_1}\cdots v_{n_k}\cos[n_1\Psi_{n_1}+\cdots+n_k\Psi_{n_k})]$$

R. S. Bhalerao, M. Luzum and J.-Y. Ollitrault, Phys. Rev. C **84** 034910 (2011)

- Few additional remarks:
  - Cumulants of v_n and v_n^2 are in general different
  - v_n and Ψ_n have different properties (e.g. with respect to Lorentz invariance)

Cumulants in flow analyses

- Traditionally, azimuthal angles are chosen as fundamental observables in the cumulant expansion
- Based on this approach, one derives e.g. $v_n\{4\}$ observable
  - It gives an estimate for flow harmonic v_n by using 4-particle azimuthal cumulant (not 4-p azimuthal correlator!)
  - For large multiplicities, $v_n\{4\}$ suppresses well nonflow effects
- But this traditional approach yields to very weird and inconsistent results when applied to the correlations of different flow amplitudes

Why the traditional cumulant approach with azimuthal angles which worked so well in the past fails when applied in the studies of correlations of different flow amplitudes?

New paradigm in flow analyses

- Example: General 2-particle cumulant

$$\langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$$

  - Traditional approach: fundamental observable is an angle

$$X_1 \to e^{in\varphi_1}\,, \quad X_2 \to e^{-in\varphi_2}$$

  - New approach: fundamental observable is a flow amplitude

$$X_1 \to v_n^2\,, \quad X_2 \to v_m^2$$

- Two approaches yield **accidentally the same results** for lower order Symmetric Cumulants SC(k,l), but differ for higher orders SC(k,l,m), SC(k,l,m,n), ...
  - Which one is correct?

Reconciliation (1/2)

- From the fundamental properties of cumulants (statistical independence, reduction, semi-invariance, homogeneity, multilinearity, additivity, etc.), we have established the following two simple necessary conditions:

1. We take temporarily that in the definition of $\lambda(X_1, \ldots, X_N)$ all observables $X_1, \ldots, X_N$ are statistically independent and factorize all multivariate averages into the product of single averages $\Rightarrow$ the resulting expression must reduce identically to 0;
2. We set temporarily in the definition of $\lambda(X_1, \ldots, X_N)$ all observables $X_1, \ldots, X_N$ to be the same and equal to X $\Rightarrow$ for the resulting expression it must hold that

$$\lambda(aX + b) = a^N \lambda(X)\,, \tag{23}$$

where a and b are arbitrary constants, and N is the number of observables in the starting definition of $\lambda(X_1, \ldots, X_N)$.

Multivariate observable is a multivariate cumulant only if it satisfies both above requirements (arXiv:2101.05619)

Reconciliation (2/2)

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- New flow observables ('the next generation') which do satisfy all formal mathematical properties of cumulants:
  - 'Symmetric and Asymmetric Cumulants' (genuine multiharmonic correlations of flow amplitudes)
    - See arXiv:1901.06968 and Sec. V in arXiv:2101.05619
  - 'Cumulants of symmetry plane correlations'
    - See Sec. VI in arXiv:2101.05619
  - 'Event-by-event cumulants of azimuthal angles'
    - See Sec. IV in arXiv:2101.05619 and arXiv: 2106.05760
    - **The main topic of today's talk**

Example #1: Higher-order SC

- New paradigm:

  1/ Cumulant expansion directly on flow amplitudes v^2:

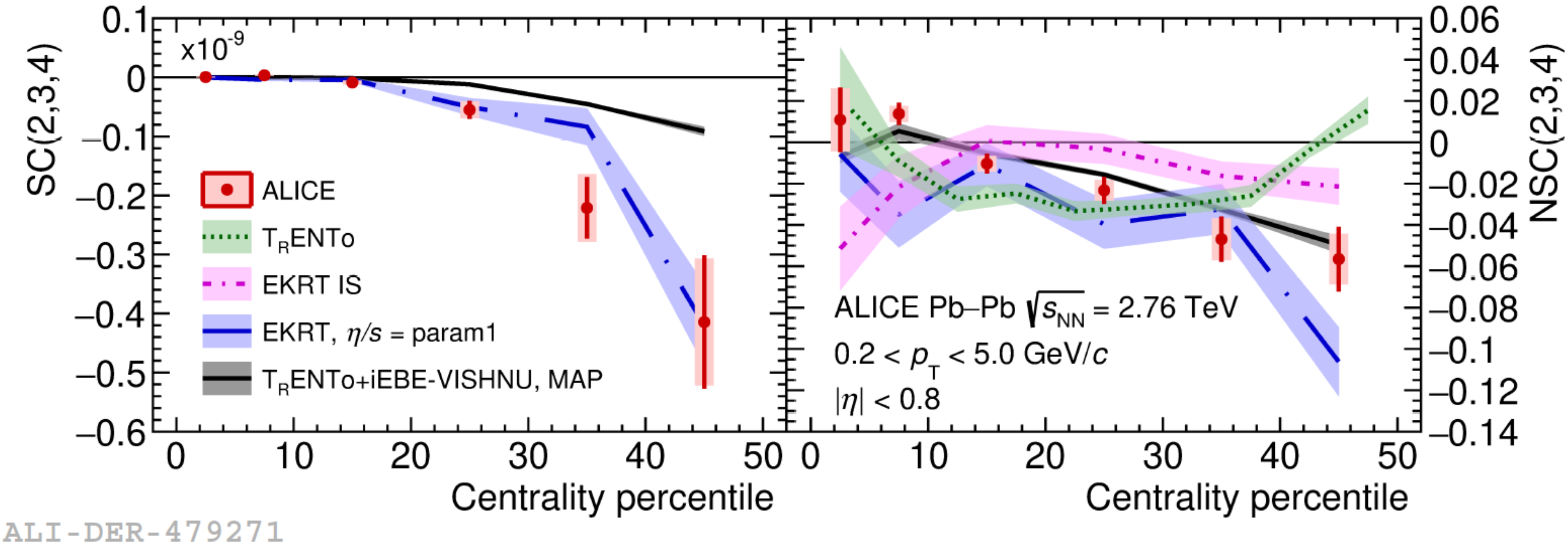
$$\mathrm{SC}(k,l,m) \equiv \left\langle v_k^2 v_l^2 v_m^2 \right\rangle - \left\langle v_k^2 v_l^2 \right\rangle \left\langle v_m^2 \right\rangle - \left\langle v_k^2 v_m^2 \right\rangle \left\langle v_l^2 \right\rangle - \left\langle v_l^2 v_m^2 \right\rangle \left\langle v_k^2 \right\rangle + 2 \left\langle v_k^2 \right\rangle \left\langle v_l^2 \right\rangle \left\langle v_m^2 \right\rangle$$

