New results on exact solutions of relativistic, dissipative hydrodynamics

TAMÁS CSÖRGŐ, GÁBOR KASZA

DAY OF FEMTOSCOPY, GYÖNGYÖS

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arXiv:2003.08859

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Introduction

We provide **new, exact solutions of relativistic Navier-Stokes and Isreal-Stewart equations**

Relativistic Navier-Stokes and Israel-Stewart theory: possible theoretical formulations of dissipative, relativistic hydrodynamics

In these solutions: great amount of freedom in the choice of shear and bulk viscosity, and heat conductivity coefficients

Special velocity profile: spherically symmetric Hubble flow

If $\tau >> \tau_0$, **both of the new solutions approach a perfect fluid solution**:

arXiv:nucl-th/0306004 (T. Csörgő, L. P. Csernai, Y. Hama, T. Kodama)

The seeking of these solutions were motivated by:

arXiv:1909.02498 (M. Nagy, M. Csanád, Z. Jiang, T. Csörgő)

Relativistic hydrodynamics (Navier-Stokes)

Local conservation of the four momentum and the particle number:

$$\partial_\mu \left(n u^\mu \right) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

The energy-momentum tensor is:

$$T^{\mu\nu} = (\varepsilon + p)\, u^\mu u^\nu - p g^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu} - \Delta^{\mu\nu} \Pi$$

The heat current (with the heat conductivity λ):

$$q^\mu = \lambda \left(g^{\mu\nu} - u^\mu u^\nu \right) \left(\partial_\nu T - T u^\rho \partial_\rho u_\nu \right)$$

ζ: bulk viscosity

η: shear viscosity

The following terms describes the viscous effects:

$$\pi^{\mu\nu} = \eta \left[\Delta^{\mu\rho} \partial_\rho u^\nu + \Delta^{\nu\rho} \partial_\rho u^\mu \right] - \frac{2}{d} \eta \Delta^{\mu\nu} \partial_\rho u^\rho$$

$$\Pi = -\zeta \partial_\rho u^\rho$$

Relativistic hydrodynamics (Israel-Stewart)

Local conservation of the four momentum and the particle number:

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To close the equation system:

EoS:

In this work: *κ=const.*

ζ: bulk viscosity

η: shear viscosity

Hubble-type solutions: scale variable

Hubble-type velocity field: $u^\mu = \frac{x^\mu}{\tau} = \gamma\left(1, \frac{r_x}{t}, \frac{r_y}{t}, \frac{r_z}{t}\right)$

Scale equation: $u^\mu \partial_\mu s = 0$

Directional scale variables: $s_x = \frac{r_x}{t},\ s_y = \frac{r_y}{t},\ s_z = \frac{r_z}{t}$

Satisfy the scale equation separately: $u^\mu \partial_\mu s_i = \partial_\tau s_i = 0$

Hubble-type solutions: equations to solve

Navier-Stokes theory

Continuity equation: $\partial_\tau n + \frac{d}{\tau} n = 0$

Energy conservation: $\partial_\tau p + \left(1 + \frac{1}{\kappa}\right) \frac{d}{\tau} p = \frac{d^2}{\tau^2} \frac{\zeta}{\kappa}$

Euler-equation: $p\tau - \zeta d = \phi(\tau)$

Entropy equation: $\partial_\tau \sigma + \frac{d}{\tau} \sigma = \frac{d^2}{\tau^2} \frac{\zeta}{T} \geq 0$

M. Nagy, M. Csanád, Z. Jiang, T. Csörgő: arXiv:1909.02498

Heat conductivity and shear viscosity cancelled!

Israel-Stewart theory

Continuity equation: $\partial_\tau n + \frac{d}{\tau} n = 0$

Energy conservation: $\partial_\tau p + \left(1 + \frac{1}{\kappa}\right) \frac{d}{\tau} p = -\frac{d}{\tau} \frac{\Pi}{\kappa}$

Bulk pressure: $\Pi = -\zeta \frac{d}{\tau} - \tau_\Pi \dot{\Pi}$

Euler-equation: $p + \Pi = \Psi(\tau)$

Entropy equation: $\partial_\tau \sigma + \frac{d}{\tau} \sigma = -\frac{d}{\tau} \frac{\Pi}{T} \geq 0$

T. Csörgő, G. K.: arXiv:2003.08859

Hubble-type solutions: equations to solve

Navier-Stokes theory $\tau_\Pi \to 0$, or Π is constant

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T. Csörgő, G. K.: arXiv:2003.08859

Hubble-type solutions: ansatz for the bulk viscosity

Navier-Stokes theory: ζ ~ p

$$\zeta = \zeta_0 \frac{p}{p_0}$$

Isreal-Stewart theory: ζ ~ Π

$$\zeta = \Pi \frac{\zeta_0}{\Pi_0}$$

Navier-Stokes theory

Continuity equation: $\partial_\tau n + \frac{d}{\tau} n = 0$

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Analytic solutions of NS equations, with κ=const

