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# The Standard Model physics at high-energy muon colliders

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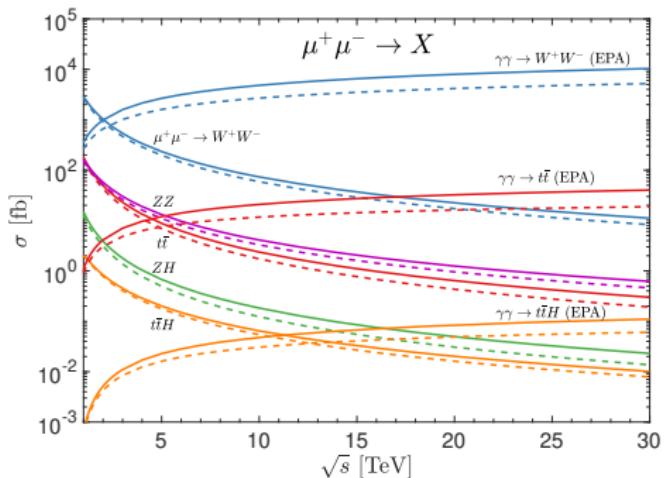
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In collaboration with Tao Han and Yang Ma  
2007.14300 and ongoing

# The SM processes at high-energy muon colliders

[T. Han, Y. Ma, KX 2007.14300]



- The annihilations decreases as  $1/s$
- The fusions increase as  $\log^p(s)$ , which take over at high energies
- **Question:** How to treat photon as a parton properly at high energies when EW gauge bosons ( $W/Z$ ) become active?

# EW physics at high energies

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- At high energies, every particle become massless

$$\frac{v}{E} : \frac{v}{100 \text{ TeV}} \sim \frac{\Lambda_{\text{QCD}}}{100 \text{ GeV}}, \frac{v}{E}, \frac{m_t}{E}, \frac{M_W}{E} \rightarrow 0!$$

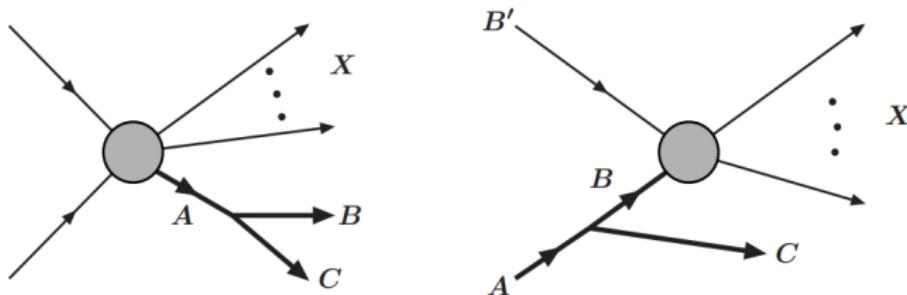
- The splitting phenomena dominate due to large log enhancement
- The EW symmetry is restored:  $SU(2)_L \times U(1)_Y$  unbroken
- Goldstone Boson Equivalence:

$$\epsilon_L^\mu(k) = \frac{E}{M_W}(\beta_W, \hat{k}) \simeq \frac{k^\mu}{M_W} + \mathcal{O}\left(\frac{M_W}{E}\right)$$

The violation terms is power counted as  $v/E \rightarrow$  QCD higher twist effects  
 $\Lambda_{\text{QCD}}/Q$  [[G. Cuomo, A. Wulzer, arXiv:1703.08562; 1911.12366](#)].

- We mainly focus on the **splitting phenomena**, which can be factorized and resummed as the **EW PDFs** in the ISR, and the **Fragementaions/Parton Shower** in the FRS.
- Other interesting aspects: the polarized EW boson scattering, top-Yukawa coupling effect

# Factorization of the EW splittings



$$d\sigma \simeq d\sigma_X \times d\mathcal{P}_{A \rightarrow B+C}, \quad E_B \approx zE_A, \quad E_C \approx \bar{z}E_A, \quad k_T \approx z\bar{z}E_A \theta_{BC}$$

$$\frac{d\mathcal{P}_{A \rightarrow B+C}}{dz dk_T^2} \simeq \frac{1}{16\pi^2} \frac{z\bar{z}|\mathcal{M}^{(\text{split})}|^2}{(k_T^2 + \bar{z}m_B^2 + zm_C^2 - z\bar{z}m_A^2)^2}, \quad \bar{z} = 1 - z$$

- The dimensional counting:  $|\mathcal{M}^{(\text{split})}|^2 \sim k_T^2$  or  $m^2$
- To validate the factorization formalism
  - The observable  $\sigma$  should be **infra-red safe**
  - Leading behavior comes from the **collinear splitting**

[Ciafaloni et al., hep-ph/0004071; 0007096; C. Bauer, Ferland, B. Webber et al., arXiv:1703.08562; 1808.08831]

[A. Manohar et al., 1803.06347; T. Han, J. Chen & B. Tweedie, arXiv:1611.00788]

# Splitting functions: EW

- Starting from the unbroken phase: all massless

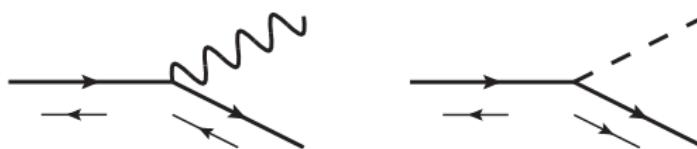
$$\mathcal{L}_{SU(2) \times U(1)} = \mathcal{L}_{gauge} + \mathcal{L}_\phi + \mathcal{L}_f + \mathcal{L}_{Yukawa}$$

- Particle contents:

- Chiral fermions  $f_{L,R}$
- Gauge bosons:  $B, W^{0,\pm}$

- Higgs  $H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(h - i\phi^0) \end{pmatrix}$

- Splitting functions [See Ciafaloni et al. hep-ph/0505047, Chen et al. 1611.00788 for complete lists.]