  2/ Azimuthal angles are used merely to build an estimator for the above observable:

$$\begin{aligned}
\mathrm{SC}(k,l,m) = &\, \langle\langle\cos[k\varphi_1+l\varphi_2+m\varphi_3-k\varphi_4-l\varphi_5-m\varphi_6]\rangle\rangle \\
&- \langle\langle\cos[k\varphi_1+l\varphi_2-k\varphi_3-l\varphi_4]\rangle\rangle \langle\langle\cos[m(\varphi_5-\varphi_6)]\rangle\rangle \\
&- \langle\langle\cos[k\varphi_1+m\varphi_2-k\varphi_5-m\varphi_6]\rangle\rangle \langle\langle\cos[l(\varphi_3-\varphi_4)]\rangle\rangle \\
&- \langle\langle\cos[l\varphi_3+m\varphi_4-l\varphi_5-m\varphi_6]\rangle\rangle \langle\langle\cos[k(\varphi_1-\varphi_2)]\rangle\rangle \\
&+ 2\,\langle\langle\cos[k(\varphi_1-\varphi_2)]\rangle\rangle \langle\langle\cos[l(\varphi_3-\varphi_4)]\rangle\rangle \langle\langle\cos[m(\varphi_5-\varphi_6)]\rangle\rangle
\end{aligned}$$

$\mathbf{SC}(k, l, m)$ in ALICE

- Run 1 (2010) Pb–Pb collisions at $\sqrt{s_{\mathrm{NN}}}$ = 2.76 TeV



Comparison with the state-of-the-art models: Development of genuine multiharmonic correlations during hydrodynamic evolution

ALICE Collaboration, arXiv 2101.02579

Example #2: Asymmetric Cumulants (AC)

- Generalization of Symmetric Cumulants
- Fundamental observable is v^2
  - Choice driven by experiment: The simplest flow moment which can be estimated experimentally with azimuthal correlations
  - All terms are preserved in the cumulant expansion

$$\mathrm{AC}_{2,1}(m,n) = \langle v_m^4 v_n^2\rangle - \langle v_m^4\rangle\langle v_n^2\rangle - 2\langle v_m^2 v_n^2\rangle\langle v_m^2\rangle + 2\langle v_m^2\rangle^2\langle v_n^2\rangle,$$

$$\begin{aligned}\mathrm{AC}_{3,1}(m,n) = {}& \langle v_m^6 v_n^2\rangle - \langle v_m^6\rangle\langle v_n^2\rangle - 3\langle v_m^2 v_n^2\rangle\langle v_m^4\rangle - 3\langle v_m^4 v_n^2\rangle\langle v_m^2\rangle \\ &+ 6\langle v_m^4\rangle\langle v_m^2\rangle\langle v_n^2\rangle + 6\langle v_m^2 v_n^2\rangle\langle v_m^2\rangle^2 - 6\langle v_m^2\rangle^3\langle v_n^2\rangle,\end{aligned}$$

$$\begin{aligned}\mathrm{AC}_{4,1}(m,n) = {}& \langle v_m^8 v_n^2\rangle - \langle v_m^8\rangle\langle v_n^2\rangle - 4\langle v_m^2 v_n^2\rangle\langle v_m^6\rangle - 6\langle v_m^4 v_n^2\rangle\langle v_m^4\rangle \\ &+ 6\langle v_m^4\rangle^2\langle v_n^2\rangle - 4\langle v_m^6 v_n^2\rangle\langle v_m^2\rangle + 8\langle v_m^6\rangle\langle v_m^2\rangle\langle v_n^2\rangle \\ &+ 24\langle v_m^2 v_n^2\rangle\langle v_m^4\rangle\langle v_m^2\rangle + 12\langle v_m^4 v_n^2\rangle\langle v_m^2\rangle^2 \\ &- 36\langle v_m^4\rangle\langle v_m^2\rangle^2\langle v_n^2\rangle - 24\langle v_m^2 v_n^2\rangle\langle v_m^2\rangle^3 + 24\langle v_m^2\rangle^4\langle v_n^2\rangle,\end{aligned}$$

$$\begin{aligned}\mathrm{AC}_{2,1,1}(k,l,m) = {}& \langle v_k^4 v_l^2 v_m^2\rangle - \langle v_k^4 v_l^2\rangle\langle v_m^2\rangle - \langle v_k^4 v_m^2\rangle\langle v_l^2\rangle - \langle v_k^4\rangle\langle v_l^2 v_m^2\rangle \\ &+ 2\langle v_k^4\rangle\langle v_l^2\rangle\langle v_m^2\rangle - 2\langle v_k^2 v_l^2\rangle\langle v_k^2 v_m^2\rangle - 2\langle v_k^2 v_l^2 v_m^2\rangle\langle v_k^2\rangle \\ &+ 4\langle v_k^2 v_l^2\rangle\langle v_k^2\rangle\langle v_m^2\rangle + 4\langle v_k^2 v_m^2\rangle\langle v_k^2\rangle\langle v_l^2\rangle \\ &+ 2\langle v_k^2\rangle^2\langle v_l^2 v_m^2\rangle - 6\langle v_k^2\rangle^2\langle v_l^2\rangle\langle v_m^2\rangle.\end{aligned}$$

