High-energy scattering at next-to-leading order

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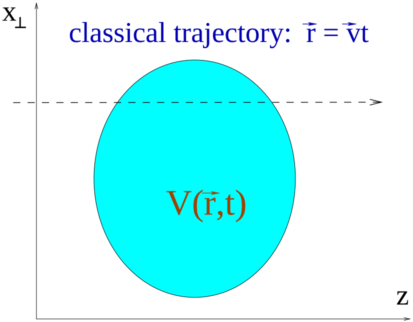
Low x in Kavala 23 June 2010

Outline

- High-energy scattering and Wilson lines in quantum mechanics, QED and QCD.
- Light-cone OPE versus OPE in color dipoles.
- The LO evolution of color dipoles: BK equation.
- NLO amplitudes and NLO BK equation in $\mathcal{N}$=4 SYM.
- NLO amplitudes for deep inelastic scattering in QCD.
- Conclusions.

"Wilson lines" in quantum mechanics



WKB approximation: $\Psi \sim e^{\frac{i}{\hbar}S}$

$$S = \int (pdz - Edt)$$

$$= -Et + \int^z dz' \sqrt{2m(E - V(z')}$$

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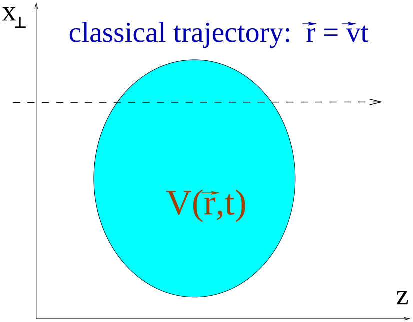

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$$\Psi(\vec{r},t) = e^{-\frac{i}{\hbar}(Et-kx)}\ e^{-\frac{i}{v\hbar}\int_{-\infty}^{z} dz' V(z')}$$

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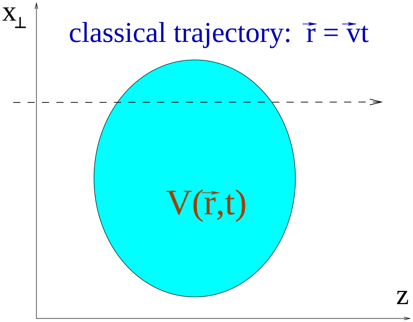

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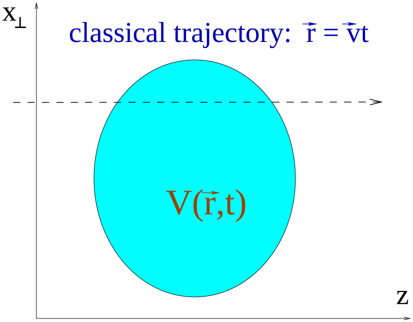

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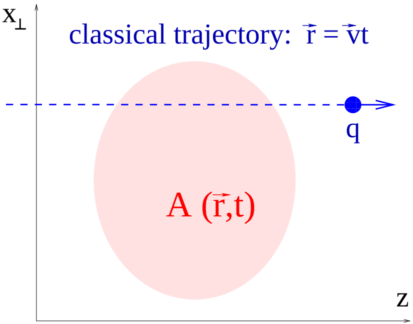
The scattering amplitude is proportional to $\Psi(t=\infty)$ defined by

$$U(x_\perp) \ = \ e^{-\frac{i}{v\hbar}\int_{-\infty}^{\infty} dz' V(z' + x_\perp)}$$

Glauber formula: $\sigma_{\text{tot}} = 2\int d^2x_\perp\ [1 - \Re U(x_\perp)]$

High-energy phase factor in QED and QCD



$$S_e = \int dt \left\{ - mc^2 \sqrt{1 - \frac{v^2}{c^2}} - e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A} \right\}$$
$$= S_{\text{free}} + \int dt \left(-e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A} \right)$$