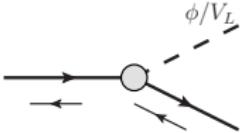
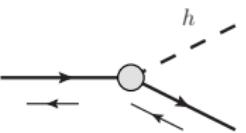
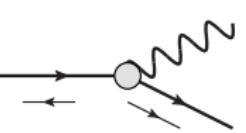


	$\frac{1}{8\pi^2} \frac{1}{k_T^2} \frac{1+\bar{z}^2}{z}$	$\frac{1}{8\pi^2} \frac{1}{k_T^2} \frac{z}{2}$
$f_{s=L,R} \rightarrow$	$V_T f_s^{(')}$	$[BW]_T^0 f_s$
	$g_V^2 (Q_{f_s}^V)^2$	$g_1 g_2 Y_{f_s} T_{f_s}^3$
		$y_{f_R^{(')}}^2$
	Infrared, collinear singularities ( $P_{gq}$ )	Collinear singularity chirality-flip, Yukawa

# Corrections to the GET in the EWSB

- New fermion splitting:  $P \sim \frac{v^2}{k_T^2} \frac{dk_T^2}{k_T^2}$

- $V_L$  is of IR,  $h$  has no IR

		
$\frac{1}{16\pi^2} \frac{v^2}{k_T^4} \frac{1}{z}$	$\frac{1}{16\pi^2} \frac{v^2}{k_T^4}$	$\frac{1}{16\pi^2} \frac{v^2}{k_T^4}$
$f_s \rightarrow V_L f_s^{(1)} (V \neq \gamma)$	$h f_s$	$V_T f_{-s}^{(1)}$
Chirality conserving non-zero for massless $f$		Chirality flipping $\sim m_f$

- The PDFs for  $W_L/Z_L$  behaves as constants, which does not run at the leading log: "Bjorken scaling" restoration (higher-twist effects)

$$f_{V_L/f}(x, Q^2) \sim \alpha \frac{1-x}{x}$$

# PDFs and Fragmentations (parton showers)

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- Initial state radiation (ISR), PDFs (DGLAP):

$$f_B(z, \mu^2) = \sum_A \int_z^1 \frac{d\xi}{\xi} f_A(\xi) \int_{m^2}^{\mu^2} dz d\mathcal{P}_{A \rightarrow B+C}(z/\xi, k_T^2)$$

$$\frac{\partial f_B(z, \mu^2)}{\partial \mu^2} = \sum_A \int_z^1 \frac{d\xi}{\xi} \frac{d\mathcal{P}_{A \rightarrow B+C}(z/\xi, \mu^2)}{dz dk_T^2} f_A(\xi, \mu^2)$$

- Final state radiation (FSR): Fragmentations (parton showers):

$$\Delta_A(t) = \exp \left[ - \sum_B \int_{t_0}^t dz \mathcal{P}_{A \rightarrow B+C}(z) \right],$$

$$f_A(x, t) = \Delta_A(t) f_A(x, t_0) + \int_{t_0}^t \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \mathcal{P}_{A \rightarrow B+C}(z) f_A(x/z, t')$$

- Very important formulation for the LHC physics, and future colliders.

# The novel features of the EW PDFs

- The EW PDFs must be polarized due to the chiral nature of the EW theory

$$f_{V_+/A_+} \neq f_{V_-/A_-}, \quad f_{V_+/A_-} \neq f_{V_-/A_+},$$

$$\hat{\sigma}(V_+B_+) \neq \hat{\sigma}(V_-B_-), \quad \hat{\sigma}(V_+B_-) \neq \hat{\sigma}(V_-B_+)$$

We are not able to factorize the cross sections in an unpolarized form.

$$\sigma \neq f_{V/A} \hat{\sigma}(VB), \quad f_{V/A} = \frac{1}{2} \sum_{\lambda, s_1} f_{V_\lambda/A_{s_1}}, \quad \hat{\sigma}(VB) = \frac{1}{4} \sum_{\lambda, s_2} \hat{\sigma}(V_\lambda B_{s_2})$$

- The interference gives the mixed PDFs [Bauer '17,

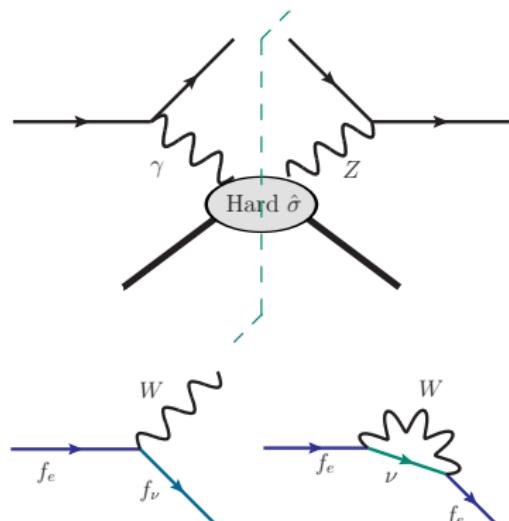
'18, Manohar '18 , Tao '16.]

