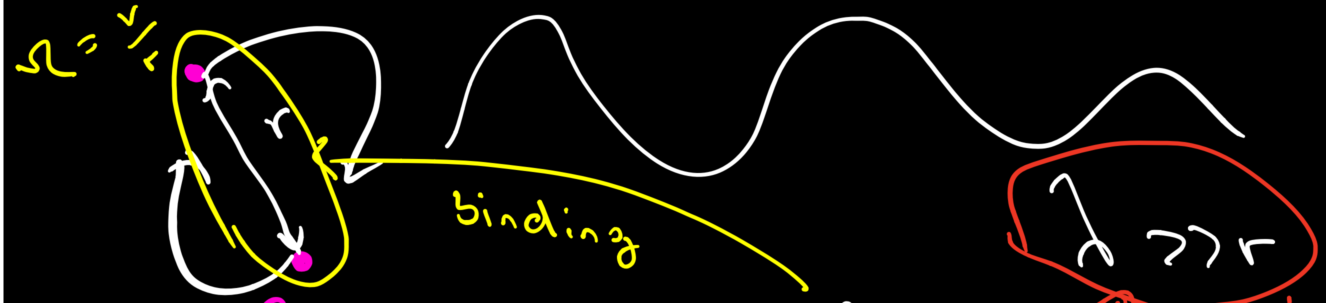
III

2-body problem

(NRGR)
Goldberger
Rothstein



WL theory
(Love, etc)

$$\overset{0}{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

"Method of regions" $\Longrightarrow$

$H_{\mu\nu}$ potential $\left(\partial_0 \sim \frac{v}{r};\ \partial_i \sim \frac{1}{r}\right)$

$\bar{h}_{\mu\nu}$ radiation $\partial_\mu \sim \left(\frac{v}{r}, \frac{v}{r}\right)$

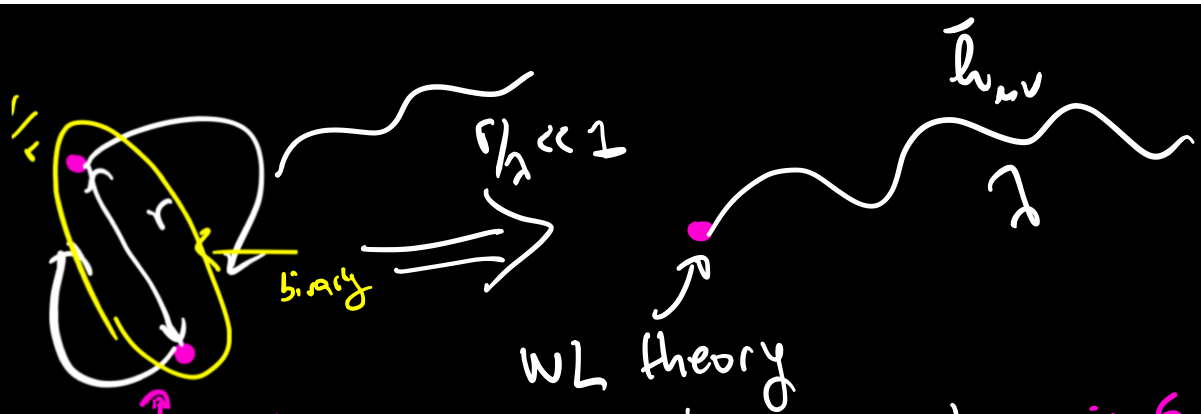
on-shell
$i\varepsilon$ matters!

$\Downarrow$ off-shell

$$\frac{1}{p_0^2 - \vec{p}^{\,2} \cancel{+ i\varepsilon}} \simeq \frac{-1}{\vec{p}^{\,2}}\left(1 + \frac{p_0^2}{\vec{p}^{\,2}} + \cdots\right) \qquad p^0 \ll |\vec{p}|$$

*(decoupling in space only!)

For the long-distance theory the "potentials" are UV physics $\Longrightarrow$ integrate them out!



WL theory for the binary!