$\Rightarrow$ phase factor for the high-energy scattering is

$$U(x_\perp, v) = e^{-\frac{ie}{\hbar c}\int_{-\infty}^{\infty} dt(-e\Phi + \frac{e}{c}\vec{v}\cdot\vec{A})}$$
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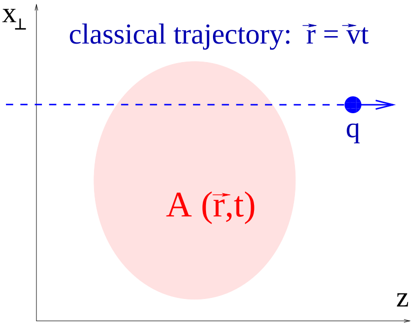

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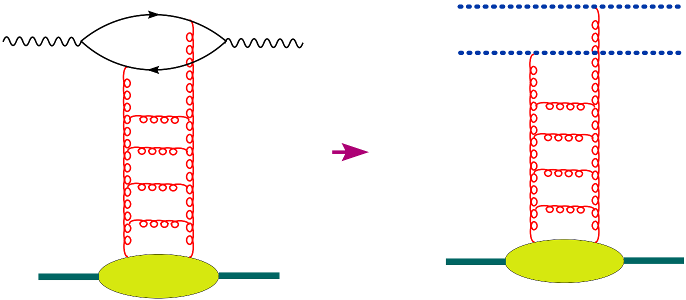
In QCD $e \to -g$, $A_\mu \to A_\mu \equiv A^a_\mu \frac{\lambda^a}{2}$

$$\Rightarrow\ U(x_\perp, v) = P\exp\{\frac{ig}{\hbar c}\int_{-\infty}^{\infty} dt\ \dot{x}_\mu A^\mu(x(t))\}\qquad \text{Wilson} - \text{line}\ \ \text{operator}$$

(Later $\hbar = c = 1$)

- At high energies, particles move along straight lines $\Rightarrow$ the amplitude of $\gamma^* A \to \gamma^* A$ scattering reduces to the matrix element of a two-Wilson-line operator (color dipole):



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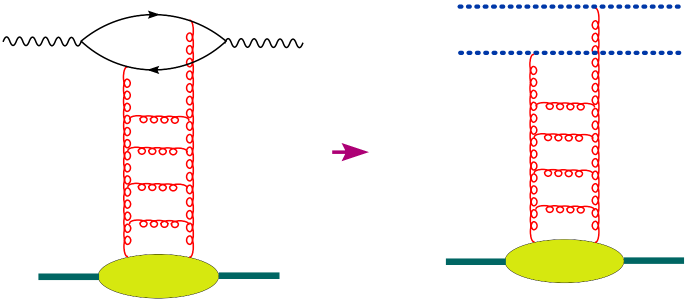

$$A(s) = \int \frac{d^2 k_\perp}{4\pi^2} I^A(k_\perp) \langle B | \mathrm{Tr}\{U(k_\perp) U^\dagger(-k_\perp)\} | B \rangle$$

Formally, ➤ means the operator expansion in Wilson lines

Four steps of an OPE

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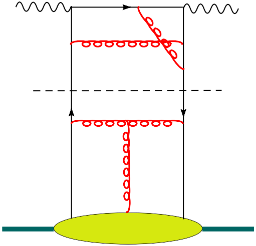
- To convolute the solution with the initial conditions for the evolution and get the amplitude

Light-cone expansion and DGLAP evolution



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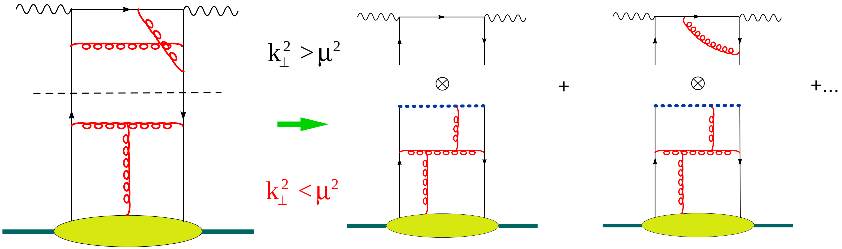
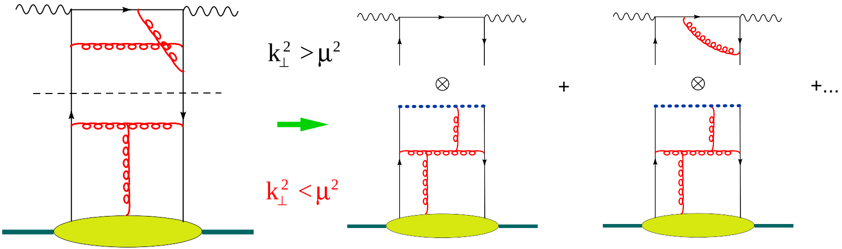


