Integrable sigma models

2d theories are q unusual. They often have hidden symmetries: bosonization, T-duality, $SO(2)_1$...

One such property is integrability or the presence of extra conserved quantities

For example, the principal chiral model (PCM)

$$S_{PCM} = \frac{1}{2}\int \mathrm{Tr}\left(\partial_\mu g\, \partial^\mu g^{-1}\right) \qquad g(x_\mu) \in G \text{ some Lie grp}$$

$$= \frac{1}{2}\int \mathrm{Tr}\left(g g^{-1}\partial_\mu g\, \partial^\mu g^{-1}\right)$$

$$= -\frac{1}{2}\int \mathrm{Tr}\left(g^{-1}\partial_\mu g\; g^{-1}\partial^\mu g\right)$$

$\partial_\mu(g^{-1}g) = 0$

so $\partial_\mu g^{-1} g = -g^{-1}\partial_\mu g$

Its eom is

$0 = \delta(g^{-1}g)$

so $\delta g^{-1} = -g^{-1}\delta g g^{-1}$

$$\delta S_{PCM} = -\int \mathrm{Tr}\left[\delta(g^{-1}\partial g)\, g^{-1}\partial g\right]$$

$$= -\int \mathrm{Tr}\left[-g^{-1}\delta g\, \underline{g^{-1}\partial g\, g^{-1}}\partial g + g^{-1}\partial\delta g\, g^{-1}\partial g\right]$$

$$\overset{IBP}{=} -\int \mathrm{Tr}\left[g^{-1}\delta g\, (\partial g^{-1})\partial g - \delta g\, \partial(g^{-1}\partial g\, g^{-1})\right]$$

$$= -\int \mathrm{Tr}\, \delta g\left[\partial g^{-1}\, \partial g g^{-1} - \partial(g^{-1}\partial g g^{-1})\right]$$

$$= \int \mathrm{Tr}\left[\delta g\, g^{-1}\, \partial(\partial g g^{-1})\right]$$

completness of Tr on Lie alg $\Rightarrow$ $\partial_\mu(\partial^\mu g g^{-1}) = 0$

Can show that we also have $\partial_\mu(g^{-1}\partial^\mu g) = 0$

Ex Show that we also have $\partial_\mu (g^{-1} \partial^\mu g) = 0$

The eoms are just Noether current eqs

$$\partial_\mu J_L^\mu = 0 \qquad \partial_\mu J_R^\mu = 0$$

Ex Derive $J_L^\mu = \partial^\mu g \, g^{-1}$, $J_R^\mu = g^{-1} \partial^\mu g$

using Noether procedure from invariance

$$g \to h_L g \,, \qquad g \to g h_R$$

These currents encode the global target-space symmetry of PCM $G_L \times G_R$

Not only are they conserved, they are FLAT

$$d J_R = d(g^{-1} dg) = \underbrace{dg^{-1}} \wedge dg + g^{-1} \cancel{d^2}^{\,=0} g$$

$$= -(g^{-1} dg \, g^{-1}) dg$$

i.e.

$$0 = D J_R \equiv d J_R + J_R \wedge J_R$$

Maurer-Cartan integrability eq

Ex Show that in components

$$\partial_0 J_1^a - \partial_1 J_0^a + f^a{}_{bc} J_0^b J_1^c = 0$$

$f^a{}_{bc}$ structure constants of Lie alg

It is this FLATNESS that is key to extra conserved quantities.

Compute: $d \int^{x_1} j^0(x_0, y)\,dy = (dx_0 \partial_0 + dx_1 \partial_1) \int^{x_1} j^0$

$$= dx_0 \int^{x_1} \partial_0 j^0 + dx_1\, j^0(x_0, x_1)$$

$$= -dx_0 \int^{x_1} \partial_1 j^1 + dx_1\, j^0 = -dx_0\, j^1 + dx_1 j^0$$

$$= * j$$

using this it is simple to check

$$\hat{j}_\mu = * j_\mu - \left(\int^{x_1} j^0\right) j_\mu$$

is a conserved current $\partial_\mu \hat{j}^\mu = 0$

with corresp non-local charge

$$\hat{Q} = \int_{-\infty}^{\infty} j^1\, dx_1 - \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_1} dy_1 \left[j^0(x_0, x_1), j^0(x_0, y_1) \right]$$

