κ symmetry

κ symmetry is a local fermionic symmetry

Its form for superstrings is quite intricate ...

Superparticle

$\mu = 0, \dots, D-1$

Worldline fields $\quad x^\mu(\tau)$

$\Theta^{Aa}(\tau) \qquad A = 1, \dots, N$

a spinor index

usual susy

$$\delta\Theta^A = \epsilon^A$$
$$\delta\bar\Theta^A = \bar\epsilon^A$$
$$\delta x^\mu = i\bar\epsilon^A \Gamma^\mu \theta^A$$
$$\delta e = 0$$

ϵ cst spinor

$\bar\epsilon = \epsilon^\dagger C$

we have

$$\delta\dot\Theta^A = \dot\epsilon^A = 0$$
$$\delta(\underbrace{\dot x^\mu - i\bar\Theta^A\Gamma^\mu\dot\Theta^A}) = i(\bar\epsilon^A\Gamma^\mu\Theta^A)^{\cdot} - i\bar\epsilon^A\Gamma^\mu\dot\Theta^A = 0$$

So $S(\dot\Theta^A, p^\mu)$ is susy

e.g.
$$\boxed{S = \frac{1}{2}\int e^{-1}\, p^2\, d\tau \qquad p^\mu \equiv \dot x^\mu - i\bar\Theta^A\Gamma^\mu\Theta^A}$$

is N-susy $\quad$ (for $A = 1, \dots, N$ arbitrary)

eoms

$$\delta_e : \quad p^2 = 0$$
$$\delta_x : \quad \dot p^\mu = 0$$
$$\delta_\Theta : \quad p_\mu \Gamma^\mu \dot\Theta^A = 0$$

The Θ eom is weird: it is not a std kinetic term (+ interactions) on spinor indices it is

$$\textcircled{*} \quad \boxed{M_a{}^b \, \dot\theta_b{}^A = 0} \qquad\qquad M_a{}^b = p\cdot(\Gamma)_a{}^b$$

where $\quad M^2 = p\cdot\Gamma \; p\cdot\Gamma = -p^2 \, \mathbb{1}_a{}^b = 0$

use δe eom

So M has Jordan block decomposition

$$M = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^{[\dim \text{spinor}]/2} \qquad \forall \lambda = 0$$

M removes $\frac{1}{2}$ of components of θ in $\textcircled{*}$

these dofs <u>decouple</u> from theory

S has extra local symmetry $\qquad$ Siegel '83

$$\delta\theta^A = \eta^A \equiv i\, p\cdot\Gamma \kappa^A \qquad \kappa^A = \kappa^A(\tau) \text{ arb. fn}$$

$$\delta x^\mu = i\,\bar\theta^A \Gamma^\mu \delta\theta^A = -\bar\theta\,\Gamma^\mu p\cdot\Gamma\kappa^A$$

let's check this explicitly $\quad$ (I supress A index)

$$\begin{aligned}
\delta p^\mu &= \delta(\dot x^\mu - i\bar\theta\Gamma^\mu\dot\theta) \\
&= i(\bar\theta\Gamma^\mu\delta\theta)^{\cdot} - i\bar\theta\Gamma^\mu\delta\dot\theta + i\,\delta\bar\theta\,\Gamma^\mu\dot\theta \\
&= i\,\dot{\bar\theta}\Gamma^\mu\delta\theta + i\,\delta\bar\theta\Gamma^\mu\dot\theta = 2i\,\dot{\bar\theta}\,\Gamma^\mu\delta\theta
\end{aligned}$$

(Majorana spinors) easily generalize to others

$$\begin{aligned}
\text{So} \quad \delta p^2 &= 2p\cdot\delta p \\
&= 4i\,\dot{\bar\theta}\,p\cdot\Gamma\,\delta\theta = -4\,\dot{\bar\theta}\,p\cdot\Gamma\,p\cdot\Gamma\,\kappa \\
&= 4p^2\,\dot{\bar\theta}\,\kappa
\end{aligned}$$

So

$$\delta S = \frac{1}{2}\int e\, \delta p^2 = \frac{1}{2}\int e\, 4p^2 \bar{\theta} \kappa \neq 0$$

to make S invariant we declare

$$\delta e^{-1} = -4e^{-1}\, \dot{\bar{\theta}}\, \kappa$$

$0 = \delta(e^{-1}e) = \delta e^{-1} e + e^{-1}\delta e$

So $$\delta e = 4e\, \dot{\bar{\theta}}\, \kappa$$

κ is <u>local</u> but is <u>not</u> w-line susy (no w-line spinors!)