$$f_{\gamma Z} \sim \langle \Omega | A^{\mu\nu} Z_{\mu\nu} | \Omega \rangle + \text{h.c.},$$

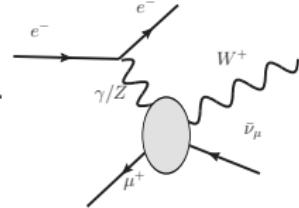
similarly for  $f_{hZ_L}$ .

- Bloch-Nordsieck theorem violation due to the non-cancelled divergence in  $f \rightarrow f' V$ : cutoff  $M_V/Q$  or redefinition

$$f_1 \sim f_e + f_\nu, \quad f_3 \sim f_e - f_\nu$$



# A toy example: $e^- \mu^+ \rightarrow e^- W^+ \bar{\nu}_\mu$



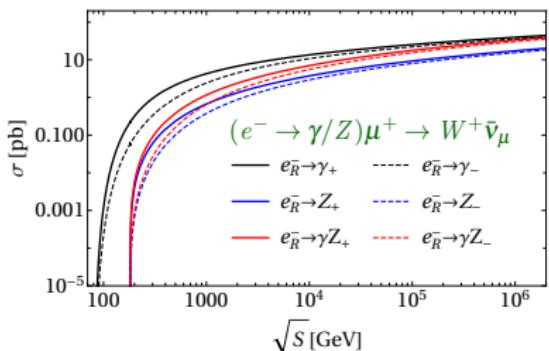
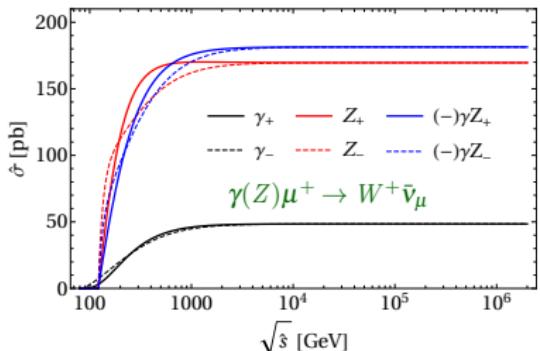
- EWA:  $f_{V_\lambda/e_s^\pm}(x, Q) = \frac{1}{8\pi^2} g_1 g_2 P_{V_\lambda/e_s^\pm}(x) \log(Q^2/m_Z^2)$

$$g_L = \frac{g_2}{c_W} \left( -\frac{1}{2} + s_W^2 \right) < 0, \quad g_R = \frac{g_2}{c_W} s_W^2 > 0, \quad g_e = -e$$

	$e_L^-$	$e_R^-$	$e_L^+$	$e_R^+$
$Z_-$	$g_L^2 \frac{1}{x}$	$g_R^2 \frac{(1-x)^2}{x}$	$g_L^2 \frac{(1-x)^2}{x}$	$g_R^2 \frac{1}{x}$
$Z_+$	$g_L^2 \frac{(1-x)^2}{x}$	$g_R^2 \frac{1}{x}$	$g_L^2 \frac{1}{x}$	$g_R^2 \frac{(1-x)^2}{x}$
$\gamma Z_-$	$g_e g_L \frac{1}{x}$	$g_e g_R \frac{(1-x)^2}{x}$	$g_e g_L \frac{(1-x)^2}{x}$	$g_e g_R \frac{1}{x}$
$\gamma Z_+$	$g_e g_L \frac{(1-x)^2}{x}$	$g_e g_R \frac{1}{x}$	$g_e g_L \frac{1}{x}$	$g_e g_R \frac{(1-x)^2}{x}$

- The contribution of the mixed PDF  $f_{\gamma Z}$  can be either **constructive or destructive**

$$\sigma = \sum_{\lambda, s_1, s_2} f_{V_\lambda/e_s^-} \hat{\sigma}(V_\lambda \mu_{s_2}^+ \rightarrow W^+ \bar{\nu}_\mu)$$



# The PDF evolution

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- The DGLAP equations

$$\frac{df_i}{d\log \mu^2} = \sum_I \frac{\alpha_I}{2\pi} \sum_j P_{ij}^I \otimes f_j$$

- The initial conditions

$$f_{\ell/\ell}(x, m_\ell^2) = \delta(1-x)$$

- Three regions and two matchings

- $m_\ell < \mu < \mu_{\text{QCD}}$ : QED
- $\mu = \mu_{\text{QCD}} \lesssim 1 \text{ GeV}$ :  $f_q \propto P_{q\gamma} \otimes f_\gamma, f_g = 0$
- $\mu_{\text{QCD}} < \mu < \mu_{\text{EW}}$ : QED  $\otimes$  QCD
- $\mu = \mu_{\text{EW}} = M_Z$ :  $f_v = f_t = f_W = f_Z = f_{\gamma Z} = 0$
- $\mu_{\text{EW}} < \mu$ : EW  $\otimes$  QCD.

$$\begin{pmatrix} f_B \\ f_{W^3} \\ f_{BW^3} \end{pmatrix} = \begin{pmatrix} c_W^2 & s_W^2 & -2c_W s_W \\ s_W^2 & c_W^2 & 2c_W s_W \\ c_W s_W & -c_W s_W & c_W^2 - s_W^2 \end{pmatrix} \begin{pmatrix} f_\gamma \\ f_Z \\ f_{\gamma Z} \end{pmatrix}$$

- We work in the  $(B, W)$  basis. The technical details can be referred to the backup slides.