("Love", etc)

$$S_{\text{eff}} = -M\int d\tau + \frac{1}{2}\int d\tau\, Q^{ab}\,(\gamma)\,E_{ab} + Q^{abc}\,\boxed{\nabla_c E_{ab}}$$

$\frac{1}{2}\omega_0^{ij}L^{ij}$

binding mass

multipoles

$\propto \left(\frac{r}{\lambda}\right)^n$

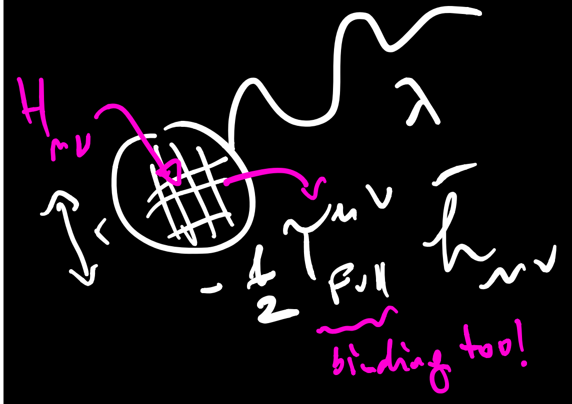
TIME DOESN'T DECOUPLE!

Now $\langle Q^{ab}\rangle_{\text{short}} \leftarrow 1+2+$ "potentials"

is the quadrupole of the binary

Once again we have to MATCH.

BUT! We know the "UV PHYSICS"



$-\frac{1}{2}T^{\mu\nu}_{\text{full}}\,\bar{h}_{\mu\nu}$

binding too!

$\vec{r}\cdot\vec{\partial} \ll 1$

$\Longrightarrow$

$$-M + \frac{1}{2}Q^{ij}E^{\text{long}}_{ij} + \dots$$

$\propto \bar{h}$

$\mathcal{O}\left(\frac{r}{\lambda}\right)^n$

$Q^{ij}_N = m x x \simeq \sim r$

scale r inside the multipoles.

Let's start with the general structure.

$$-\frac{1}{2}\int \overset{\partial_\mu T^{\mu\nu}=0}{\overset{\downarrow}{T^{\mu\nu}_{(t,\vec{x})}}}\,\underset{\bar{h}\to\partial_{(\mu}\varepsilon_{\nu)}}{\bar{h}_{\mu\nu}(t,\vec{x})}\,d^4x \qquad \bar{h}_{\mu\nu}(t,\vec{x}) = \bar{h}_{\mu\nu}(t,\vec{0}) + \partial_i \bar{h}_{\mu\nu}(t,\vec{0})\,x^i + \dots$$

$$\Rightarrow -\frac{1}{2}\int \boxed{T^{\mu\nu}(t,\vec{x})\,\bar{h}_{\mu\nu}(t,\vec{0})}\,d^3\vec{x}\,dt + \dots$$

* LO in derivatives

$$\int\left[\bar{h}_{\mu\nu}(t,0)\int T^{\mu\nu}(t,\vec{x})\,d^3\vec{x}\right]^{*} dt$$

this already looks like a local term

<u>Matching</u>: We use on-shell modes $\bar{h}^{TT}_{ij}$ (trace-less and transverse)

$\uparrow$ space

$$* \int T^{ij}(t,\vec{x})\,d^3\vec{x} \underset{(\partial_\mu T^{\mu\nu}=0)}{\overset{\nearrow}{=}} \frac{1}{2}\partial_0^2 \int \underbrace{T^{00}(t,\vec{x})\,x^i x^j\,d^3\vec{x}}_{\text{quadrupole!}}$$

$$\frac{1}{2}\int \underline{dt} \quad T^{00} x^i x^j d^3\vec{x}\,\Big]\, \frac{\partial_0^2}{2}\bar{h}^{TT}_{ij}(t,\vec{0}) \qquad (\text{IBP})$$

$$E^{TT}_{ij} \underset{\substack{\uparrow \\ \text{on-shell}}}{=} -\frac{1}{2}\partial_0^2 \bar{h}^{TT}_{ij}$$