The solution of the pressure is: $$p(\tau) = p_0 \left(\frac{p_A}{p_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa}\right)}, \quad \frac{p_A}{p_0} = f_{A,0} = \exp\left[\frac{d^2\zeta_0}{\kappa_0 p_0 \tau_0}\right]$$

The temperature has a generalized form: $$T = T_0 \left(\frac{T_A}{T_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right)$$

T. Csörgő, G. K.: arXiv:2003.08859

Conserved charge, μ > 0

$$p = nT$$

$$\frac{T_A}{T_0} = \frac{p_A}{p_0} = f_{0,A} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0}\right)$$

$$n = n_0 \left(\frac{\tau_0}{\tau}\right)^d \mathcal{V}\left(s_x, s_y, s_z\right)$$

No conserved charge, μ = 0

$$p = \frac{T\sigma}{1+\kappa}$$

$$\frac{T_A}{T_0} = \left(\frac{\sigma_A}{\sigma_0}\right)^{\kappa_0} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0}\frac{1}{1+\kappa_0}\right)$$

$$\sigma = \sigma_0 \left(\frac{\sigma_A}{\sigma_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^d \mathcal{V}\left(s_x, s_y, s_z\right), \quad \frac{\sigma_A}{\sigma_0} = f_{0,A}^{\frac{\kappa_0}{1+\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{p_0 \tau_0}\frac{1}{1+\kappa_0}\right)$$

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T. Csörgő, G. K.: [arXiv:2003.08859](https://arxiv.org/abs/2003.08859)

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No conserved charge, μ = 0

$$\boxed{\mathcal{T}\left(s_x, s_y, s_z\right) = \frac{1}{\mathcal{V}\left(s_x, s_y, s_z\right)}}$$

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Analytic solutions of IS equations, with κ=const

Bulk viscosity:
$$\Pi(\tau) = \Pi_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\tau_\Pi}\frac{\zeta_0}{\Pi_0}} \exp\left(-\frac{\tau-\tau_0}{\tau_\Pi}\right)$$

Pressure:
$$p(\tau) = p_A \left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa}\right)} \left[1 + \frac{p_0 - p_A}{p_A} \cdot \frac{\Gamma\left(B, \frac{\tau}{\tau_\Pi}\right)}{\Gamma\left(B, \frac{\tau_0}{\tau_\Pi}\right)}\right]$$

Constants:
$$p_A = p_0 - \frac{\Pi_0 d}{\kappa}\left(\frac{\tau_0}{\tau_\Pi}\right)^{-B} \exp\left(\frac{\tau_0}{\tau_\Pi}\right) \Gamma\left(B, \frac{\tau_0}{\tau_\Pi}\right)$$
$$B = d\left(1 + \frac{1}{\kappa} - \frac{\zeta_0}{\Pi_0}\frac{1}{\tau_\Pi}\right)$$

T. Csörgő, G. K.: arXiv:2003.08859

Asymptotically perfect fluid solutions

In the $\tau >> \tau_0$ limit, both the NS and IS cases lead to the same asymptotic perfect fluid temperature profile and pressure:

$$T \sim T_A \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right) \qquad p \sim p_A \left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa_0}\right)}$$

If μ=0 the entropy density asymptotically equals to a perfect fluid form (and if μ≠0 the particle density is unchanged):

$$\sigma \sim \sigma_A \left(\frac{\tau_0}{\tau}\right)^{d} \mathcal{V}\left(s_x, s_y, s_z\right)$$

The bulk viscosity is absorbed to the asymptotic normalization constants!

The effect of bulk viscosity cancels!