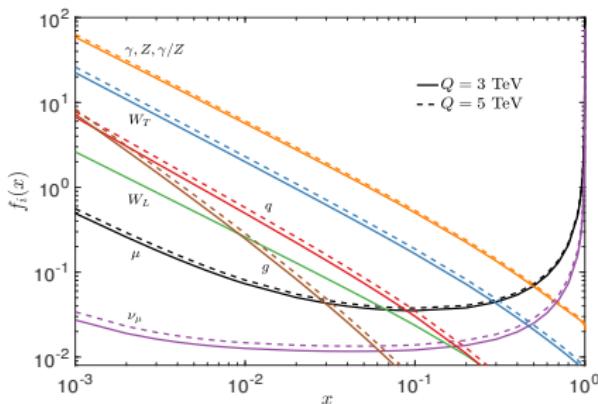
# EWPDFs at a muon collider

- The sea leptonic and quark PDFs

$$v = \sum_i (v_i + \bar{v}_i), \quad \ell_{\text{sea}} = \bar{\mu} + \sum_{i \neq \mu} (\ell_i + \bar{\ell}_i), \quad q = \sum_{i=d}^t (q_i + \bar{q}_i)$$

- The averaged momentum fractions:  $\langle xf_i \rangle = \int xf_i(x)dx$

$Q$	$\mu$	$\gamma, Z, \gamma Z$	$W^\pm$	$v$	$\ell_{\text{sea}}$	$q$	$g$
$M_Z$	97.9	2.06	0	0	0.028	0.035	0.0062
3 TeV	91.5	3.61	1.10	3.59	0.069	0.13	0.019
5 TeV	89.9	3.82	1.24	4.82	0.077	0.16	0.022



- $W_L$  does not evolve, reflecting the residue of the EW broken (high-twist) effects
- We have neutrinos, quarks, and gluon, and everything as partons [Tao, Yang, Xie, 2007.14300].

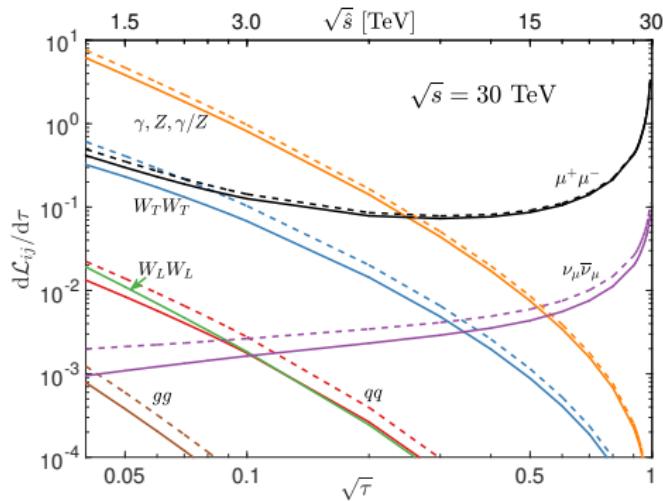
# The parton luminosities

- Production cross sections

$$\sigma(\ell^+\ell^- \rightarrow F + X) = \int_{\tau_0}^1 d\tau \sum_{ij} \frac{d\mathcal{L}_{ij}}{d\tau} \hat{\sigma}(ij \rightarrow F), \quad \tau = \hat{s}/s$$

- Partonic luminosities

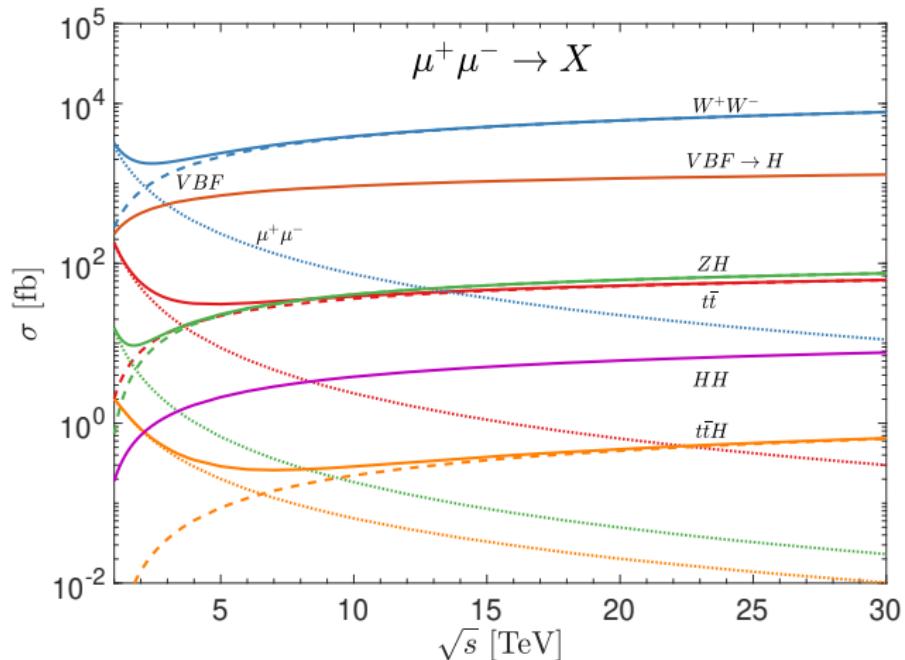
$$\frac{d\mathcal{L}_{ij}}{d\tau} = \frac{1}{1 + \delta_{ij}} \int_{\tau}^1 \frac{d\xi}{\xi} \left[ f_i(\xi, Q^2) f_j \left( \frac{\tau}{\xi}, Q^2 \right) + (i \leftrightarrow j) \right]$$