(other terms (traces) can also be obtained in this fashion)

$$\hookrightarrow \quad Q^{ij}_E(t) = \int \left[T^{00}_{(t,\vec{x})} x^i x^j\right]^{TF} d^3\vec{x}$$

at LO in derivatives

the next term is $\left[\underset{\uparrow}{T^{ij}} x^k\right]\partial_k h_{ij}$

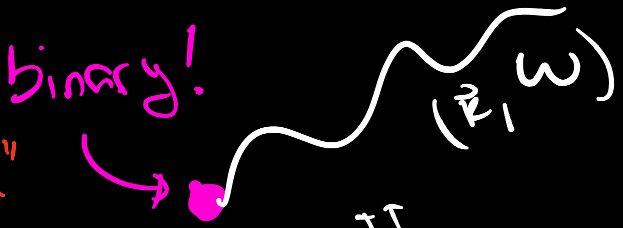
$$\underset{\uparrow}{J^{ij}}\,\underset{\substack{\uparrow \\ \epsilon W \checkmark}}{B_{ij}} \qquad 2\otimes 1 = \underset{\text{oct.}}{3_E} \oplus \underset{\text{current}}{2_M} + \text{trace}$$

$$J^{ij}(t) = -\frac{1}{2}\int d^3x\, \epsilon^{ik\ell}\left[T^{0k}_{(t,\vec{x})} x^j x^\ell\right]^{TF} + (i\leftrightarrow j)$$

our task now reduces to group theory and then MATCHING! the thry with the EFT

Before we do that, let's compute the radiated power from the EFT

(this is generic!)



"unphysical"

$$-\int M \underbrace{\sqrt{\boxed{1+h_{00}}}}_{d\tau}\left(\frac{1}{2}Q^{ij}\overset{\partial h_{ij}^{TT}}{E_{ij}}+\frac{4}{3}J^{ij}B_{ij}+\dots\right)$$

rate

$$iA_{h=\pm 2}(\omega) \rightarrow d\Gamma_h = \frac{1}{\boxed{T}}\frac{d^3\vec k}{(2\pi)^3\, 2|\vec k|}\left|A_h\right|^2$$

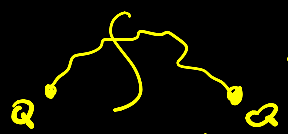
amplitude

$k^0 = |\,|$

long-time

$$\hookrightarrow \dot E = \sum_{h=\pm 2}\int |\vec k|\, d\Gamma_h(\omega = |\vec k|, \vec k)$$

opt. thm.

$D\bar h \rightarrow$  $\Rightarrow \langle QQ\rangle\ \mathrm{Im}\langle EE\rangle$

$\hookrightarrow \delta(p^2)$

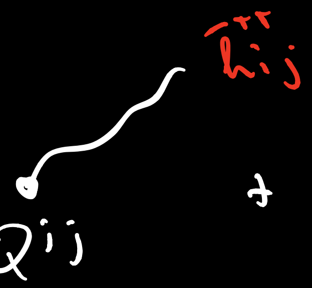
Imaginary part $\rightarrow \Big|$  $\Big|^2$

$\langle in|out\rangle^J$

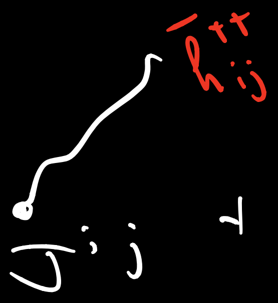
b.c. !

In the EFT side:

$iA^{(TT)}_{EFT} =$



$+$



$+ \dots$

on-shell ($\omega = |\vec{k}|$)

$$= \frac{i}{4m_{Pl}} \epsilon^{*}_{ij}(\tilde{k}, h) \left[\omega^2 Q^{ij} + \frac{4}{3} |\tilde{k}| k^l \varepsilon^{ikl} J^{jk} + \dots \right]$$

$\rightarrow$ using $\sum_h \varepsilon^{ij} \varepsilon^{*kl} = \frac{1}{2} \left(\delta\delta + \dots + \frac{1}{k^2} \left(\delta k k + \dots \right) + \frac{1}{k^4} k_i k_j k_k k_l \right)$

$$\hookrightarrow \dot{E} = \frac{G}{\pi T} \int_0^\infty d\omega \left[\frac{\omega^6}{5} |Q^{ij}|^2 + \frac{16}{45} \omega^6 |J^{ij}|^2 + \dots \right.$$

$$\hookrightarrow \dot{E} = \frac{G}{5} \left\langle \left(\dddot{Q}^{ij} \right)^2 \right\rangle + \dots \quad (*)$$

celebrated formula!