In fact, can construct ∞ tower of con. curs

potential

Since $\partial^\mu j_\mu^{(n)} = 0 \implies j_\mu^{(n)} = \epsilon_{\mu\nu} \partial^\nu \chi^{(n)}$

then $$J_\mu^{(n+1)} \equiv D_\mu \chi^{(n)} = \partial_\mu \chi^{(n)} + J_\mu \chi^{(n)}$$

is also conserved

$$\partial^\mu J_\mu^{(n+1)} = \partial^\mu D_\mu \chi^{(n)} = D_\mu \partial^\mu \chi^{(n)} = D_\mu \epsilon^{\mu\nu} J_\nu^{(n)}$$

$$= \epsilon^{\mu\nu} D_\mu D_\nu \chi^{(n-1)} = 0$$

Ex Show $[\partial_\mu, D^\mu] = \partial_\mu J^\mu = 0$

$$[D_\mu, D_\nu] = \partial_{[\mu} J_{\nu]} + [J_\mu, J_\nu] = 0$$

show that with $\chi^{(-1)} = 1$, $J^{(0)} \equiv J^\mu_R$

These currents give corresponding conserved charges $Q^{(n)}$ combined into

$$T(u) = \sum_{k=-1}^{\infty} u^{-k-1} Q^{(k)} = 1 + \frac{1}{u} Q^{(0)} + \frac{1}{u^2} Q^{(1)} + \dots$$

This MONODROMY MATRIX can be defined more abstractly using LAX CONNECTION

$$\mathcal{L}_\mu \equiv \frac{1}{u^2-1} \left(J_\mu + u \, \epsilon_\mu{}^\nu J_\nu \right)$$

$$\mathcal{D}_\mu \equiv \partial_\mu - \mathcal{L}_\mu$$

Flatness of LAX $\Longleftrightarrow$ eoms + Maurer Cartan

$$[\partial_\mu - \mathcal{L}_\mu , \partial_\nu - \mathcal{L}_\nu] = -\partial_\mu \mathcal{L}_\nu + \partial_\nu \mathcal{L}_\mu + [\mathcal{L}_\mu , \mathcal{L}_\nu]$$

$$= \frac{1}{u^2-1}\left(-\partial_{[\mu} J_{\nu]} - u\,\partial_\mu \epsilon_{\nu\rho} J^\rho + u\,\partial_\nu \epsilon_{\mu\rho} J^\rho + \frac{1}{u^2-1}\left([J_\mu , J_\nu] + u^2 \epsilon_{\mu\rho}\epsilon_{\nu\sigma}[J^\rho , J^\sigma] \right) \right)$$

$$= \frac{1}{u^2-1}\left(-\underbrace{\partial_{[\mu} J_{\nu]}}_{MC} + 2u\,\underbrace{\partial_\mu J^\mu}_{con} + \underbrace{\frac{1-u^2}{u^2-1}[J_\mu , J_\nu]}_{MC} \right)$$

$$= 0$$

The flatness of Lax connection $\mathcal{D}_\mu$ $\Rightarrow$

aux lin system

$$\mathcal{D}_\mu \Phi(x_\mu ; u) = 0$$

is consistent. Formally, we solve it

"initial" condition

$$\Phi = \underbrace{P\exp\left(\int_{\tilde{x}_1}^{x_1} dy_1\, \mathcal{L}_1(x_0, y_1 ; u) \right)}_{\equiv T(x_0, \tilde{x}_1, x_1 ; u)} \Phi(x_0, \tilde{x}_1 ; u)$$

Formally $\mathcal{D}_1 T = \partial_1 T - \mathcal{L}_1 T = \mathcal{L}_1 T - \mathcal{L}_1 T = 0$

($\mathcal{D}_0$ eq is compatible)