T. Csörgő, L. P. Csernai, Y. Hama, T. Kodama: arXiv:nucl-th/0306004

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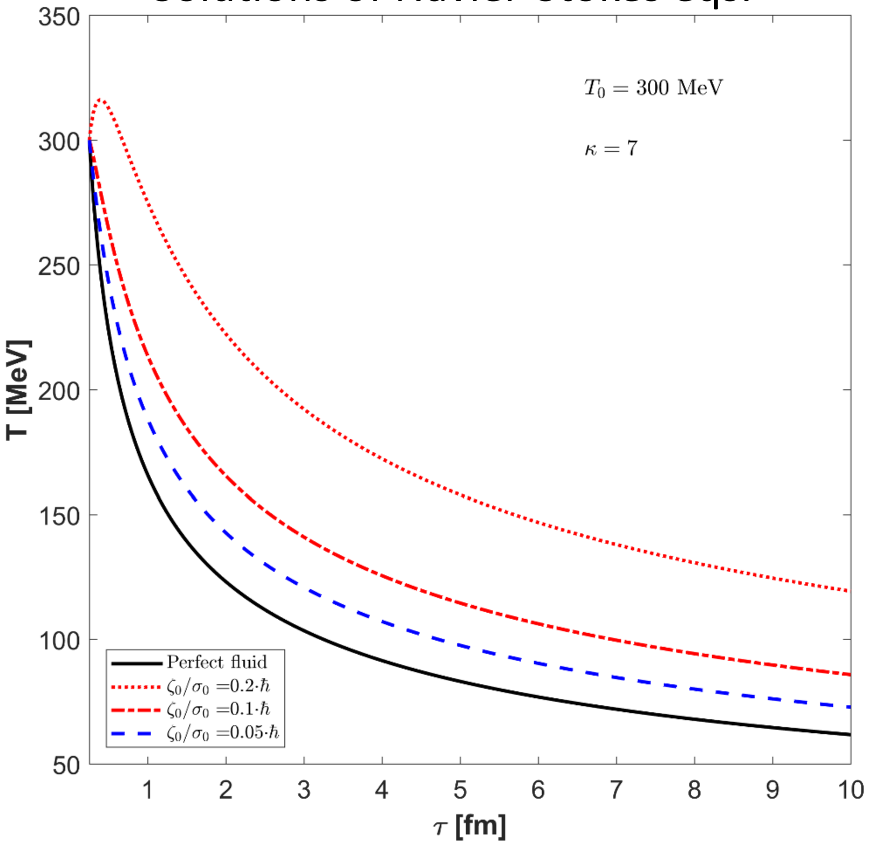
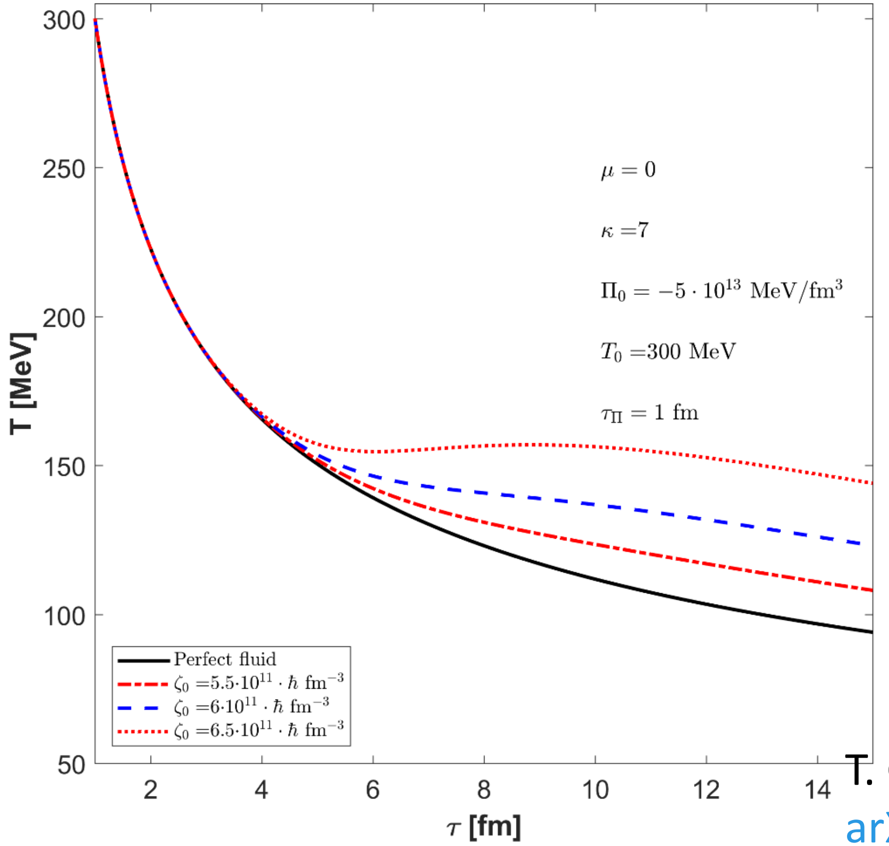
Evolution of the temperature: same initial conditions

Solutions of Navier-Stokes eqs.



Solutions of Isreal-Stewart eqs.