μ^2 - factorization scale (normalization point)

$k_\perp^2 \ > \ \mu^2$ - coefficient functions
$k_\perp^2 \ < \ \mu^2$ - matrix elements of light-ray operators (normalized at μ^2)



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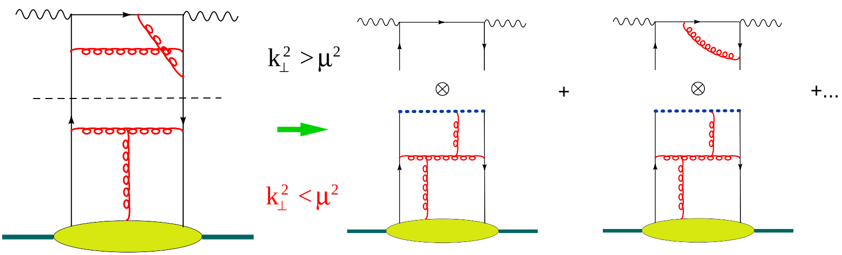
OPE in light-ray operators $(x-y)^2 \to 0$

$$T\{j_\mu(x)j_\nu(y)\} = \frac{(x-y)_\xi}{2\pi^2(x-y)^4}\Big[1+\frac{\alpha_s}{\pi}(\ln(x-y)^2\mu^2+C)\Big]\bar\psi(x)\gamma_\mu\gamma^\xi\gamma_\nu[x,y]\psi(y)$$

$[x,y] \ \equiv \ Pe^{ig\int_0^1 du \ (x-y)^\mu A_\mu(ux+(1-u)y)}$ - gauge link

Light-cone expansion and DGLAP evolution



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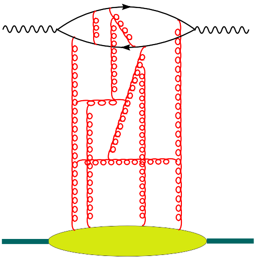
Renorm-group equation for light-ray operators $\Rightarrow$ DGLAP evolution of parton densities $\qquad (x-y)^2 = 0$

$$\mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x,y]\psi(y) \ = \ K_{\rm LO}\bar{\psi}(x)[x,y]\psi(y) + \alpha_s K_{\rm NLO}\bar{\psi}(x)[x,y]\psi(y)$$

High-energy expansion in color dipoles



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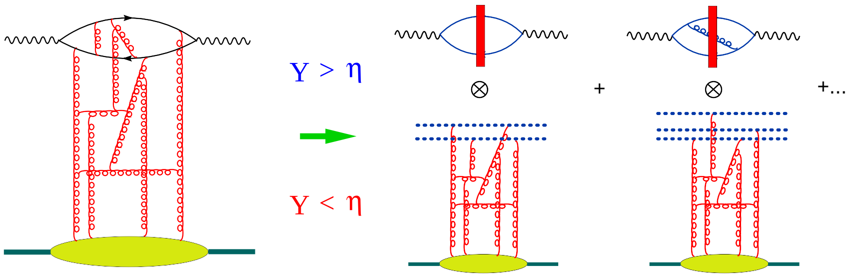

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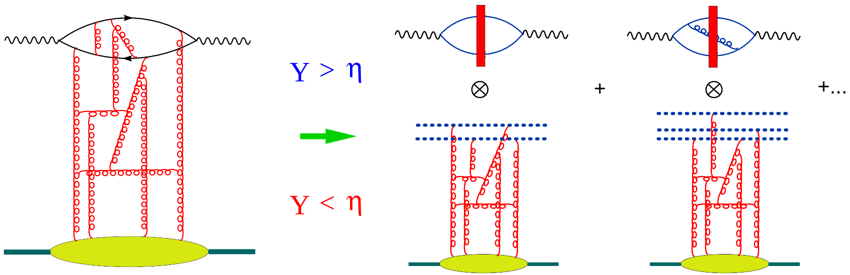