Event-by-event cumulants of azimuthal angles

A. Bilandzic, arXiv:2106.05760, prepared for **‘Offshell-2021’**

Main idea

- Traditionally, cumulants of azimuthal angles are defined in terms of all-event averages:

$$c_n\{2\} \equiv \langle\langle e^{in(\varphi_1-\varphi_2)}\rangle\rangle - \langle\langle e^{in\varphi_1}\rangle\rangle\langle\langle e^{-in\varphi_2}\rangle\rangle$$

  - Due to underlying symmetries, all terms which are not isotropic, are averaged out to 0 => fundamental properties of cumulants are lost

- New approach: cumulants of azimuthal angles are defined in terms of single-event averages:

$$\kappa_{11} \equiv \langle e^{in(\varphi_1-\varphi_2)}\rangle - \langle e^{in\varphi_1}\rangle\langle e^{-in\varphi_2}\rangle$$

  - **‘Event-by-event cumulants of azimuthal angles’**
  - Despite underlying symmetries, all terms in the expansion are kept
  - Interpretation and meaning of cumulants is completely different

Role of symmetries

- Cumulant is identically 0 if one of the variables in it is statistically independent of the others
  - This holds true over the whole phase space
- Reflection symmetry
  - Cumulant can be accidentally 0 due to symmetry $f(x,y) = f(x,-y)$ but in this case they are never 0 over the whole phase space
- Permutation symmetry
  - Marginal distributions of different variables are the same
- Frame independence
- Relabelling
  - Azimuthal correlators of different variables are estimated from exactly the same sample => properties of cumulants are lost

Sec. II in arXiv:2106.05760

Event-by-event cumulants of azimuthal angles

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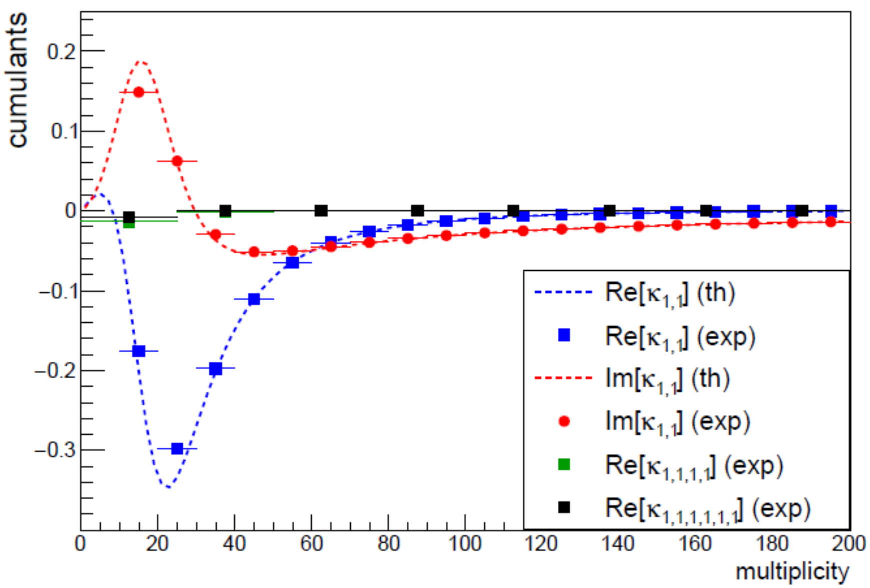
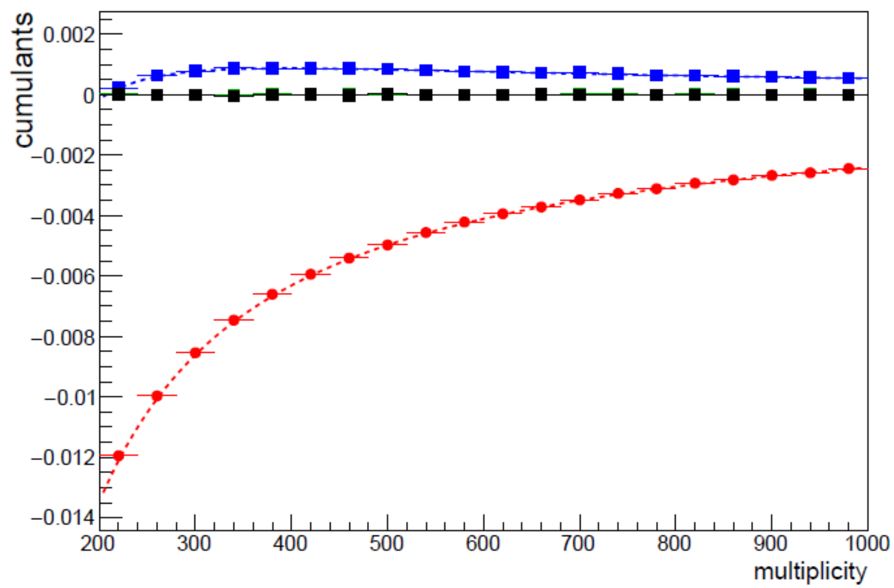
- Toy Monte Carlo study: Azimuthal angles are sampled pair-wise => only 2-particle correlations are present
  - New 2-particle cumulants correctly recover the theoretical input
  - New 4- and 6-particle cumulants are identically 0





- Works only if we have full handle over combinatorial background

Sec. IV in arXiv:2101.05619

Role of combinatorial background

- The origin of the problem: The dataset is randomized
  - Particles emitted in the same process: ‘signal’
  - Particles taken from different processes: ‘background’
- In most analyses in high-energy physics, ‘signal’ and ‘background’ are separated by using mixed-event technique
  - Not applicable for azimuthal angles, due to random event-by-event fluctuations of impact parameter vector
- Can we instead analytically solve the problem of combinatorial backround?