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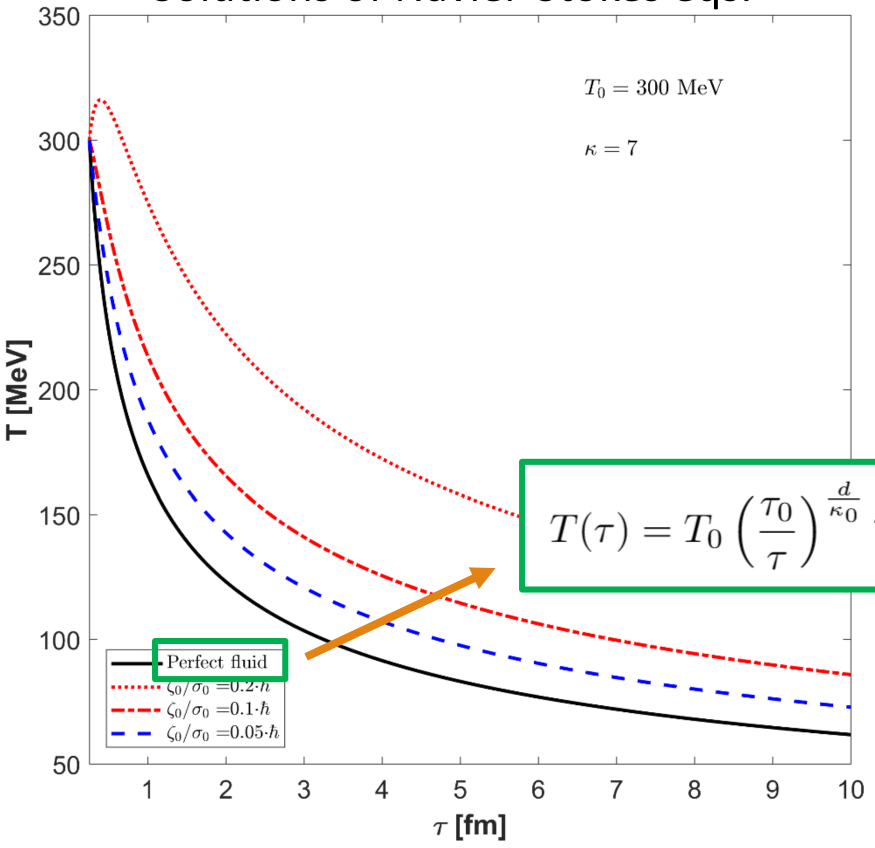

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right)$$

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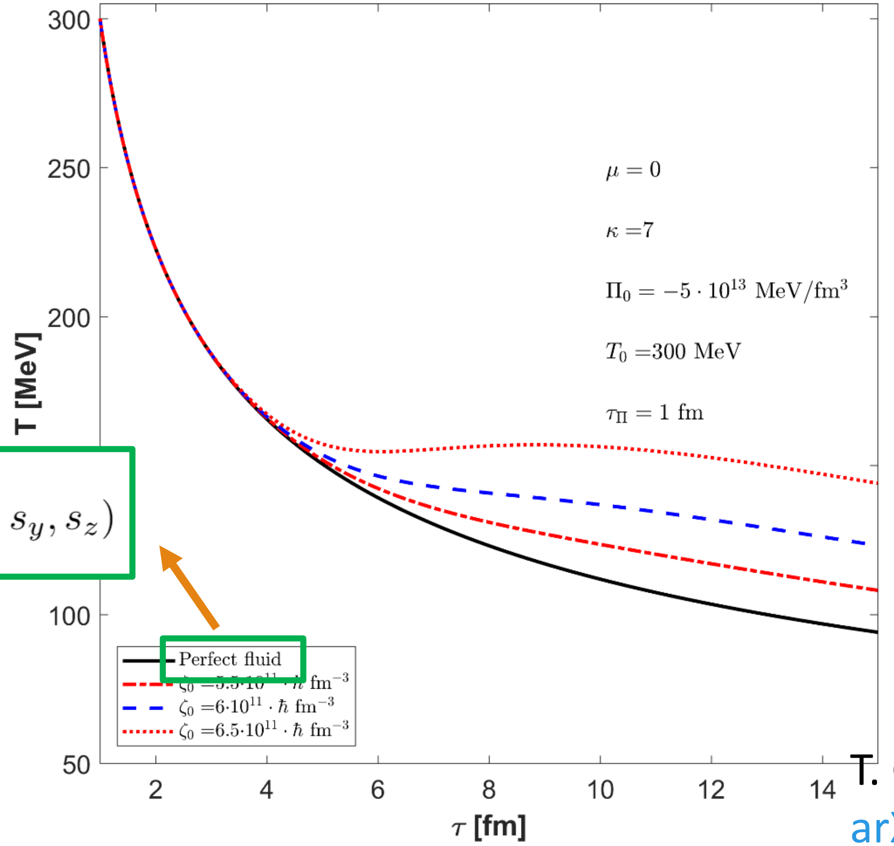