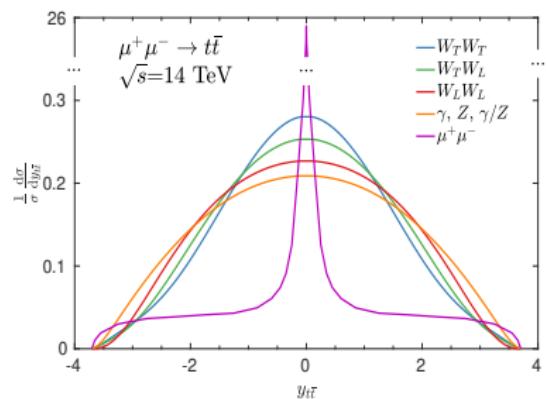
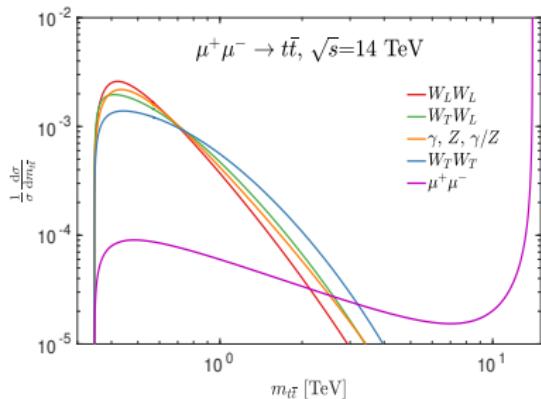
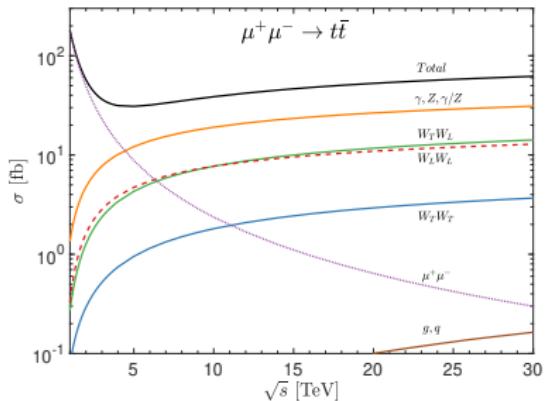
# Semi-inclusive processes

Just like in hadronic collisions:

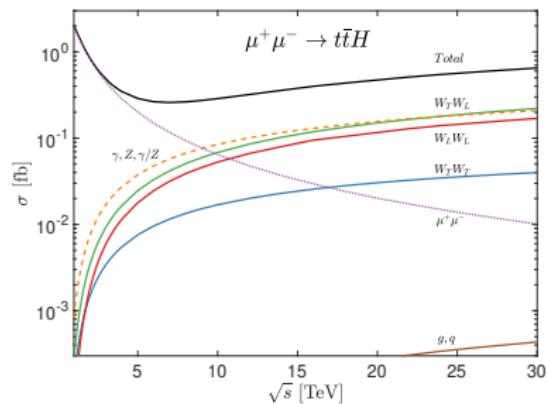
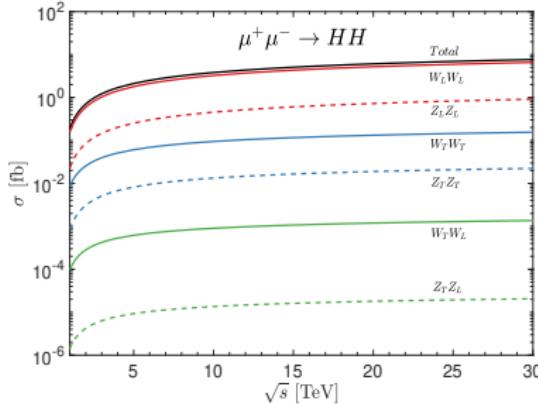
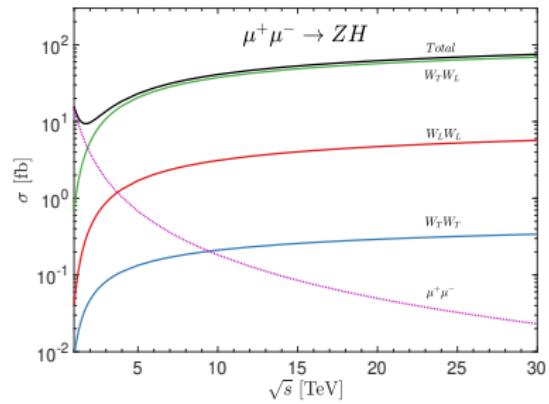
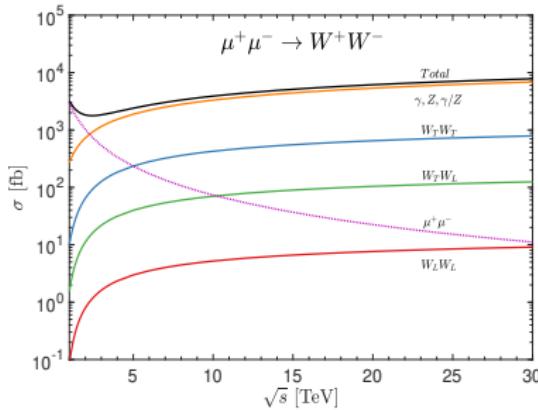
$$\mu^+ \mu^- \rightarrow \text{exclusive particles} + \text{remnants}$$