<u>Ex:</u> 1) Show using eoms that

$$[\delta_{\kappa_1}, \delta_{\kappa_2}]\, \theta = i\, p\cdot\Gamma\, \kappa^A$$

for $\kappa^A \equiv 4\left(\kappa_2^A\, \bar{\theta}^B \kappa_1^B - 1 \leftrightarrow 2\right)$

So commutator of two κ symms is a κ symm

2) No conserved quantities (vanish on-shell)

κ-Symmetry: Superstring

The same game can be played for Green-Schwarz superstring

BUT — $N=1,2$ only (to write down "WZ" term)

— Need Fierz id (for susy: MW spinors in 3, 4, 6, 10 dims)

$$S_{GS} = -\frac{1}{2\pi}\int d^2\sigma \, \gamma^{\alpha\beta} \Pi^\mu_\alpha \Pi^\nu_\beta \, \eta_{\mu\nu} \qquad \gamma^{\alpha\beta} \equiv \sqrt{h}\, h^{\alpha\beta}$$

$$-\frac{1}{\pi}\int d^2\sigma \, \epsilon^{\alpha\beta}\Big[i\,\partial_\alpha X^\mu \left(\bar\Theta^1 \Gamma_\mu \partial_\beta \Theta^1 - \bar\Theta^2 \Gamma_\mu \partial_\beta \Theta^2\right) + \bar\Theta^1 \Gamma^\mu \partial_\alpha \Theta^1 \, \bar\Theta^2 \Gamma_\mu \partial_\beta \Theta^2 \Big]$$

where $$\Pi_\alpha{}^\mu \equiv \partial_\alpha X^\mu - i\,\bar\Theta^A \Gamma^\mu \partial_\alpha \Theta^A$$

Θ^{Aa} $\quad A = 1, 2$

$a \in$ MW spinor in $D = 1+9$

1) S_{GS} is supersymmetric

$$\delta\Theta^A = \epsilon^A \qquad \epsilon^A \text{ cst spinors}$$

$$\delta X^\mu = i\,\bar\epsilon^A \Gamma^\mu \Theta^A$$

Need Fierz $\quad \bar\epsilon\, \Gamma_\mu \psi_{[1} \bar\psi_2 \Gamma^\mu \psi_{3]} = 0$

2) S_{GS} is κ-symmetric

$$\delta\Theta^A = 2i\, \Gamma^\mu \Pi_{\mu\alpha} \kappa^{A\alpha}$$

$$\delta X^\mu = i\,\bar\Theta^A \Gamma^\mu \delta\Theta^A$$

$$\delta\gamma^{\alpha\beta} = -16\sqrt{h}\left(P_-^{\alpha\gamma}\bar{\kappa}^{1\beta}\partial_\gamma\theta^1 + P_+^{\alpha\gamma}\bar{\kappa}^{2\beta}\partial_\gamma\theta^2\right)$$

where $P_\pm \equiv \frac{1}{2}\left(h^{\alpha\beta} \pm \epsilon^{\alpha\beta}/\sqrt{h}\right)$

projects reducible 2d vector onto (anti) s.d. 1d irrep

Ex: • Show that 1) + 2) hold (GSW vol1 5.1.2, 5.1.3)

• Show that κ symmetry can be used to fix

$$(\Gamma^0 + \Gamma^9)\,\theta^A = 0$$

• using this κ-gauge and

$$X^+ = x^+ + p^+\tau$$

show that $S_{G.S.}$ leads to eoms for

$$(\partial_\sigma^2 - \partial_\tau^2)X^i = 0 \qquad i = 1 \ldots 8$$

$$(\partial_\tau + \partial_\sigma)S^{1a} = 0 \qquad a = 1 \ldots 8$$

$$(\partial_\tau - \partial_\sigma)S^{2a} = 0$$

Coset Superstring

GS superstring action is formally known in any sugra bkd.

Soln of sugra $\Longleftrightarrow$ κ symmetry

In practice, need to find full sugra superfields (solve torsion constraints) NOT just spacetime fields...

For some bkds algebraic coset formulation instead.

e.g. – Flat space Henneaux Mezincescu 84

– $AdS_5 \times S^5$ Metsaev Tseytlin 98a

– plane-wave

The target spacetime is a symmetric space

e.g.

$$AdS_5 \times S^5 \cong \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$$

Symmetric spaces: G/H group coset with Lie alg relations

$$[H, H] \in H$$

$$[G\backslash H, H] \in$$

$$\underbrace{[G\backslash H, G\backslash H] \in H \qquad \Omega(G\backslash H) = -G\backslash H}$$

$\mathbb{Z}_2$ automorphism

$$\Omega(H) = H$$

Exercise: $S^5 \cong \frac{SO(6)}{SO(5)}$ with Lie algebra $SO(5) = \{M_{ij}\}$

$SO(6)\backslash SO(5) = \{M_{i6}\}$

$i, j = 1, \dots, 5$

1) Show that the commutation rels take the symmetric space form.

