4d Chern-Simons theory

3d CS Lagrangian

$$L_{cs} = \kappa \, \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

I will often drop tr

$A = A_\mu dx^\mu$ $\qquad$ $A_\mu = A_\mu^a T^a$

$\mu = 0, 1, 2$ $\qquad$ T^a generators of Lie algebra $\mathfrak{g}$

is a "simple" gauge theory. $\qquad$ κ coupling cst

1) Equations of motion:

$$\delta L = \delta A \wedge dA + A \wedge d\delta A + 2\, \delta A \wedge A \wedge A$$
$$\overset{\mathrm{IBP}}{=} 2\, \delta A \wedge (dA + A \wedge A)$$

$$F = 0$$

solution pure gauge $A_\mu = g^{-1} \partial_\mu g$ $\qquad$ $g(x) \in G$ (group)

no local degrees of freedom

2) Gauge transformations

- under infinitesimal gauge transforms

$$\delta A_\mu = \partial_\mu \epsilon + [A_\mu, \epsilon]$$

the Lagrangian is invariant (up to tot der)

δL_{cs} = surface term

- under general gauge transforms

$$A_\mu \to g^{-1} \partial_\mu g + g^{-1} A_\mu g$$

[∞-imal: $g = e^{\epsilon} \sim 1 + \epsilon$

… is not gauge invariant

Lagrangian is not gauge invariant

$$L_{cs} \longrightarrow L_{cs} - \frac{\kappa}{3}\,\underbrace{\mathrm{tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right)}_{\text{winding number density for } g(x).}$$

With suitable b.c. for $g(x)$ and normalization winding number is an integer n

$$S_{cs} \longrightarrow S_{cs} - 8\pi^2 \kappa\, n$$

The path integral has $e^{iS_{cs}}$ so will be gauge invariant for

In perturb. th. use κ as coupling constant

$$\kappa = \frac{k}{4\pi}$$

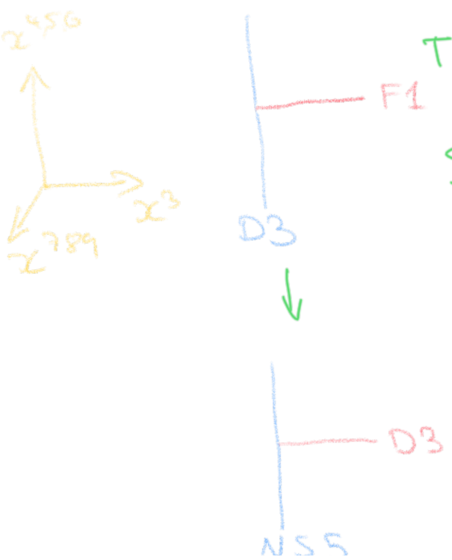
k is called the LEVEL

3) CS couplings in String theory

an open string ending on a D3

		0	1	2	3	4	5	6	7	8	9
	F1	x			x						
S	D3	x				x	x	x			
	D1	x			x						
T^{12}	D3	x				x	x	x			
	D3	x	x	x	x						
S	D5	x	x	x		x	x	x			
	D3	x	x	x	x						
	NS5	x	x	x		x	x	x			

ND=4



D3 ending on an NS5

The WZ coupling of D3 gauge field to RR potentials includes

$$S_{WZ} = \int d^4x\, C \wedge e^F = \int d^4x\, C_4 + \underbrace{C_0 F \wedge F}$$

For constant θ angle C_0 we can IBP this term

$$\propto C_0 \int A dA + \frac{2}{3} A^3$$

check $d(AdA + A^3) \propto F \wedge F$

These configurations discussed in Gaiotto-Witten

4d Chern-Simons

Consider

$$S_{4d} = h \int \omega \wedge L_{CS}$$

where ω is a 1-form. Integrand is 4 form ✓

We integrate over $\Sigma \times C$

Riemann surface $w, \bar{w}$ — complex curve $z, \bar{z}$

usually $\mathbb{R}^2 \times \mathbb{C}$

We align

$$\omega = \omega_z dz$$

ω has only dz component

$$A = A_w dw + A_{\bar{w}} d\bar{w} + A_{\bar{z}} d\bar{z} + A_z dz$$

$$\omega \wedge A = \omega_z dz \wedge \left(A_w dw + A_{\bar{w}} d\bar{w} + A_{\bar{z}} d\bar{z} + \cancel{A_z dz} \right)$$

A_z is NOT present in S_{4d}

For now take $\omega \equiv dz$

Alternatively, have extra gauge inv $A \to A + \chi\, dz$ $\forall \chi$

We have defined a 4d gauge theory with coupling constant $\hbar$, partial connection A and 2-d Lorentz invariance.