# The decomposition and distributions



# Other processes: $W^+ W^-$ , $ZH$ , $HH$ , $t\bar{t}H$



## Summary and prospects

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- At high energies, all particles become **massless**. The EW symmetry is asymptotically restored.
- The **splitting** phenomena dominate at high energies. The ISR can be factorized as the **PDFs**, the FSR as **Fragmentations (parton shower)**.
- The EW PDFs are **polarized**, as well as the hard partonic cross sections, because of the chiral nature of the EW theory.
- The interference gives **mixed** PDFs, which can be either **positive or negative**. The contribution can be either **constructive or destructive**.
- Near the threshold (at low energies), the factorization breaks down. We need to **match** to the fixed-order calculation.
- The longitudinal PDFs ( $f_{V_L}$ ) do not run at the leading log. But the contribution is very important due to the large Yukawa coupling.
- Bloch-Nordsieck theorem violation: Factorization breaks down for the insufficiently inclusive processes.
  - Cutoff ( $M_V/Q$ ) to regulate the divergence,
  - Fully inclusive to cancel all the divergence.

# Decomposition into singlet and non-singlet PDFs

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The singlets

$$f_L = \sum_{i=e,\mu,\tau} (f_{\ell_i} + f_{\bar{\ell}_i}), \quad f_U = \sum_{i=u,c} (f_{u_i} + f_{\bar{u}_i}), \quad f_D = \sum_{i=d,s,b} (f_{d_i} + f_{\bar{d}_i})$$

The non-singlets

- The only non-trivial singlet  $f_{e,NS} = f_e - f_{\bar{e}}$
- the leptons

$$f_{\ell_i,NS} = f_{\ell_i} - f_{\bar{\ell}_i} \quad (i = 2, 3), \quad f_{\ell,12} = f_{\bar{e}} - f_{\bar{\mu}}, \quad f_{\ell,13} = f_{\bar{e}} - f_{\bar{\tau}};$$

- the up-type quarks

$$f_{u_i,NS} = f_{u_i} - f_{\bar{u}_i}, \quad f_{u,12} = u - c;$$

- and the down-type quarks

$$f_{d_i,NS} = f_{d_i} - f_{\bar{d}_i}, \quad f_{d,12} = d - s, \quad f_{d,13} = d - b.$$

Reconstruction:

$$f_e = \frac{f_L + (2N_\ell - 1)f_{e,NS}}{2N_\ell}, \quad f_{\bar{e}} = f_{\bar{\mu}} = f_{\bar{\mu}} = f_{\tau} = f_{\bar{\tau}} = \frac{f_L - f_{e,NS}}{2N_\ell}.$$

$$f_u = f_{\bar{u}} = f_c = f_{\bar{c}} = \frac{f_U}{2N_u}, \quad f_d = f_{\bar{d}} = f_s = f_{\bar{s}} = f_b = f_{\bar{b}} = \frac{f_D}{2N_d}.$$

# The QED $\otimes$ QCD DGLAP evolution

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- The singlets and gauge bosons

$$\begin{pmatrix} f_L \\ f_U \\ f_D \\ f_\gamma \\ f_g \end{pmatrix} = \frac{d}{dL} \begin{pmatrix} P_{\ell\ell} & 0 & 0 & 2N_\ell P_{\ell\gamma} & 0 \\ 0 & P_{uu} & 0 & 2N_u P_{u\gamma} & 2N_u P_{ug} \\ 0 & 0 & P_{dd} & 2N_d P_{d\gamma} & 2N_d P_{dg} \\ P_{\gamma\ell} & P_{\gamma u} & P_{\gamma d} & P_{\gamma\gamma} & 0 \\ 0 & P_{gu} & P_{gd} & 0 & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} f_L \\ f_U \\ f_D \\ f_\gamma \\ f_g \end{pmatrix}$$

- The non-singlets

$$\frac{d}{dL} f_{NS} = P_{ff} \otimes f_{NS}.$$

## The EW isospin (T) and charge-parity (CP) basis

- The leptonic doublet and singlet in the (T,CP) basis

$$f_\ell^{0\pm} = \frac{1}{4} [(f_{v_L} + f_{\ell_L}) \pm (f_{\bar{v}_L} + f_{\bar{\ell}_L})], \quad f_\ell^{1\pm} = \frac{1}{4} [(f_{v_L} - f_{\ell_L}) \pm (f_{\bar{v}_L} - f_{\bar{\ell}_L})].$$
$$f_e^{0\pm} = \frac{1}{2} [f_{e_R} \pm f_{\bar{e}_R}]$$

- Similar for the quark doublet and singlets.