↳ 1) who is this!??

2) how does it evolve ?!?

Task: Match $T^{\mu\nu}_{UV}(x_1, x_2, \overset{\text{potentials}}{H})$ $\to$ T^{00} (binding), T^{ij} (flux)

saddle-point approx

background field $\nabla^{(H)} H_{\mu\nu}$

external source (non-prop.)

"J"

$Z[J]$

$$\int \underbrace{DH_{\mu\nu}}_{\text{classical}} \exp\left[i\left(S_{EH}(\underset{UV}{H_{\mu\nu}} + \underset{IR}{\bar h_{\mu\nu}}) + \underbrace{S^{(H)}_{GF}(H)} \right) + \sum_j S_{WL_j} \right]$$

UV from R scale

$$\underset{\langle in|out\rangle^J}{=} \exp iS_{eff}\left[\bar h_{\mu\nu}, \underbrace{M, Q^{ij}, \dots}_{\substack{\text{functions of} \\ x_1(t),\, x_2(t)}} \right]$$

Binding

$$\int dt \left(\int d^3\vec x\, T^{00}(t, \vec x) \right) \frac{h_{00}(t,0)}{2} \to -\int M \underbrace{\sqrt{1 + h_{00}}}\, d\tau$$

$K + V$ ← EOM

$$\Rightarrow M(t) \equiv \int T^{00}(t, \vec x)\, d^3\vec x$$

$\dot M$ = flux! ⊛

from $\partial_\mu T^{\mu\nu}(\bar h) = 0$

But we can also match the "vacuum"!

(no closed-loops & connected)

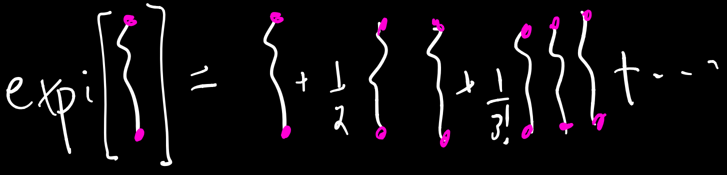
$\mathcal{O}\left(\frac{S_{eff}}{\hbar}\right)$

$\hbar$-kicks

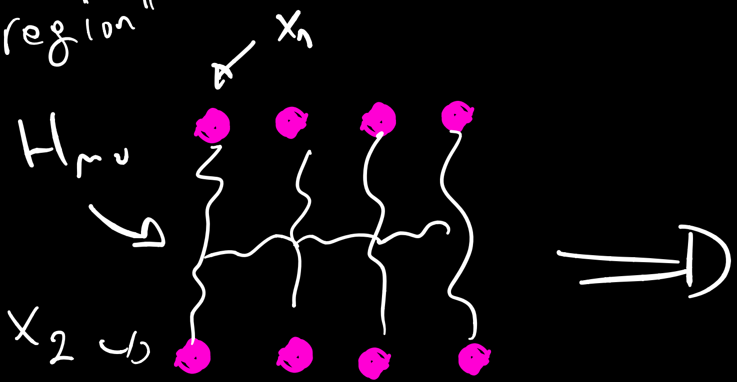
r build-up

exponentiates!

$$\left[e^{iW[J]} = Z[J]\right]$$



"potential region"

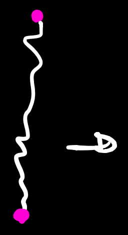


binding

$$-\int V dt$$

Potentials: PN expand!

$\mathcal{O}(G)$

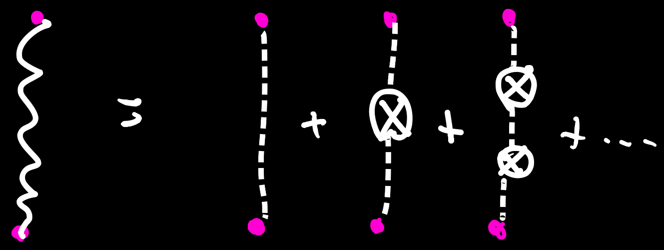


$$- \frac{m_1 m_2}{4 m_{pl}^2} \int dt_1 dt_2 \boxed{\langle H_{\mu\nu}(1) H_{\alpha\beta}(2) \rangle} v_1^\mu v_1^\nu v_2^\alpha v_2^\beta$$

(s=t; $v^0 = 1$)

& $\displaystyle\int \frac{d^4p}{(2\pi)^4} \frac{i}{p_0^2 - \vec{p}^{\,2}} e^{ip\cdot(x_1 - x_2)}$

Potentials: PN expand!