The Monodromy matrix

$$T \equiv T(x_0 , \tilde{x}_1 = -\infty , x_1 = \infty ; u)$$

$$= P\exp\left(\int^{\infty} \mathcal{L}_1 \right)$$

$\left. \right._{-\infty}$

Ex: show that T has expansion

$$T = 1 + \frac{1}{u} Q^{(0)} + \frac{1}{u^2} Q^{(1)} + \dots$$

Hint: $\alpha_\mu \Big|_{u=0} = 0 \qquad \partial_u \alpha_\mu \Big|_{u=0} = * j_\mu \qquad \partial_u^2 \alpha_\mu \Big|_{u=0} = 2 j_\mu$

How do these charges transform under

Lorentz $\qquad x^\mu \to x^\mu + \lambda^\mu{}_\nu x^\nu \qquad \lambda^{\mu\nu} = -\lambda^{\nu\mu}$

boosts in particular

Ex: Show

$$0 = \partial_\mu (b^\mu)^{\rho\sigma} = \partial_\mu \left(x^{[\rho} T^{\sigma]\mu} \right)$$

in particular boost "charge" is

$$B = \int dx_1 \, x_1 T^{00} = \int dx_1 \, x_1 H$$

For PCM (& other models) Poisson bracket analysis

$$\{ B, Q^{(0)} \}_{PB} = 0 \qquad \{ B, Q^{(1)} \} = 0$$

In quantum theory this is deformed
(coupling h)

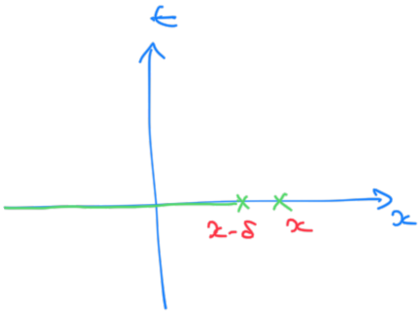
$$[B, Q^{(0)}] = 0$$

$$[B, Q^{(1)}] = -\frac{\hbar C}{4\pi i} Q^{(0)}$$

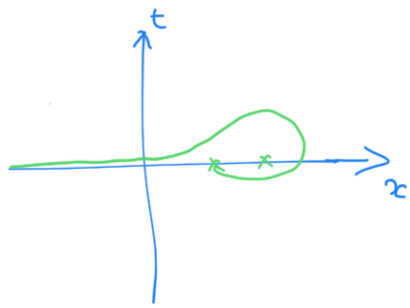
$C = T^a T^a$ is the quadratic Casimir

The deformation happens because $Q^{(1)}$ is bi-local and needs to be regularised

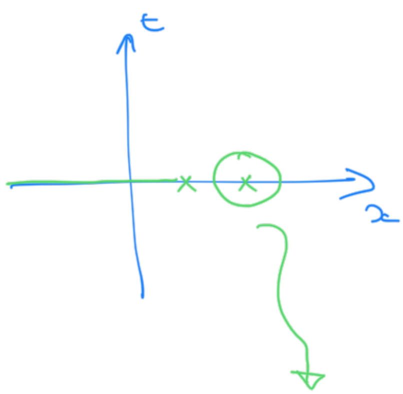
$$Q^{(1)} \sim \int j(x) \int^x dy\, j(y) \sim \int^x dy\, \frac{j(x)}{y-x}$$



Boost by 2π



deform contour
(j flat)



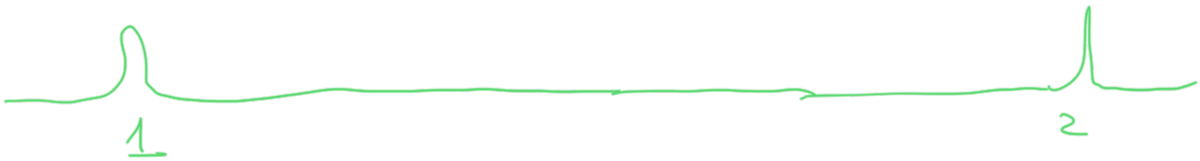
picks up residue

$$J(x)\,J(y) \sim \frac{J(x)}{x-y}$$

Consider evaluating the bilocal charge

$$Q^{(1)a} = f^{a}{}_{bc} \int dx\, J_0^b \int^x dy\, J_0^c$$

on two distant excitations



The integration splits into:

$$\underbrace{\int j \int j}_{\text{both near 1}} + \underbrace{\int j \int j}_{\text{both near 2}} + \underbrace{\underbrace{\int j}_{\text{near 1}}}_{Q^{(0)}} \; \underbrace{\underbrace{\int j}_{\text{near 2}}}_{Q^{(0)}}$$

so the charge will be

$$Q^{(1)} = Q^{(1)} \otimes 1 + 1 \otimes Q^{(1)} + f_{abc}\, Q^{(0)} \otimes Q^{(0)}$$