- The bosonic

$$f_B^{0\pm} = f_{B_+} \pm f_{B_-}, \quad f_{BW}^{1\pm} = f_{BW_+} \pm f_{BW_-},$$
$$f_W^{0\pm} = \frac{1}{3} \left[ \left( f_{W_+^+} + f_{W_+^-} + f_{W_+^3} \right) \pm \left( f_{W_-^+} + f_{W_-^-} + f_{W_-^3} \right) \right],$$
$$f_W^{1\pm} = \frac{1}{2} \left[ \left( f_{W_+^+} - f_{W_+^-} \right) \mp \left( f_{W_-^+} - f_{W_-^-} \right) \right],$$
$$f_W^{2\pm} = \frac{1}{6} \left[ \left( f_{W_+^+} + f_{W_+^-} - 2f_{W_+^3} \right) \pm \left( f_{W_-^+} + f_{W_-^-} - 2f_{W_-^3} \right) \right].$$

# The EW PDFs in the singlet/non-singlet basis

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Construct the singlets and non-singlets

- Singlets

$$f_L^{0,1\pm} = \sum_i^{N_g} f_\ell^{0,1\pm}, \quad f_E^{0\pm} = \sum_i^{N_g} f_e^{0\pm},$$

- Non-singlets

$$f_{L,NS}^{0,1\pm} = f_{\ell_1}^{0,1\pm} - f_{\ell_2}^{0,1\pm}, \quad f_{E,NS}^{0\pm} = f_{e_1}^{0\pm} - f_{e_2}^{0\pm}$$

- The trivial non-singlets

$$f_{L,23}^{0,1\pm} = f_{E,23}^{0\pm} = 0$$

Reconstruct the PDFs for each flavors

- The leptonic PDFs

$$f_{\ell_1}^{0,1\pm} = \frac{f_L^{0,1\pm} + (N_g - 1)f_{L,NS}^{0,1\pm}}{N_g}, \quad f_{\ell_2}^{0,1\pm} = f_{\ell_3}^{0,1\pm} = \frac{f_L^{0,1\pm} - f_{L,NS}^{0,1\pm}}{N_g},$$

$$f_{e_1}^{0\pm} = \frac{f_E^{0\pm} + (N_g - 1)f_{E,NS}^{0\pm}}{N_g}, \quad f_{e_2}^{0\pm} = f_{e_3}^{0\pm} = \frac{f_E^{0\pm} - f_{E,NS}^{0\pm}}{N_g}.$$

- The quark components can be constructed as singlets/non-singlets, and reconstructed correspondingly as well.

# The DGLAP in the singlet and non-singlet basis

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$$\frac{d}{dL} \begin{pmatrix} f_L^{0\pm} \\ f_Q^{0\pm} \\ f_E^{0\pm} \\ f_U^{0\pm} \\ f_D^{0\pm} \\ f_B^{0\pm} \\ f_W^{0\pm} \\ f_g^{0\pm} \end{pmatrix} = \begin{pmatrix} P_{LL}^{0\pm} & 0 & 0 & 0 & 0 & P_{LB}^{0\pm} & P_{LW}^{0\pm} & 0 \\ 0 & P_{QQ}^{0\pm} & 0 & 0 & 0 & P_{QB}^{0\pm} & P_{QW}^{0\pm} & P_{Qg}^{0\pm} \\ 0 & 0 & P_{EE}^{0\pm} & 0 & 0 & P_{EB}^{0\pm} & 0 & 0 \\ 0 & 0 & 0 & P_{UU}^{0\pm} & 0 & P_{UB}^{0\pm} & 0 & P_{Ug}^{0\pm} \\ 0 & 0 & 0 & 0 & P_{DD}^{0\pm} & P_{DB}^{0\pm} & 0 & P_{Dg}^{0\pm} \\ P_{BL}^{0\pm} & P_{BQ}^{0\pm} & P_{BE}^{0\pm} & P_{BU}^{0\pm} & P_{BD}^{0\pm} & P_{BB}^{0\pm} & 0 & 0 \\ P_{WL}^{0\pm} & P_{WQ}^{0\pm} & 0 & 0 & 0 & 0 & P_{WW}^{0\pm} & 0 \\ 0 & P_{gQ}^{0\pm} & 0 & P_{gU}^{0\pm} & P_{gD}^{0\pm} & 0 & 0 & P_{gg}^{0\pm} \end{pmatrix} \otimes \begin{pmatrix} f_L^{0\pm} \\ f_Q^{0\pm} \\ f_E^{0\pm} \\ f_U^{0\pm} \\ f_D^{0\pm} \\ f_B^{0\pm} \\ f_W^{0\pm} \\ f_g^{0\pm} \end{pmatrix}$$