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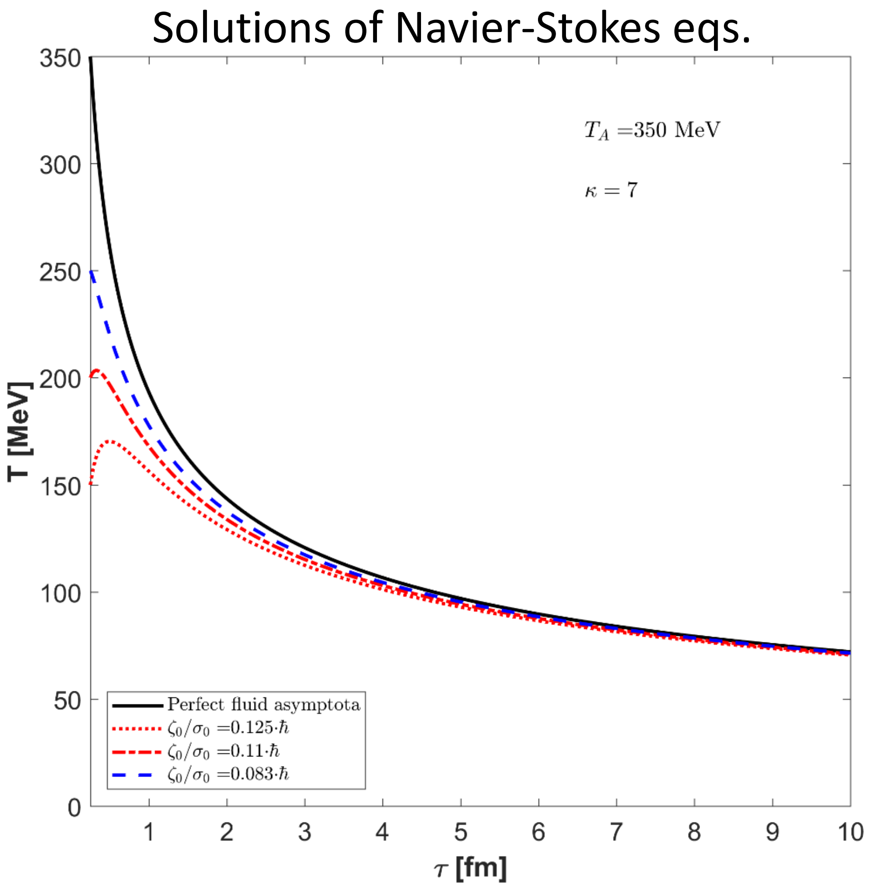
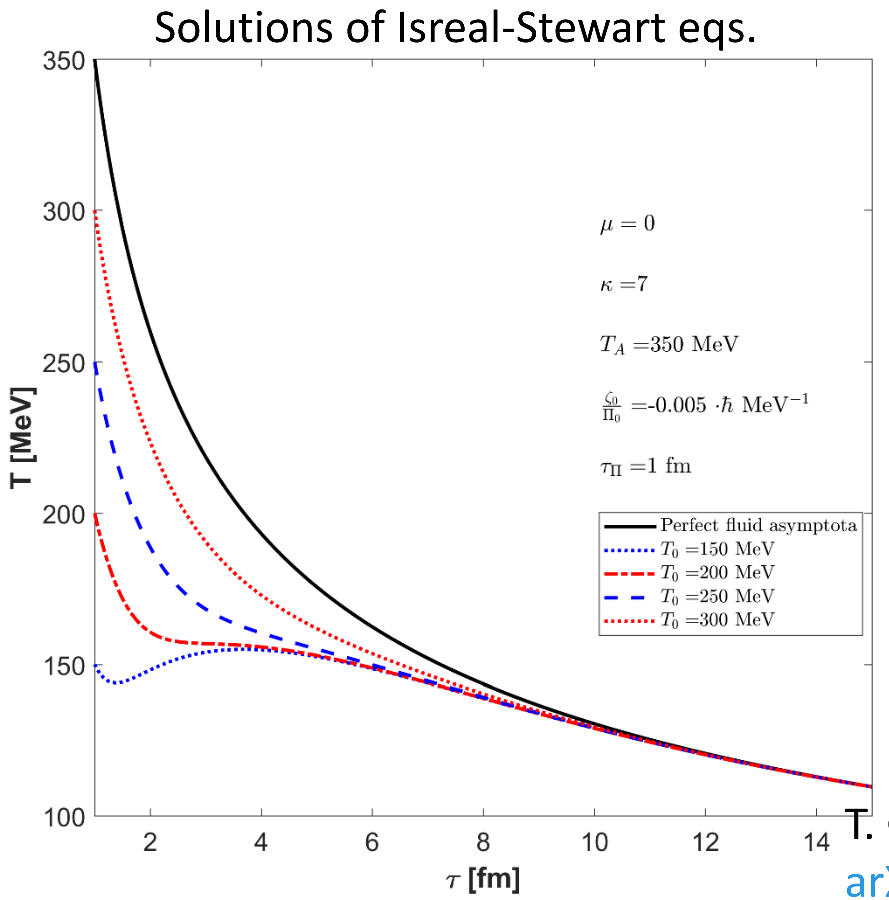
Evolution of the temperature: fluct. initial conditions