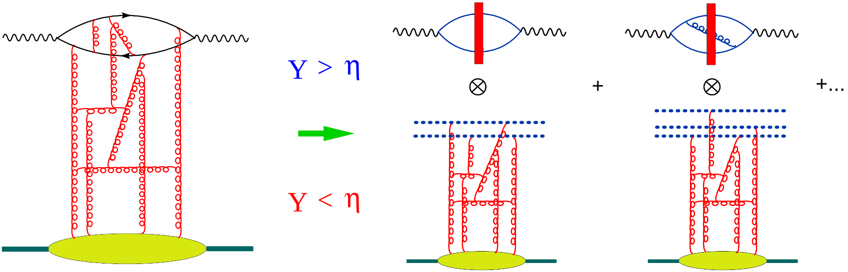
η - rapidity factorization scale

Rapidity Y > η - coefficient function (“impact factor”)

Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

$$U_x^\eta = \mathrm{Pexp}\left[ig\int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_\perp)\right]$$

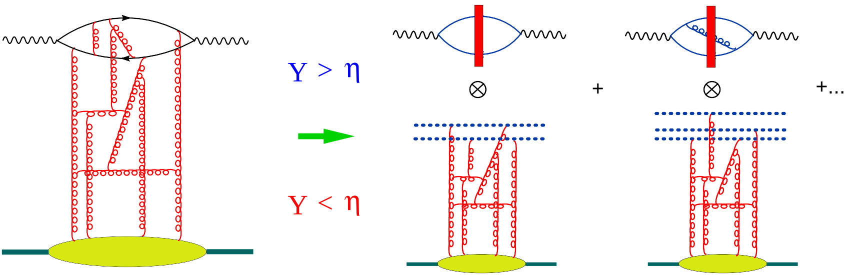
$$A_\mu^\eta(x) = \int \frac{d^4k}{(2\pi)^4}\theta(e^\eta - |\alpha_k|)e^{-ik\cdot x}A_\mu(k)$$



The high-energy operator expansion is

$$T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2\ I^{\rm LO}_{\mu\nu}(z_1,z_2,x,y){\rm Tr}\{\hat{U}^\eta_{z_1}\hat{U}^{\dagger\eta}_{z_2}\}$$
$$+\ \text{NLO contribution}$$

High-energy expansion in color dipoles



η - rapidity factorization scale

Evolution equation for color dipoles

$$\frac{d}{d\eta}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} = \frac{\alpha_s}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2}[\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} - N_c\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\}] + \alpha_s K_{\mathrm{NLO}}\mathrm{tr}\{U^\eta_x U^{\dagger\eta}_y\} + O(\alpha_s^2)$$

(Linear part of K_{NLO} = $K_{\mathrm{NLO\ BFKL}}$)

Spectator frame: propagation in the shock-wave background.

Each path is weighted with the gauge factor $Pe^{ig\int dx_\mu A^\mu}$. Quarks and gluons do not have time to deviate in the transverse space $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.

[$x \to z$: free propagation]$\times$
[$U^{ab}(z_\perp)$ - instantaneous interaction with the $\eta < \eta_2$ shock wave]$\times$
[$z \to y$: free propagation]

Evolution equation for color dipoles

To get the evolution equation, consider the dipole with the rapidies up to η_1 and integrate over the gluons with rapidities $\eta_1 > \eta > \eta_2$. This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidities up to η_2).

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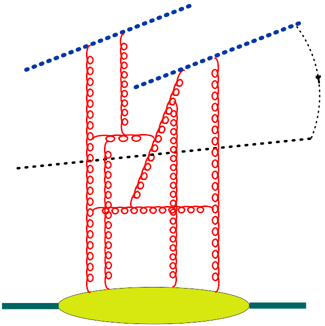

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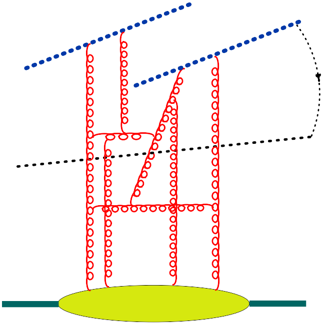