$$\frac{d}{dL} \begin{pmatrix} f_L^{1\pm} \\ f_Q^{1\pm} \\ f_W^{1\pm} \\ f_{BW}^{1\pm} \end{pmatrix} = \begin{pmatrix} P_{LL}^{1\pm} & 0 & P_{LW}^{1\pm} & P_{LM}^{1\pm} \\ 0 & P_{QQ}^{1\pm} & P_{QW}^{1\pm} & P_{QM}^{1\pm} \\ P_{WL}^{1\pm} & P_{WQ}^{1\pm} & P_{WW}^{1\pm} & 0 \\ P_{ML}^{1\pm} & P_{MQ}^{1\pm} & 0 & P_{MM}^{1\pm} \end{pmatrix} \otimes \begin{pmatrix} f_L^{1\pm} \\ f_Q^{1\pm} \\ f_W^{1\pm} \\ f_{BW}^{1\pm} \end{pmatrix}$$

$$\frac{d}{dL} f_W^{2\pm} = P_{WW}^{2\pm} \otimes f_{WW}^{2\pm}$$

The splitting functions can be constructed in terms of Refs. [Chen et al. 1611.00788, Bauer et al.

# Mellin transformation and inversion

- The Mellin transformation converts the convolution into the multiplication

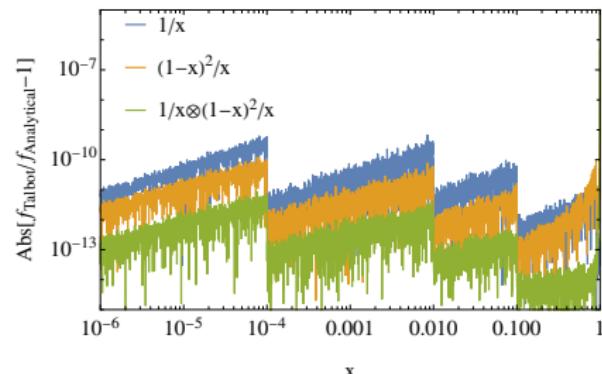
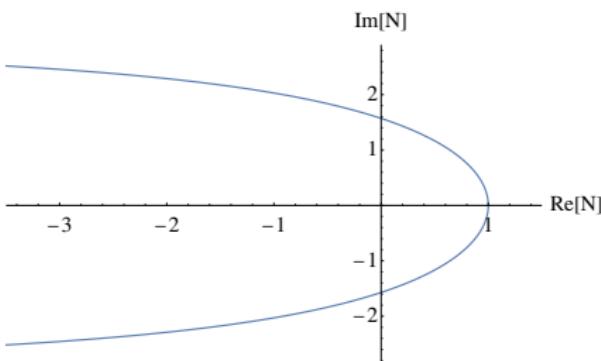
$$\tilde{f}(N) = \int_0^1 dx x^N f(x) \Leftrightarrow f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dN x^{-N} \tilde{f}(N)$$

$$[f \otimes g](x) = \int_x^1 \frac{d\xi}{\xi} f(\xi) g\left(\frac{x}{\xi}\right) \Leftrightarrow [\widetilde{f \cdot g}](N) = \tilde{f}(N) \cdot \tilde{g}(N)$$

- The Talbot algorithm:

- The contour  $N(\theta) = r\theta(\cot\theta + i)$
- The integration

$$f(x) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} d\theta \frac{dN}{d\theta} x^{-N(\theta)} \tilde{f}(N(\theta))$$



# Solving the DGLAP equations iteratively

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- DGLAP equations in the Mellin space:

$$\frac{d}{dL} \tilde{f}_i = \sum_I a_I \sum_j \tilde{P}_{ij}^I \tilde{f}_j,$$

- Numerical equations

$$\frac{\Delta \tilde{f}}{\Delta L} = \frac{\tilde{f}_k - \tilde{f}_{k-1}}{h} = a_{k-1} \tilde{P} \tilde{f}_{k-1} + \mathcal{O}(h).$$

- The recursion

$$\tilde{f}_k = \tilde{f}_{k-1} + h a_{k-1} \tilde{P} \tilde{f}_{k-1} + \mathcal{O}(h^2), \quad \tilde{f} = \tilde{f}_N + \mathcal{O}(h).$$

- The 2nd order of Runge-Kutta algorithm

$$k_1 = a_{k-1} \tilde{P} \tilde{f}_{k-1}, \quad k_2 = a_k \tilde{P} (\tilde{f}_{k-1} + h k_1),$$

$$\tilde{f}_k = \tilde{f}_{k-1} + h \frac{k_1 + k_2}{2} + \mathcal{O}(h^3), \quad \tilde{f} = \tilde{f}_N + \mathcal{O}(h^2).$$

- 4th order

$$k_1 = a_{k-1} \tilde{P} \tilde{f}_{k-1}, \quad k_2 = a_{k-1/2} \tilde{P} (\tilde{f}_{k-1} + h k_1 / 2),$$

$$k_3 = a_{k-1/2} \tilde{P} (\tilde{f}_{k-1} + h k_2 / 2), \quad k_4 = a_k \tilde{P} (\tilde{f}_{k-1} + h k_3),$$

$$\tilde{f}_k = \tilde{f}_{k-1} + \frac{1}{6} h (k_1 + 2k_2 + 2k_3 + k_4) + \mathcal{O}(h^5), \quad \tilde{f} = \tilde{f}_N + \mathcal{O}(h^4),$$

# The negligible contribution from $\nu\bar{\nu}$ and $q,g$

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