Statistical independence

- If two random observables, x and y, are statistically independent, then their joined 2-particle probability density function (p.d.f.) fully factorizes into **marginal p.d.f.'s**:

$$f_{xy}(x,y) = f_x(x) f_y(y)$$

- Two marginal p.d.f.'s are defined as:

$$\begin{aligned} f_x(x) &\equiv \int_Y f_{xy}(x,y)\,dy \\ f_y(y) &\equiv \int_X f_{xy}(x,y)\,dx \end{aligned}$$

- In general, $f_x(x)$ and $f_y(y)$ are two different p.d.f.'s

Two-particle correlations

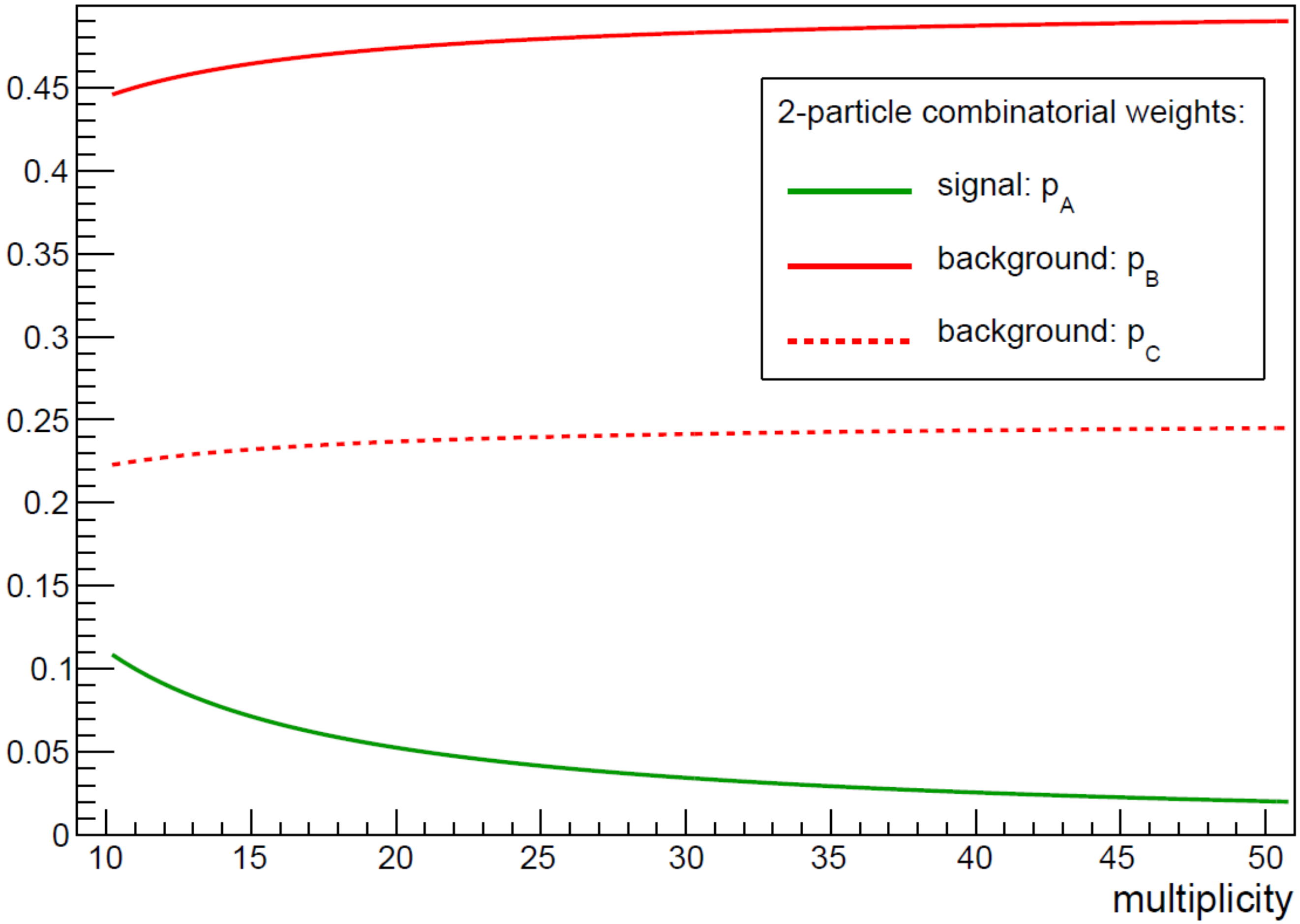
- If particles are emitted from p.d.f. $f(x,y)$, and if the resulting sample is randomized, what is the p.d.f. $w(x,y)$ which describes the final randomized sample?
- The most general result:

$$w(x,y) = p_A f_{xy}(x,y) + p_B f_x(x) f_y(y) + p_C \left[f_x(x) f_x(y) + f_y(x) f_y(y) \right]$$

- Universal combinatorial weights: p_A, p_B, p_C
- Marginal p.d.f.'s: $f_x(x)$, $f_y(y)$