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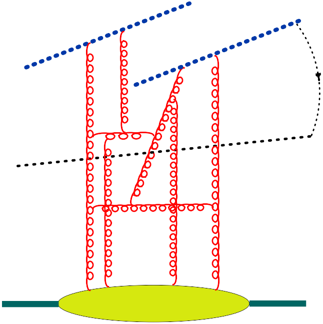
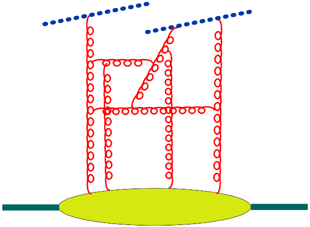


$\rightarrow$

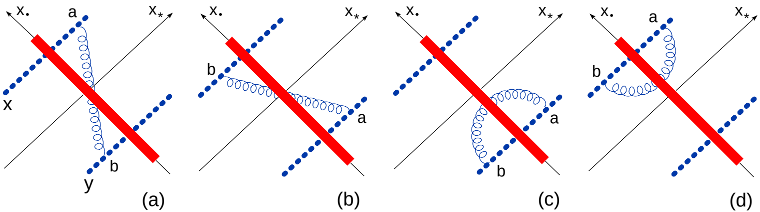
$$\alpha_s(\eta_1 - \eta_2)K_{\text{evol}} \quad \otimes$$



Evolution equation in the leading order

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} = K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\} + \ldots \quad \Rightarrow$$

$$\frac{d}{d\eta}\langle\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}} = \langle K_{\mathrm{LO}}\mathrm{Tr}\{\hat{U}_x\hat{U}^\dagger_y\}\rangle_{\mathrm{shockwave}}$$



$$U^{ab}_z = \mathrm{Tr}\{t^a U_z t^b U^\dagger_z\} \quad \Rightarrow (U_x U^\dagger_y)^{\eta_1} \rightarrow (U_x U^\dagger_y)^{\eta_1} + \alpha_s(\eta_1 - \eta_2)(U_x U^\dagger_z U_z U^\dagger_y)^{\eta_2}$$

$\Rightarrow$ Evolution equation is non-linear

Non-linear evolution equation

$$\hat{\mathcal{U}}(x,y) \equiv 1 - \frac{1}{N_c}\mathrm{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\}$$

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LLA for DIS in pQCD $\Rightarrow$ BFKL (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

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LLA for DIS in pQCD ⇒ BFKL eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1$)

LLA for DIS in sQCD ⇒ BK eqn (LLA: $\alpha_s \ll 1, \alpha_s\eta \sim 1,\ \alpha_s A^{1/3} \sim 1$)

(s for semiclassical)

The story of the non-linear evolution at high energies

- L.V. Gribov, E.M. Levin, M.G. Ryskin (1983) - GLR equation suggested
- A.H. Mueller, J. Qiu (1986) - DLA limit of GLR equation proved
- A.H. Mueller + Nikolaev, Zakharov (1994) - dipole model for the high-energy scattering
- I.B. (1996) - NL evolution equation for Wilson-line operators
- Yu.Kovchegov (1999)- evolution equation for the structure functions of heavy nuclei
- JIMWLK (1997-2000) - RG equation for Color Glass Condensate

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- To check that high-energy OPE works at the NLO level.
- To determine the argument of the coupling constant.
- To get the region of application of the leading order evolution equation.
- To check conformal invariance (in $\mathcal{N}$=4 SYM)

Argument of coupling constant

$$\frac{d}{d\eta}\hat{\mathcal{U}}(z_1,z_2) =$$

$$\frac{\alpha_s(?_\perp)N_c}{2\pi^2}\int dz_3\frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{\hat{\mathcal{U}}(z_1,z_3)+\hat{\mathcal{U}}(z_3,z_2)-\hat{\mathcal{U}}(z_1,z_2)-\hat{\mathcal{U}}(z_1,z_3)\hat{\mathcal{U}}(z_3,z_2)\Big\}$$