2) A useful rep of $SO(6)$ is 4×4 γ matrices γ^i, $\gamma^{ij} = \gamma^{[i}\gamma^{j]}$. Show that this does give an $SO(6)$ rep ($SO(5)$ obvious) $i, j = 1, \dots, 5$

3) $e^M = 1 + M + \frac{M^2}{2} + \dots$

use the Clifford algebra structure to find

$g = e^{x^i \gamma^i}$ for some scalars x^i

4) hence compute $j = g^{-1} dg$

Given a group manifold $g = e^{\mathbb{X}}$, Lie-alg valued current

$$j = g^{-1} dg = \left(1 - \mathbb{X} + \frac{\mathbb{X}^2}{2} + \dots\right)\left(d\mathbb{X} + \tfrac{1}{2} d\mathbb{X}\,\mathbb{X} + \tfrac{1}{2}\mathbb{X}\, d\mathbb{X} + \dots\right)$$
$$= d\mathbb{X} + \tfrac{1}{2}[d\mathbb{X}, \mathbb{X}] + \tfrac{1}{6}[[d\mathbb{X}, \mathbb{X}], \mathbb{X}] + \dots$$

On a symmetric space

$$j = \underset{\substack{\cap \\ H}}{j^{(0)}} + \underset{\substack{\cap \\ G \setminus H}}{j^{(2)}}$$

$J^{(0)}$ is spin connection ω

$J^{(2)}$ is vielbein e

$\gamma^{\alpha\beta} \equiv \sqrt{h}\, h^{\alpha\beta}$

σ-model action

$$\gamma^{\alpha\beta} e^a_\alpha e^a_\beta = \gamma^{\alpha\beta} \operatorname{Tr}\left(J^{(2)}_\alpha J^{(2)}_\beta\right)$$

Exercise: 5) Using Ex.4, show that

$$\operatorname{Tr}\left(J^{(2)} J^{(2)}\right) = ds^2_{S^5}$$

Supersymmetry $\rightsquigarrow$ semi-symmetric spaces

These have a $\mathbb{Z}_4$ automorphism $(1, i, -1, -i)$

$$J = \underset{\text{bos}}{J^{(0)}} + \underset{\text{ferm}}{J^{(1)}} + \underset{\text{bos}}{J^{(2)}} + \underset{\text{ferm}}{J^{(3)}}$$

On such bkds can define action

$$S = \int d^2\sigma\, \gamma^{\alpha\beta} \operatorname{Tr}\left(J^{(2)}_\alpha J^{(2)}_\beta\right) + \epsilon^{\alpha\beta} \operatorname{Tr}\left(J^{(1)}_\alpha J^{(3)}_\beta\right)$$

This turns out to be the same action as the GS action for flat space, $AdS_5 \times S^5$, max susy plane wave

Exercise 6) Flat space $N=2$ susy algebra is

$$\{Q^I, Q^J\} = -2i\,\delta^{IJ} (C\Gamma^\mu) P$$

$[Q_a, Q_b] \sim (C\Gamma^\mu)_{ab} P_\mu$

$$[P_\mu, P_\nu] = 0 \qquad [P_\mu, J_{\nu\rho}] = \eta_{\mu\nu} P_\rho - \eta_{\mu\rho} P_\nu$$

$$[J_{\mu\nu}, J_{\rho\lambda}] = \eta_{\mu\rho} J_{\nu\lambda} + 3 \text{ terms}$$

$$[P, Q] = 0 \qquad [J_{\mu\nu}, Q^I] \sim \Gamma_{\mu\nu} Q^I$$

show that this has $\mathbb{Z}_4$ automorphism

Example: Flat space

$[P, Q] = 0$

Define $g = e^{x^\mu P_\mu + \Theta^I Q^I} = e^{x^\mu P_\mu} e^{\Theta^I Q^I} \equiv g_b \, g_f$

Then $g^{-1} dg = g_f^{-1} g_b^{-1} d(g_b g_f) = g_f^{-1} g_b^{-1} dg_b \, g_f + g_f^{-1} dg_f$

$$= g_f^{-1} dx^\mu P_\mu \, g_f + g_f^{-1} dg_f$$

$[P, Q] = 0$

$$= dx^\mu P_\mu + d\Theta^I Q^I + \tfrac{1}{2}[d\Theta^I Q^I, \Theta^J Q^J] + \tfrac{1}{6}[[d\Theta^I Q^I, \Theta^J Q^J], \Theta^K Q^K] + \dots$$