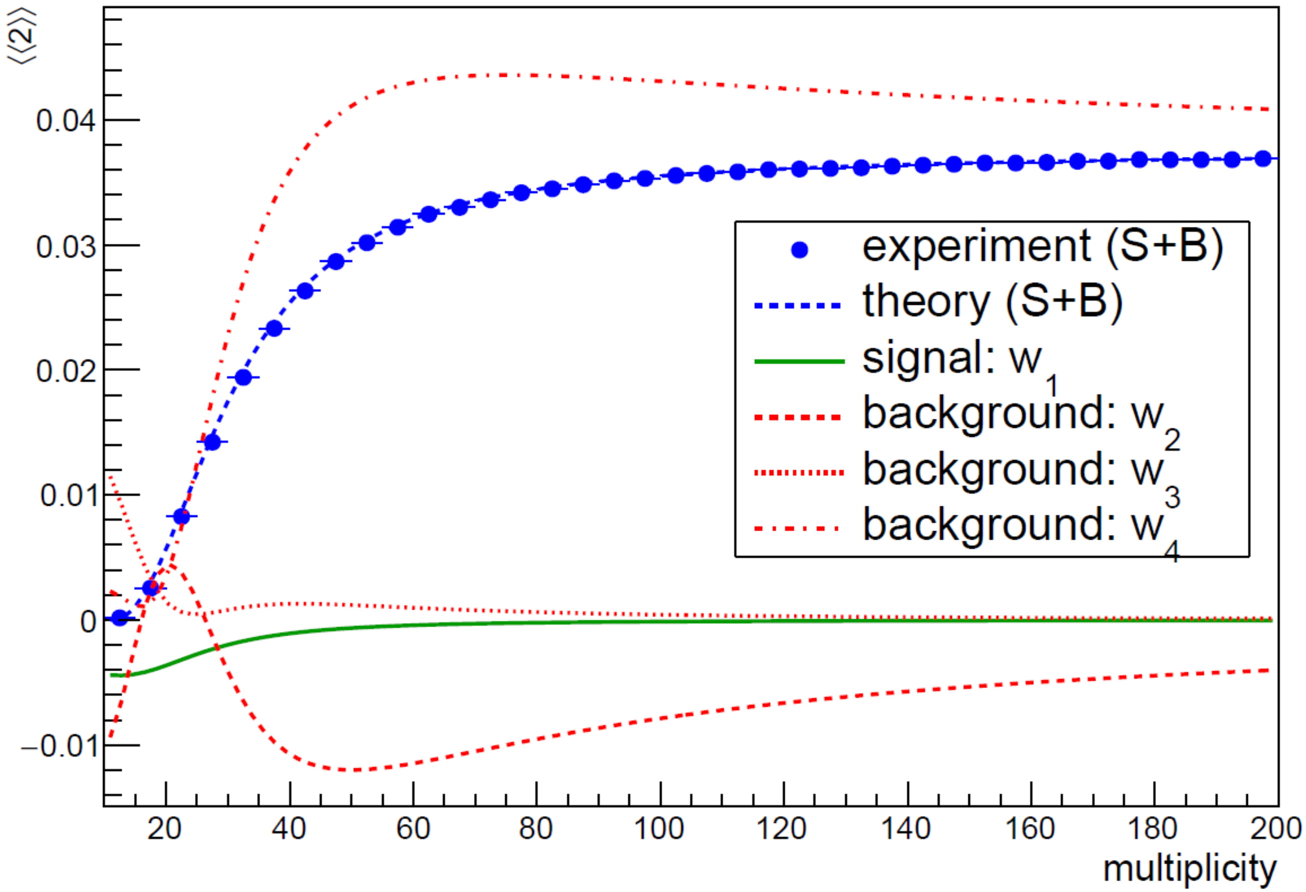
Combinatorial weights (2-particle)

- Universal and depend only on multiplicity:



Toy Monte Carlo (2-particle)

- Quantitative description of 2-particle azimuthal correlation in the randomized sample



Sec. IIIB in arXiv:2106.05760

Three-particle correlations

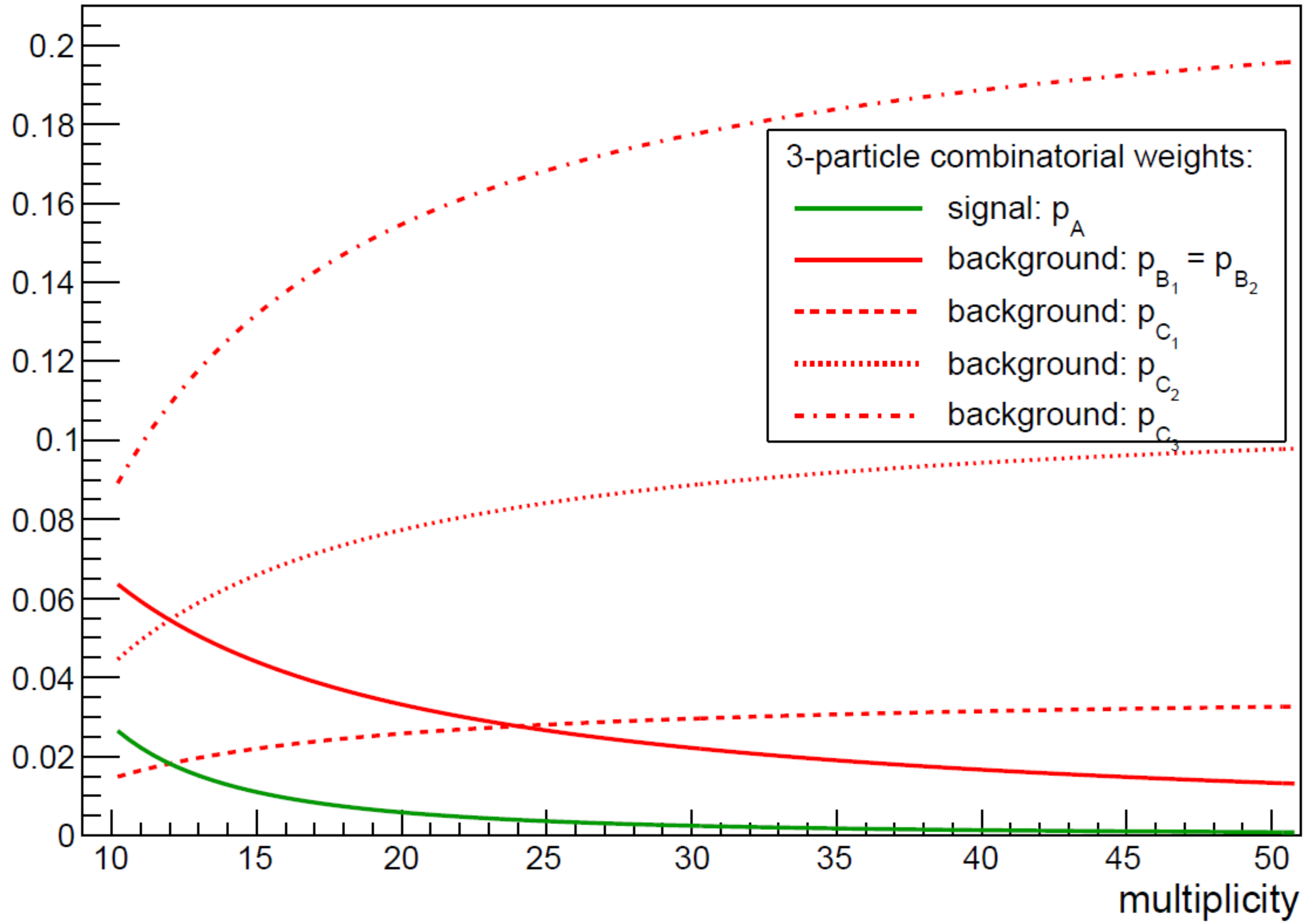
- If particles are emitted from p.d.f. $f(x,y,z)$, and if the resulting sample is randomized, what is the p.d.f. $w(x,y,z)$ which describes the final randomized sample?
- The most general result:

$$\begin{aligned} w(x,y,z) = {} & p_A f_{xyz}(x,y,z) \\ & + p_{B_1}\left[f_{xy}(x,y)f_x(z) + f_{xy}(x,y)f_y(z) + f_{xz}(x,z)f_x(y)\right. \\ & \quad \left. + f_{xz}(x,z)f_z(y) + f_{yz}(y,z)f_y(x) + f_{yz}(y,z)f_z(x)\right] \\ & + p_{B_2}\left[f_{xy}(x,y)f_z(z) + f_{xz}(x,z)f_y(y) + f_{yz}(y,z)f_x(x)\right] \\ & + p_{C_1}\left[f_x(x)f_x(y)f_x(z) + f_y(x)f_y(y)f_y(z) + f_z(x)f_z(y)f_z(z)\right] \\ & + p_{C_2}\left[f_x(x)f_x(z)f_y(y) + f_x(x)f_x(y)f_z(z) + f_y(y)f_y(z)f_x(x)\right. \\ & \quad \left. + f_y(y)f_y(x)f_z(z) + f_z(z)f_z(y)f_x(x) + f_z(z)f_z(x)f_y(y)\right] \\ & + p_{C_3} f_x(x)f_y(y)f_z(z)\,. \end{aligned}$$

- Universal combinatorial weights: $p_A, p_{B1}, p_{B2}, p_{C1}, p_{C2}, p_{C3}$
- Marginal p.d.f.'s: $f_x(x), f_y(y), f_z(z), f_{xy}(x,y), f_{xz}(x,z), f_{yz}(y,z)$

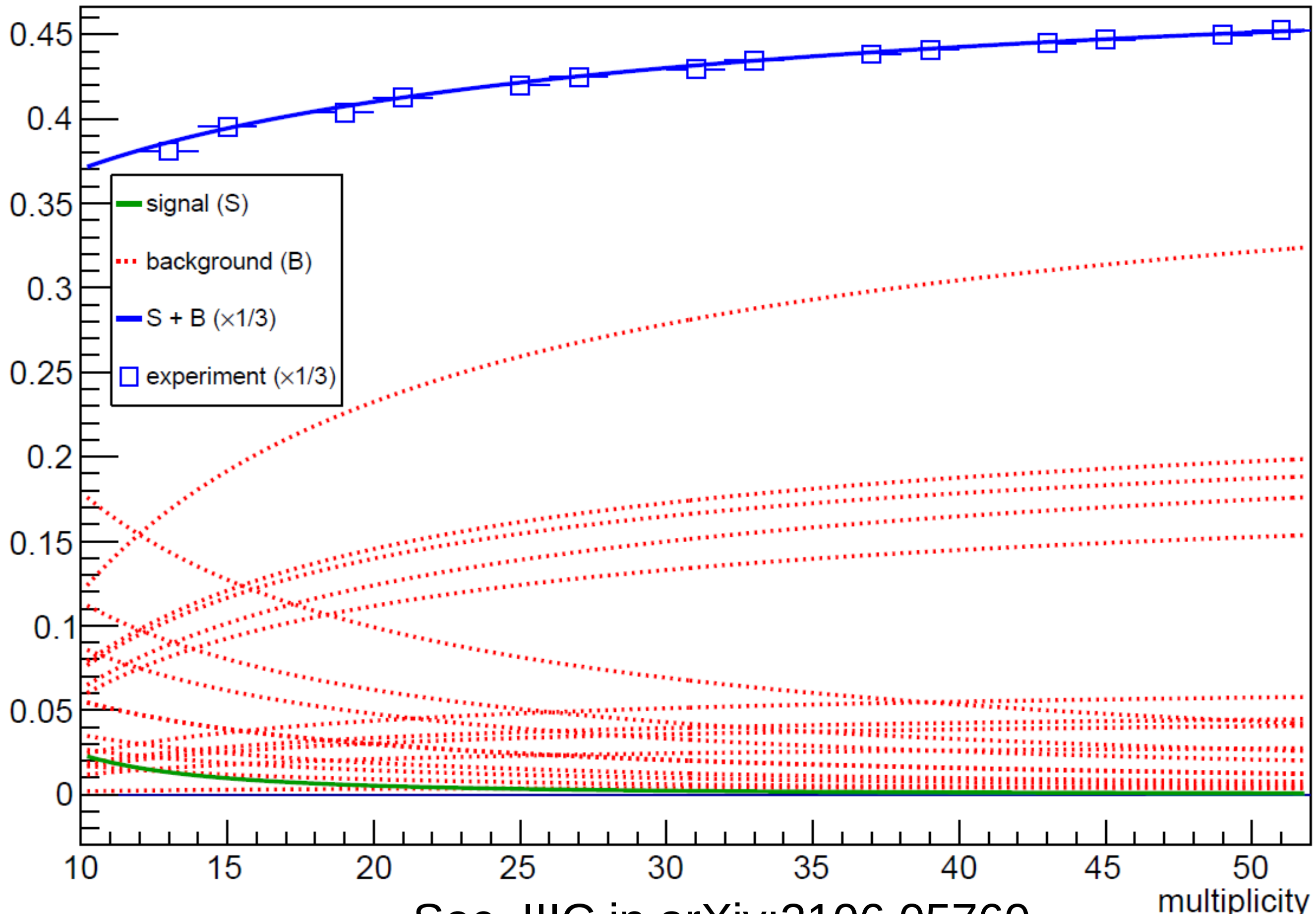
Combinatorial weights (3-particle)