$\mu\nu$ - - - - - - - - $\alpha\beta$ $\quad -\frac{i}{p^2} P_{\mu\nu\alpha\beta}$

↑ De-Donder

$\otimes \quad \frac{p_0^2}{\vec p^2} \sim v^2$

$$\frac{m_1 m_2}{4 m_{Pl}^2} \int dt_1 dt_2 \frac{P_{0000}}{p_0^2 - \vec p^2} e^{i p_0 (t_1 - t_2)} e^{-i \vec p (\vec x_1(t_1) - \vec x_2(t_2))}$$

static:

$\dot{\vec x}_1 = \dot{\vec x}_2 = 0$

$$\Rightarrow \int dt_1 e^{i p_0 t_1} \frac{dp^0}{p_0^2 - \vec p^2} \to \int \frac{\delta(p^0)}{p_0^2 - \vec p^2}$$

potential region ONLY!

$$\Rightarrow \int \frac{dt}{\vec p^2} e^{i \vec p (\vec x_1 - \vec x_2)} \propto \int \frac{dt}{|\vec x_1 - \vec x_2|}$$

$\vec v_1 \neq \vec v_2 \neq 0$

can't integrate in $dt_{1(2)}$ but we can expand!

$$\Rightarrow \int \frac{dt_1 dt_2}{\vec p^2} \underbrace{e^{i p_0 (t_1 - t_2)}}_{\text{integrate in } p^0 \to \delta(t_1 - t_2)} e^{i \vec p (\vec x_1 - \vec x_2)}$$

$$+ \int \frac{p_0^2}{\vec p^4} [\dots] = \int \frac{d}{dt_1 dt_2} \delta(t_1 - t_2)$$

↳ $\underline{\text{IBP}}$ $v_1^i v_2^j \int \boxed{\frac{p^i p^j}{\vec{p}^4}} e^{i\vec{p}\cdot\vec{r}} dt$ $\vec{r} \equiv \vec{x}_1 - \vec{x}_2$

$$\longrightarrow A\underbrace{\left(\frac{r^i r^j}{r^2} - \frac{\delta^{ij}}{3}\right)}_{\text{TF}} + B\underbrace{\delta^{ij}}_{\text{trace}}$$

$$\left(\frac{(\vec{v}_1\cdot r)(\vec{v}_2\cdot\vec{r})}{r} - \frac{\vec{v}_1\cdot\vec{v}_2}{r}\right)$$

We need $\{P_{0i0j} ; P_{ij00}\}$ couplings (also the v^2 from $\sqrt{g v v}$)

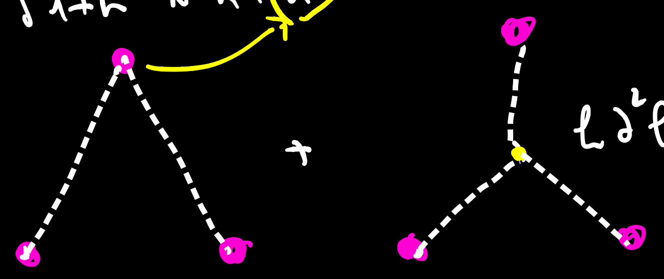
this completes the 1PN at $\mathcal{O}(G)$

What are we missing?!

Recall $\frac{GM}{r} \sim v^2 \Rightarrow \mathcal{O}(G^2 v^0) \sim \mathcal{O}(G v^2) \Leftarrow$ mixing of G's!

We need non-linear couplings!

$\sqrt{1+h} \sim h + \boxed{h^2} + \dots$



$h\partial^2 h + \boxed{h^2 \partial^2 h} + \dots$

3-graviton

$\frac{G^2 m_1 m_2^2}{r^2} + (1 \leftrightarrow 2)$

"one-loop" integrals.

Exercise: Compute the Einstein-Infeld-Hoffman potential