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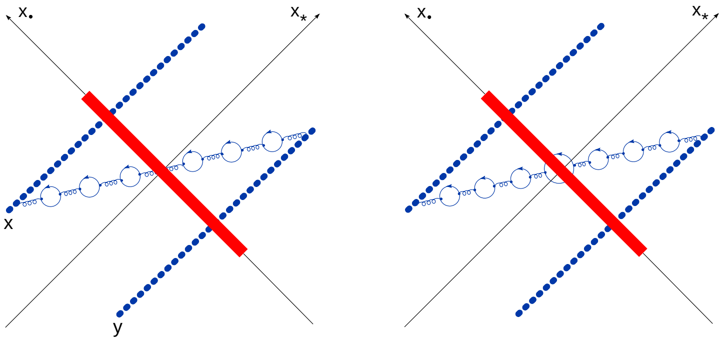
Renormalon-based approach: summation of quark bubbles



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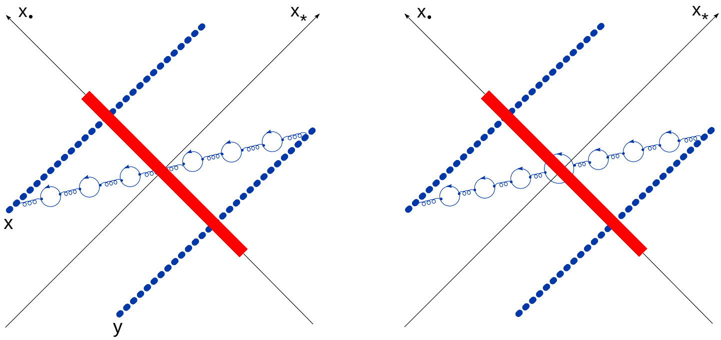

$$-\frac{2}{3}n_f \to b = \frac{11}{3}N_c - \frac{2}{3}n_f$$

Argument of coupling constant

Bubble chain sum:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} = \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\ [\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{Tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}]$$

$$\times\Big[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2}\Big(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\Big) + \frac{1}{z_{23}^2}\Big(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\Big)\Big] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

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$$\times\Big[\frac{z_{12}^2}{z_{13}^2 z_{23}^2} + \frac{1}{z_{13}^2}\Big(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\Big) + \frac{1}{z_{23}^2}\Big(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\Big)\Big] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$$\frac{\alpha_s(z_{12}^2)}{2\pi^2}\frac{z_{12}^2}{z_{13}^2 z_{23}^2} \qquad |z_{12}| \ll |z_{13}|, |z_{23}|$$

$$\frac{\alpha_s(z_{13})^2)}{2\pi^2 z_{13}^2} \qquad |z_{13}| \ll |z_{12}|, |z_{23}|$$

$$\frac{\alpha_s(z_{23})^2)}{2\pi^2 z_{23}^2} \qquad |z_{23}| \ll |z_{12}|, |z_{13}|$$

Argument of coupling constant

Bubble chain sum:

$$\frac{d}{d\eta}\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\} = \frac{\alpha_s(z_{12}^2)}{2\pi^2}\int d^2z\,\left[\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_3}\}\mathrm{Tr}\{\hat{U}_{z_3}\hat{U}^\dagger_{z_2}\} - N_c\mathrm{Tr}\{\hat{U}_{z_1}\hat{U}^\dagger_{z_2}\}\right]$$
$$\times\left[\frac{z_{12}^2}{z_{13}^2z_{23}^2} + \frac{1}{z_{13}^2}\left(\frac{\alpha_s(z_{13}^2)}{\alpha_s(z_{23}^2)} - 1\right) + \frac{1}{z_{23}^2}\left(\frac{\alpha_s(z_{23}^2)}{\alpha_s(z_{13}^2)} - 1\right)\right] + \ldots$$

I.B.; Yu. Kovchegov and H. Weigert (2006)

When the sizes of the dipoles are very different the kernel reduces to:

$$\frac{\alpha_s(z_{12}^2)}{2\pi^2}\frac{z_{12}^2}{z_{13}^2z_{23}^2} \qquad |z_{12}| \ll |z_{13}|, |z_{23}|$$

$$\frac{\alpha_s(z_{13})^2)}{2\pi^2 z_{13}^2} \qquad |z_{13}| \ll |z_{12}|, |z_{23}|$$

$$\frac{\alpha_s(z_{23})^2)}{2\pi^2 z_{23}^2} \qquad |z_{23}| \ll |z_{12}|, |z_{13}|$$

$\Rightarrow$ the argument of the coupling constant is given by the size of the smallest dipole.

To be continued in Giovanni's talk...