- Universal and depend only on multiplicity:



Toy Monte Carlo (3-particle)

- Quantitative description of 3-particle azimuthal correlation in the randomized sample



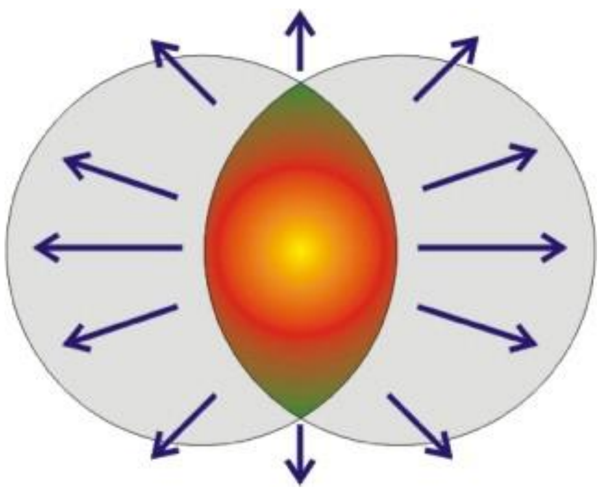
Sec. IIIC in arXiv:2106.05760

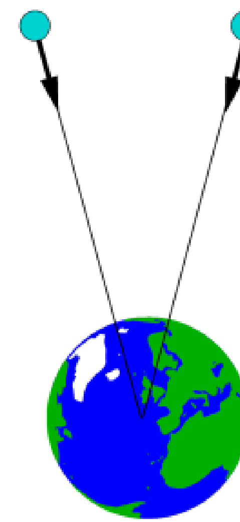
Thanks!

Backup slides

The ‘flow principle’

- Correlations among all produced particles are induced solely by correlation of each single particle to the collision geometry





- Analogy with the falling bodies in gravitational field (rhs)
- Whether or not particle are emitted simultaneously, or one by one, trajectories are the same
  - These are **statistically independent** trajectories

Example: 2-particle cumulants

- How to use this new recipe in practice?
- Reminder: General 2-particle cumulant

$$\langle X_1 X_2 \rangle_c = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$$

As an elementary example, we perform these two checks for the simplest two-variate cumulant, $\kappa(X_1, X_2) = \langle X_1 X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle$. The first check leads immediately to $\kappa(X_1, X_2) = \langle X_1 \rangle \langle X_2 \rangle - \langle X_1 \rangle \langle X_2 \rangle = 0$. Following the second check, we have that $\kappa(X) = \langle X^2 \rangle - \langle X \rangle^2$, so that:

$$\begin{aligned} \kappa(aX+b) &= \left\langle (aX+b)^2 \right\rangle - \langle aX+b \rangle^2 \\ &= a^2 \langle X^2 \rangle + 2ab \langle X \rangle + b^2 - a^2 \langle X \rangle^2 - 2ab \langle X \rangle - b^2 \\ &= a^2 \left(\langle X^2 \rangle - \langle X \rangle^2 \right) \\ &= a^2 \kappa(X) \,, \end{aligned} \tag{24}$$

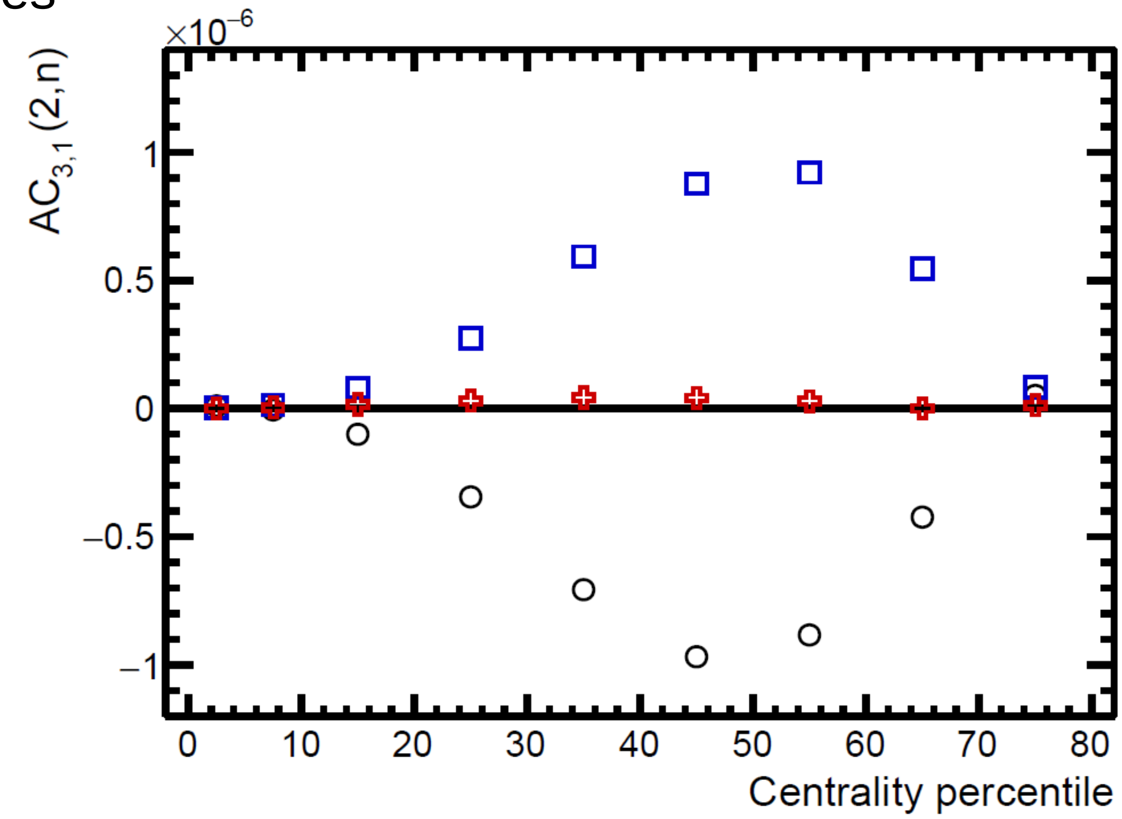
as it should be for a two-variate cumulant.