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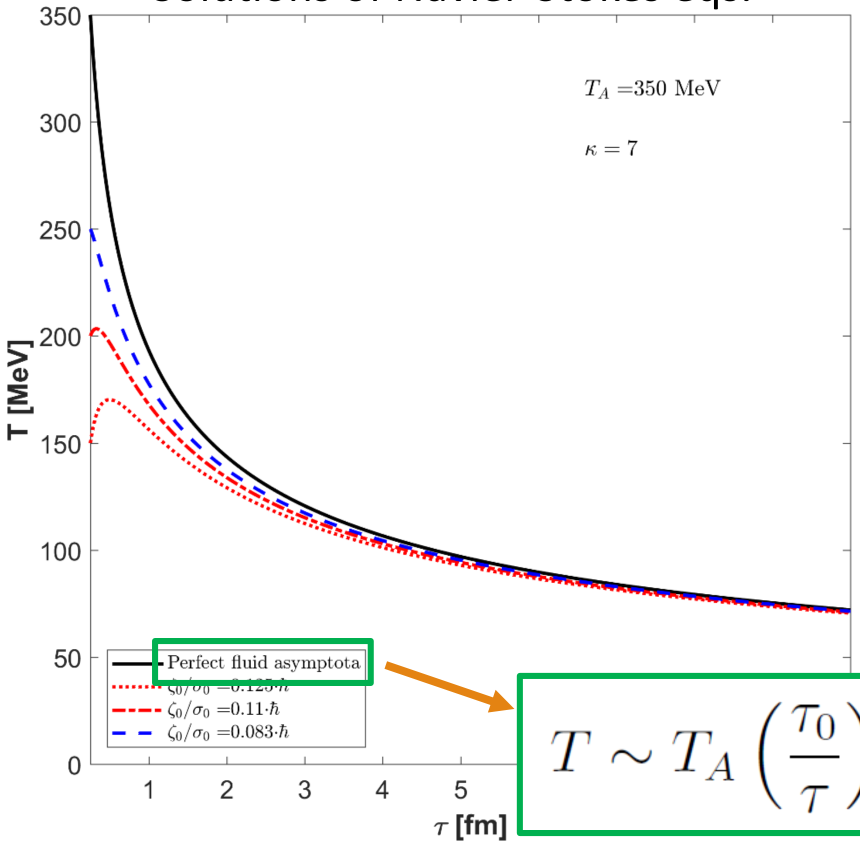

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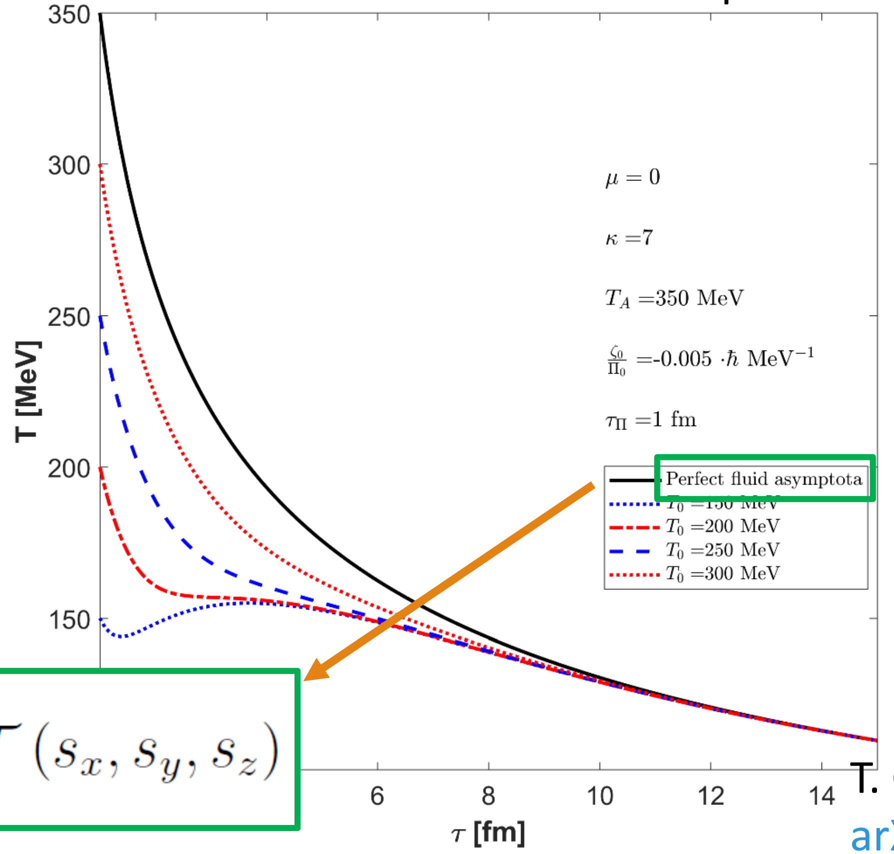