$[[Q\,Q]\,Q] \sim [P\ Q] \sim 0$

$$= \underbrace{(dx^\mu - i\, d\Theta^I C\Gamma^\mu \Theta^I)P^\mu}_{=\ J^{(2)}} + \underbrace{d\Theta^I Q^I}_{(J^{(1)} + J^{(3)})}$$

Notice that (the pull back)

$$J^{(2)}_\alpha = \partial_\alpha x^\mu - i\, \bar{\Theta}^I \Gamma^\mu \partial_\alpha \Theta^I \equiv \Pi_\alpha{}^\mu$$

from our earlier discussion

If we define bilinear form "STr"

$$\mathrm{STr}(P_\mu P_\nu) = \eta_{\mu\nu}$$

$$\mathrm{STr}\left(Q_a^I\, Q_b^J\right) = \delta^{IJ} C_{ab}$$

Then

$$\gamma^{\alpha\beta}\,\mathrm{STr}\left(J^{(2)}_\alpha\, J^{(2)}_\beta\right) = \gamma^{\alpha\beta}\,\Pi_\alpha{}^\mu\,\Pi_\beta{}^\nu\,\eta_{\mu\nu}$$

so we get kinetic term of flat space GS action!

The "WZ" term is technically a little more subtle in flat space and needs to be obtained from the 3d form of the coupling

$$\int d^3\sigma\,\mathrm{STr}\left(J^{(2)}_\mu\left[J^{(1)}_\nu, J^{(1)}_\rho\right] - J^{(2)}_\mu\left[J^{(3)}_\nu\, J^{(3)}_\rho\right]\right)\epsilon^{\mu\nu\rho}$$

Exercise: 7) Show that J satisfies the Maurer-Cartan / flatness condition

$$dJ - [J, J] = 0$$

8) Use the $\mathbb{Z}_4$ decomposition of MCeq to show that the 2d & 3d versions of the WZW coupling are equivalent.

$$d\,\mathrm{STr}\left(J^{(1)}\wedge J^{(3)}\right) = \mathrm{Str}\left(J^{(2)}\left[J^{(1)}J^{(1)}\right] - J^{(2)}\left[J^{(3)}\,J^{(3)}\right]\right)$$

Why does this fail in flat space?

Coset κ-symmetry

$\gamma^{\alpha\beta} = \sqrt{h}\, h^{\alpha\beta}$

The MT action is

$$S_{MT} = \int d^2\sigma \; \gamma^{\alpha\beta}\, \mathrm{Str}\left(J^{(2)}_\alpha J^{(2)}_\beta\right) + \epsilon^{\alpha\beta}\, \mathrm{Str}\left(J^{(1)}_\alpha J^{(3)}_\beta\right)$$

Exercises: Show that

1) S_{MT} has global $PSU(2,2|4)$ symmetry Noether ct

$$J^\alpha = g\left[\gamma^{\alpha\beta} J^{(2)}_\beta - \tfrac{1}{2}\epsilon^{\alpha\beta}\left(J^{(1)}_\beta - J^{(3)}_\beta\right)\right] g^{-1}$$

coming from left-multiplication $g \to hg$

2) Virasoro conditions $\dfrac{\delta S_{MT}}{\delta h^{\alpha\beta}}$

$$\mathrm{Str}\left(J^{(2)}_\alpha J^{(2)}_\beta\right) - \tfrac{1}{2}\gamma_{\alpha\beta}\, \mathrm{Str}\left(J^{(2)}_\delta J^{(2)\delta}\right) = 0$$

S_{MT} has κ symmetry from right local multiplication by fermionic elt

$$g \to g\, e^{\epsilon}$$

$\epsilon(x_0, x_1) \in \mathfrak{g}^{(1)} + \mathfrak{g}^{(3)}$

$\epsilon = \epsilon^{(1)} + \epsilon^{(3)}$

Currents transform as

$$\delta_\epsilon J = -d\epsilon + [J, \epsilon]$$

which can be decomposed under $\mathbb{Z}_4$ e.g.

$$\delta_\epsilon J^{(1)} = -d\epsilon^{(1)} + [J^{(0)}, \epsilon^{(1)}] + [J^{(2)}, \epsilon^{(3)}]$$

$$\delta_\epsilon J^{(2)} = [J^{(1)}, \epsilon^{(1)}] + [J^{(3)}, \epsilon^{(3)}]$$