This is how the Yangian is given in Drinfeld's first realization

we see that even if $Q^{(1)}$ is zero on a single excitation, generic states will be made up of many excitations and will necessarily have non-zero $Q^{(1)}$

Exercise: the MT superstring has a Lax connection

$$\mathcal{L}_\mu = \underbrace{J^{(0)}_\mu + \frac{1+z^2}{1-z^2} J^{(2)}_\mu + \frac{2z}{1-z^2} * J^{(2)}_\mu} + \sqrt{\frac{1+\lambda}{1-\lambda}}\, J^{(1)} + \sqrt{\frac{1-\lambda}{1+\lambda}}\, J^{(3)}$$

generalizing the symmetric space G/H Lax

Yangian structure of String theory is more complicated because of κ-symmetry. Need to gauge-fix which breaks 1+1d Lorentz invariance.

Yang Baxter equation

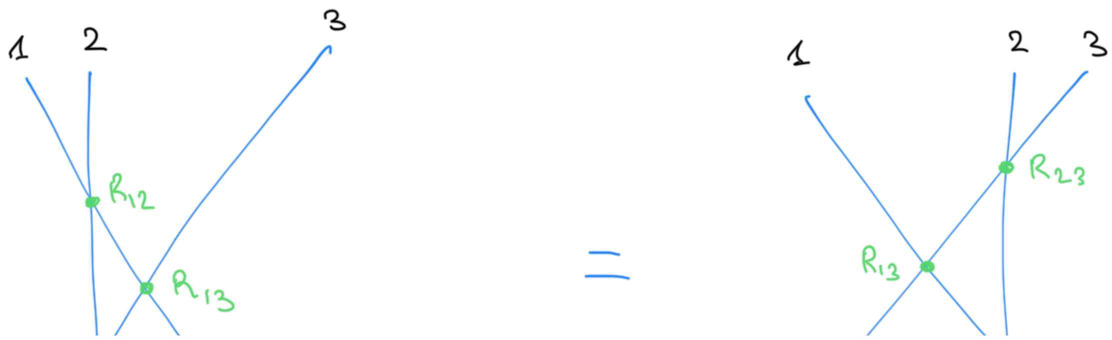
Consider a wave-function made of plane waves

$$\underset{x_1<x_2}{\Psi} \sim A\, e^{i(p_1 x_1 + p_2 x_2)} + B\, e^{i(p_2 x_1 + p_1 x_2)}$$

A/B is the S-matrix for scattering plane-wave 1 past plane-wave 2

In 1+1 d integrable theories the presence of (commuting) higher charges means we get an (infinite) set of constraints on $|out\rangle$ given some $|in\rangle$.

- no particle production
- $\{p_k^{(in)}\} = \{p_k^{(out)}\}$ (global syms also)
- factorization



$$\boxed{S_{ij} = \mathbb{P}_{ij} \cdot R_{ij}}$$

Interactions only happen $2 \to 2$ (in some frame)

different orderings must be equivalent:

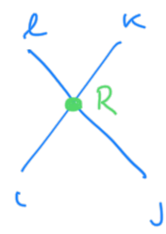
$$\boxed{R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}}$$

is the Yang Baxter equation

Typically, plane waves in rep of $Q^{(0)a} T^a$

so YBE is a matrix eq. R preserves sym

$$[R, Q^{(0)}] = 0 \qquad R_{ij}^{kl} =$$



In 1+1 d the on-shell condition means we can parametrize the dispersion relation by rapidity u

$$\begin{pmatrix} p^0 \\ p^1 \end{pmatrix} = m \begin{pmatrix} \cosh u \\ \sinh u \end{pmatrix}$$