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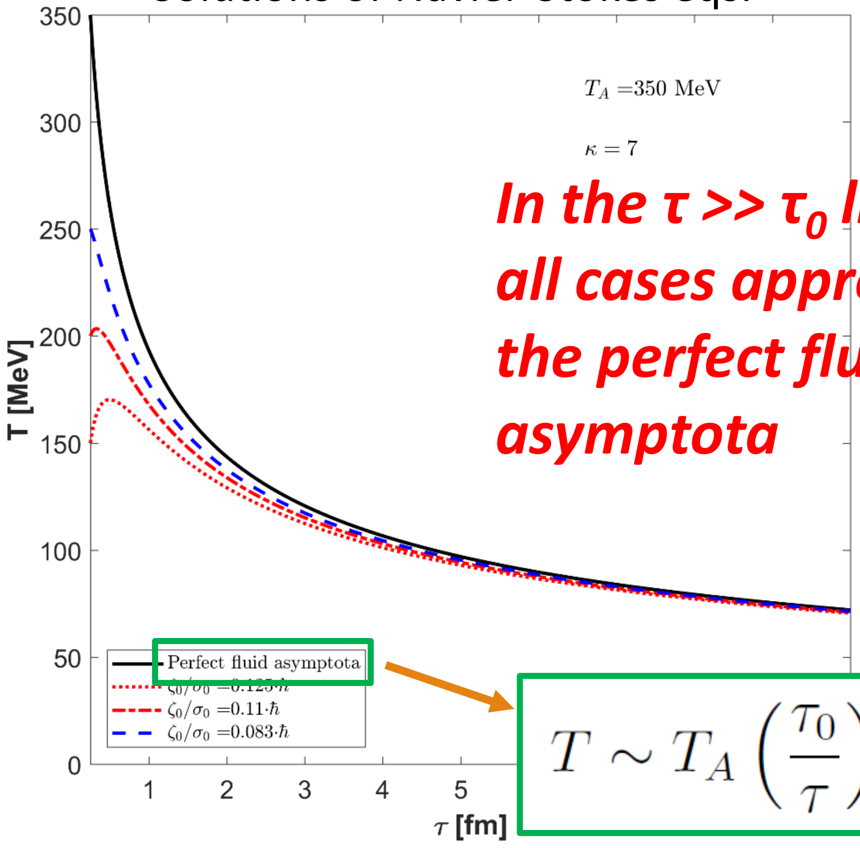

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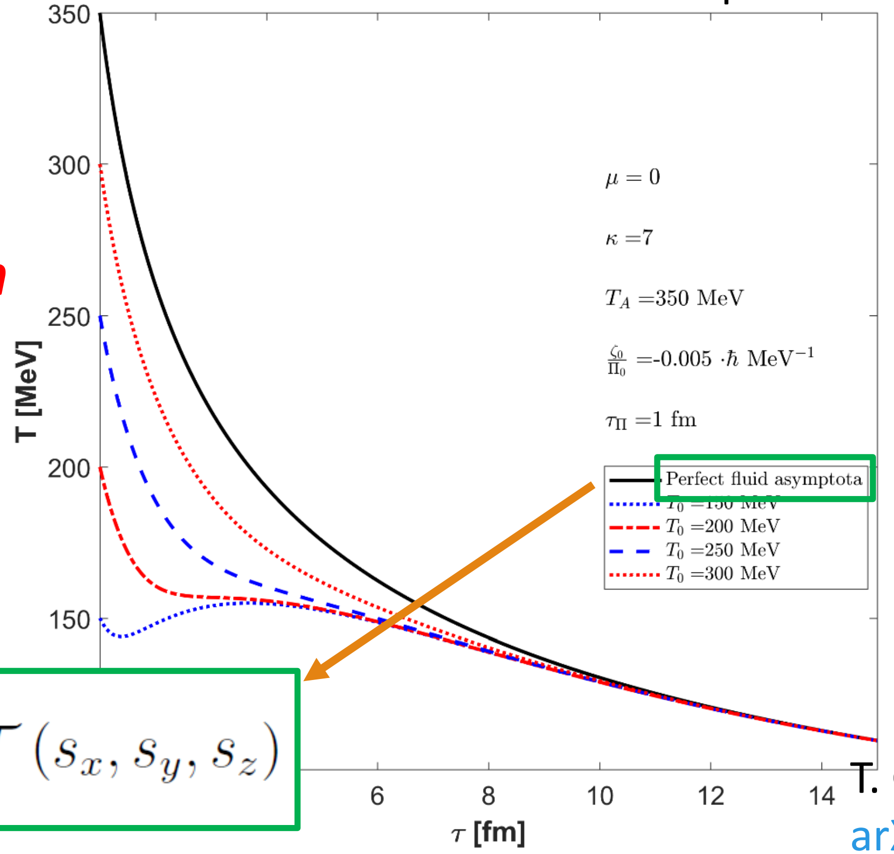