<u>Exercise:</u> Show that

$$\delta_\epsilon S_{MT} = \delta_\epsilon \gamma^{\alpha\beta}\, \mathrm{Str}\left(J^{(2)}_\alpha J^{(2)}_\beta\right)$$
$$- 4 P_+^{\alpha\beta}\, \mathrm{Str}\left([J^{(1)}_\beta, J^{(2)}_\alpha]\, \epsilon^{(1)}\right)$$
$$- 4 P_-^{\alpha\beta}\, \mathrm{Str}\left([J^{(3)}_\beta, J^{(2)}_\alpha]\, \epsilon^{(3)}\right)$$

$\delta_\epsilon \gamma^{\alpha\beta}$ undetermined as yet

recall $P_\pm^{\alpha\beta} = (\gamma^{\alpha\beta} \pm \epsilon^{\alpha\beta})/2$ projects onto (a)s.d. vector reps

<u>Hint:</u> to get the above expression IBP $d\epsilon$ terms & use MCeqs

Denoting by $V_{\alpha,\pm}$ the $P_\pm$-projected V_α we take ϵ to be of the form

$$\epsilon^{(1)} = J^{(2)}_{\alpha -} \kappa^{(1)\alpha}_{+} + \kappa^{(1)\alpha}_{+} J^{(2)}_{\alpha -}$$

$$\epsilon^{(3)} = J^{(2)}_{\alpha +} \kappa^{(3)\alpha}_{-} + \kappa^{(3)\alpha}_{-} J^{(2)}_{\alpha +}$$

$\kappa^{(1)}, \kappa^{(3)}$ are the general κ-variations

The above expression uses an explicit matrix product

$M^{(1)} M^{(2)} \in \Omega^{(1)}$

κ can be given suitable reality conditions to have $\epsilon^{(1)}, \epsilon^{(3)}$ in the Lie algebra

[illegible] ϵ into $\delta\, S$ we find

Inserting this form of ϵ into $\delta_\epsilon S_{MT}$ we find

$$
\begin{aligned}
\delta_\epsilon S_{MT} = \delta_\epsilon \gamma^{\alpha\beta}\, & \mathrm{Str}\left(J^{(2)}_\alpha J^{(2)}_\beta\right) \\
& - \tfrac{1}{2}\, \mathrm{Str}\left(J^{(2)}_{\alpha-} J^{(2)}_{\beta-}\right) \mathrm{Str}\left(\Upsilon\left[\kappa^{(1)\beta}_{+}, A^{(1)\alpha}_{+}\right]\right) \\
& - \tfrac{1}{2}\, \mathrm{Str}\left(J^{(2)}_{\alpha+} J^{(2)}_{\beta+}\right) \mathrm{Str}\left(\Upsilon\left[\kappa^{(3)\beta}_{-}, A^{(3)\alpha}_{-}\right]\right)
\end{aligned}
$$

Exercise: Above, we make use of the following relations

$$
J^{(2)}_{\alpha+} J^{(2)}_{\beta+} = \begin{pmatrix} m^i_{\alpha+} m^j_{\beta+} \gamma^i \gamma^j & 0 \\ 0 & n^i_{\alpha+} n^j_{\beta+} \gamma^i \gamma^j \end{pmatrix}
$$

(same for $+ \to -$)

$m^i_{\alpha+} m^j_{\beta+}$ is symmetric in (i,j) so $\gamma^i \gamma^j \to \frac{1}{2}\{\gamma^i, \gamma^j\}$

Then

$$
J^{(2)}_{\alpha+} J^{(2)}_{\beta+} = c\, \Upsilon + d\, \mathbb{1}
$$

The d term cancels from $\delta_\epsilon S_{MT}$ and

$$
c = \frac{1}{8} \mathrm{Str}\left(J^{(2)}_{\alpha+} J^{(2)}_{\beta-}\right)
$$

This is the $psu(2,2|4)$ version of Fierz identities

Finally if we declare

$$
\delta_\epsilon \gamma^{\alpha\beta} = \frac{1}{4} \mathrm{Str}\, \Upsilon\left(\left[\kappa^{(1)\alpha}_{+}, J^{(1)\beta}_{+}\right] + \left[\kappa^{(3)\alpha}_{-}, J^{(3)\beta}_{-}\right] + \alpha \leftrightarrow \beta\right)
$$

the rhs is symmetric, bosonic, real, $\gamma_{\alpha\beta}\, \delta\gamma^{\alpha\beta} = 0$

we find $\delta_\epsilon S_{MT} = 0$

(since $\det \gamma = 1$)