- Despite its simplicity, most of observables named cumulants in the field fail to satisfy this new recipe. What are the alternatives?

AB, M. Lesch, C. Mordasini, F. Taghavi, arXiv:2101.05619

Example Monte Carlo study

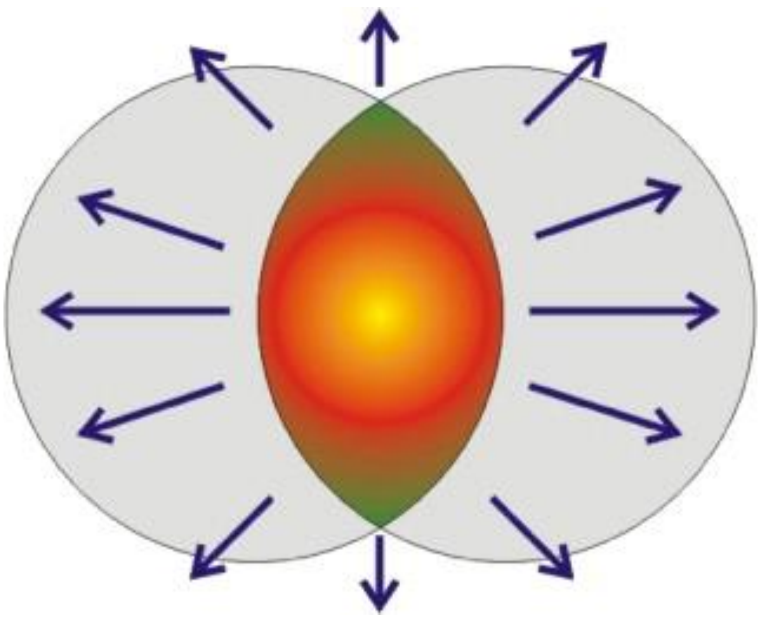
- Measurements of new AC observables is feasible, and they will provide further new and independent constraints on initial conditions and QGP properties

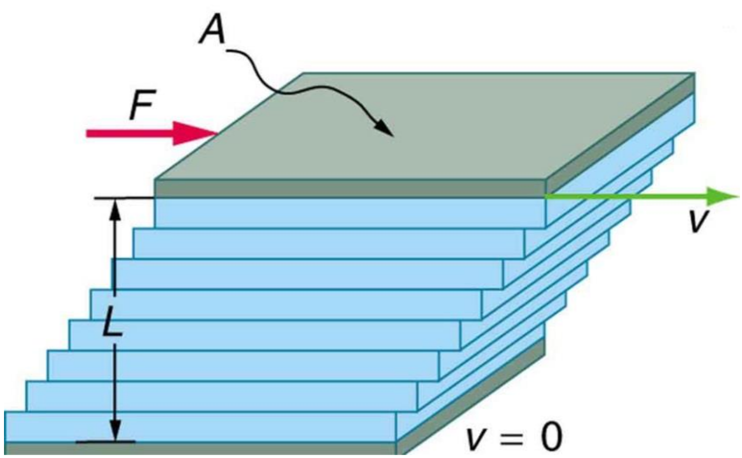


AB, M. Lesch, C. Mordasini, F. Taghavi, Section VI in arXiv:2101.05619

Hydro flow in-plane

- Non-trivial effect which is sensitive to transport coefficients of QGP (e.g. its shear viscosity)





If anisotropic flow has developed, neighboring layers are moving at different relative velocities, parallel displacement is opposed by shear viscosity

large anisotropic flow ⇔ small shear viscosity

Statistical independence, back to flow

- If anisotropic flow is the only source of correlations between produced particles, their joint n-variate p.d.f.

$$f(\varphi_1,\ldots,\varphi_n)$$

factorizes into product of n single-particle marginal p.d.f.'s:

$$f(\varphi_1,\ldots,\varphi_n) = f_{\varphi_1}(\varphi_1)\cdots f_{\varphi_n}(\varphi_n)$$

- From ‘flow principle’: All marginal p.d.f.’s are the same, and therefore parameterized by the same Fourier series:

$$\begin{aligned} f(\varphi_1,\ldots,\varphi_n) &= f(\varphi_1)\cdots f(\varphi_n) \\ f(\varphi) &= \frac{1}{2\pi}\left[1+2\sum_{n=1}^{\infty} v_n \cos[n(\varphi-\Psi_n)]\right] \end{aligned}$$