Boost: $B = \partial_u$

$e^{aB} |u\rangle = |u+a\rangle$

and the R matrix will depends only on

The Mandelstam variable depends only on

difference of rapidities

$$(p_i+p_j)^2 = 4m^2\cosh^2\left(\frac{u_i-u_j}{2}\right) \equiv 4m^2\cosh^2\frac{u_{12}}{2}$$

So $R_{ij} \equiv R_{ij}(u_{ij})$ for relativistic 1+1d th's

Yang's original solution

$$R(u) = \underbrace{\frac{u}{u+ic}}_{\text{Not fixed by YBE}}\left(\mathbb{1} - \frac{ic}{u}\mathbb{P}\right)$$

consistent with $Q^{(0)}$ symm

$\mathbb{P}_{ij}$ permutes $i \leftrightarrow j$ & for fundamental rep of $SU(N)$

$$\mathbb{P} = T^a \otimes T^a = C$$

Exercise: Check above R solves YBE

Finding / classifying solutions to YBE major enterprise

Drinfeld considered $h \to 0$ "classical limit"

$$R_{12} = 1 + h r_{12} + O(h^2)$$

Exercise Expand the YBE to $O(h^2)$ & show it implies

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

This is easily solved by quadratic Casimir

$$r_{12} \propto C = T^a \otimes T^a$$

Exercise: Check this explicitly for $SU(2)$, $SU(N)$ ...

Since r is expressed in terms of $Q^{(0)} \otimes Q^{(0)}$
can we use $Q^{(n)}$ to find higher order terms
in expansion of R ? Drinfeld $\rightsquigarrow$ YES

Need to have representations of $Q^{(n)}$

Given some such rep ρ_0 we can Boost it

$$\rho_u(Q^{(0)}) \equiv \rho_0(B(Q^{(0)})) = \rho_0(Q^{(0)})$$

$$\rho_u(Q^{(1)}) \equiv \rho_0(B(Q^{(1)})) = \rho_0(Q^{(1)} + u\, Q^{(0)})$$

$$= \rho_0(Q^{(1)}) + u\, \rho_0(Q^{(0)})$$

Drinfeld: for suitable Lie algebra reps $\rho_0(Q^{(0)})$
lifts to rep of Yangian with $\rho_0(Q^{(1)}) = 0$

and evaluation rep

$$\boxed{\rho_u(Q^{(0)}) = Q^{(0)} \quad \Big| \quad \rho_u(Q^{(1)}) = u\, \rho_0(Q^{(0)})}$$

R acts on $\rho_u(V^n) \otimes \rho_u(V^n)$
preserving the higher charges

$$\text{“}\left[R,\ \rho_u \otimes \rho_v \left(Q^{(n)}\right)\right]\text{”} = 0$$

with perturbative expansion

$$\log R = \frac{h}{u}\, T^a_{(0)} \otimes T^a_{(0)} + \frac{h^2}{u^2}\left(T^a_{(1)} \otimes T^a_{(0)} - T^a_{(0)} \otimes T^a_{(1)}\right) + h^3 \ldots$$

To make $[R, Q^{(n)}] = 0$ precise need Hopf algebra, coproducts ...
and is beyond the scope of these lectures.

A recent review of integrability: 1606.02945

Summary

① Integrable 1+1d theories (such as σ model on S^3) have extra non-local charges $Q^{(n)}$

② In quantum theory these charges transform nontrivially under Lorentz boosts e.g.

$$[\,Q^{(1)}, B\,] = h\, Q^{(0)}$$

③ On multi-excitation states $Q^{(n)}$ are necessarily non-trivial e.g.

$$\Delta(Q^{(1)}) = Q^{(1)} \otimes 1 + 1 \otimes Q^{(1)} + f^{a}{}_{bc}\, Q^{(0)b} \otimes Q^{(0)c}$$

④ Higher charges constrain Scattering of excitations (only $2 \leftrightarrow 2$ scattering)

$$[Q^{(n)}, S] = 0$$

⑤ S-matrix has perturbative expansion

$$S = 1 + h\, s_1 + h^2 s_2 + \dots$$

where s_i can be written in terms of $Q^{(n)}$