In the $\tau \gg \tau_0$ limit, all cases approach the perfect fluid asymptota

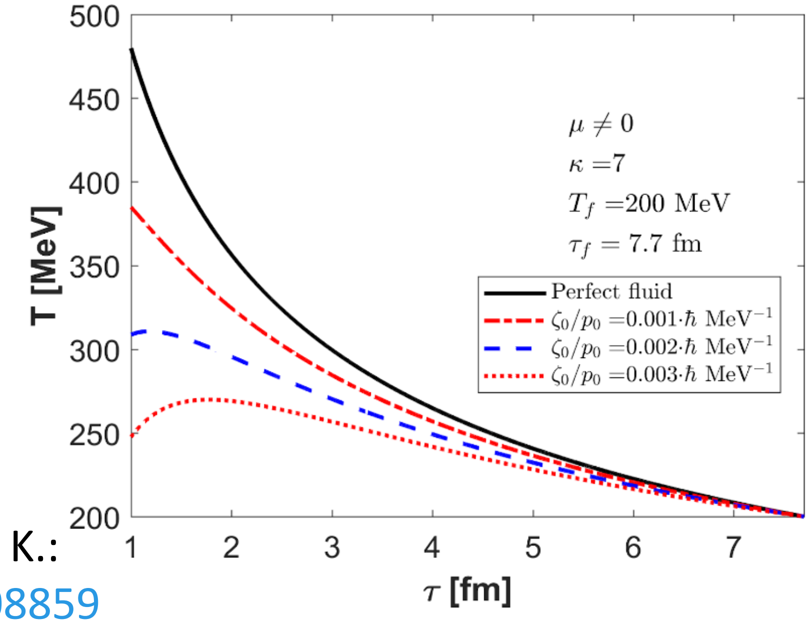
$$\boxed{T \sim T_A \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right)}$$

T. Csörgő, G. K.: arXiv:2003.08859

Application of the solutions of NS eqs.

In arXiv:1405.3877: v_2, v_3 and v_4 were reproduced for $s_{NN}^{1/2}$ = 200 GeV Au+Au collisions with τ_f=7.7 fm/c and T_f=200 MeV final state parameters

We co-varied the initial conditions so that exactly the same freeze-out parameters are obtained



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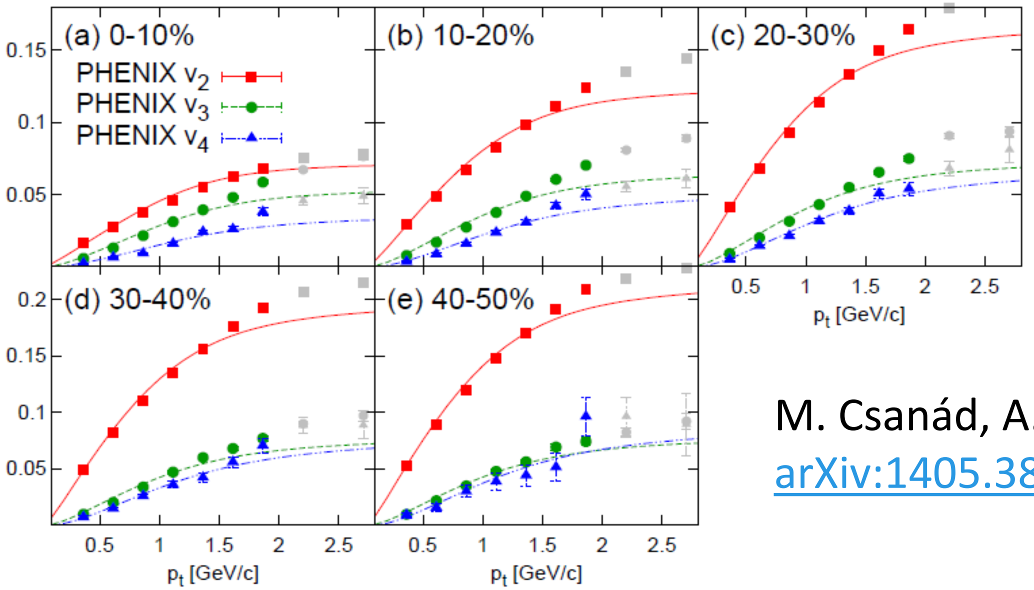

M. Csanád, A. Szabó: arXiv:1405.3877

Summary

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The effect of heat conduction and shear viscosity cancel

The solutions are causal and asymptotically perfect (the effect of bulk viscosity cancels), both for a finite and vanishing μ

These exact solutions tend to the Csörgő-Csernai-Hama-Kodama perfect fluid solution

We cannot decide from final state measurements that the medium evolved as a perfect fluid with higher initial temperature (T_A) or as a viscous fluid with lower initial temperature (T_0)

We were able to reproduce the experimental data in $s_{NN}^{1/2}$ = 200 GeV Au+Au collisions on v_2, v_3 and v_4

Thank you for your attention!

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