Eoms

$$F_{w\bar{w}} = 0$$

$$F_{w\bar{z}} = 0 = F_{\bar{w}\bar{z}}$$

The 1st eq says A is topological in $\Sigma_{(w,\bar{w})}$

The other eqs say $\mathcal{D}_{\bar{z}} A_w = 0 = \mathcal{D}_{\bar{z}} A_{\bar{w}}$

i.e. A is holomorphic in $\mathbb{C}_{(z,\bar{z})}$

Quantum

It turns out that this theory is a well-defined quantum theory to all orders in $\hbar$

Costello using BV

There are no possible gauge-invariant counterterms: they all vanish $\because$ of eoms

We can compute Feynman diagrams

Example: Propagator in "Lorenz gauge"

$$0 = d^\dagger A \equiv \partial_x A_x + \partial_y A_y + 4\partial_z A_{\bar{z}}$$

$d = dx\,\partial_x + dy\,\partial_y + d\bar{z}\,\partial_{\bar{z}}$

$d^\dagger$ adjoint

In 3d CS $\langle A(\vec{x})\, A(\vec{x}')\rangle = P(\vec{x}-\vec{x}')$ is a 2-form

which satisfies

$$dP = \delta^{(3)}(\vec{x}-\vec{x}') \qquad \text{Green's fn}$$

3 form

$$d^\dagger P = 0$$

$d^\dagger \equiv * d *$

Lorenz gauge $d^\dagger A \equiv \partial^i A_i = 0$

recall scalar Green's fn in 3d is $\frac{1}{r}$

$$\square \frac{1}{r^2} = (dd^\dagger + d^\dagger d)\frac{1}{r} = * d * d \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \underbrace{\delta(\vec{x}-\vec{x}')}_{\text{scalar}}$$

so

$$d\Big(\underbrace{* d \frac{1}{|\vec{x}-\vec{x}'|}}_{P(\vec{x}-\vec{x}')}\Big) = \delta^{(3)}(\vec{x}-\vec{x}')$$

Also

$$d^\dagger \left(* d \frac{1}{r}\right) = * d * * d\left(\frac{1}{r}\right) = * d^2 \frac{1}{r} = 0$$

as reqd

In components

$$\langle A_i(\vec{x})\, A_j(\vec{x}')\rangle = \varepsilon_{ijk}\, \partial^k \frac{1}{|\vec{x}-\vec{x}'|}$$

An analogous calculation in 4d CS gives

$$P = \left(dy \wedge d\bar{z}\, \partial_x + d\bar{z} \wedge dx\, \partial_y + 4\, dx \wedge dy\, \partial_z\right) \frac{1}{(x^2+y^2+z\bar{z})}$$

these are really $z - z'$ etc

Exercise: Check that the 4d CS propagator satisfies

$$\frac{i}{2\pi}\, dz \wedge dP = \delta^{(4)}(x, y, z, \bar{z} = 0)$$

$$d^\dagger P = 0$$

where $d^\dagger \equiv \partial_x\, i_{\partial_x} + \partial_y\, i_{\partial_y} + 4\, \partial_z\, i_{\partial_{\bar{z}}}$

<u>Notice:</u> d that appears in action is

$$d = \underbrace{dx\,\partial_x + dy\,\partial_y}_{\substack{\text{de Rham} \\ d_{(x,y)}}} + \underbrace{d\bar{z}\,\partial_{\bar{z}}}_{\substack{\text{Dolbeault} \\ \bar{\partial}_{(z,\bar{z})}}}$$

Chern-Simons observables

CS theories have no local, gauge invariant ops

$F=0$

but there are Wilson loops

$$W[A;\gamma] = \operatorname{Tr} P\exp\left(i \oint_\gamma A\right)$$

<u>Exercise:</u> Show that the operator

$$\mathcal{U}(x,y;A) \equiv P\exp \int_x^y A$$

transforms under gauge transformations

$$\mathcal{U}(x,y;A^g) = g(y)\,\mathcal{U}(x,y;A)\,g^\dagger(x)$$

hence show that $W[A;\gamma]$ is gauge-invariant

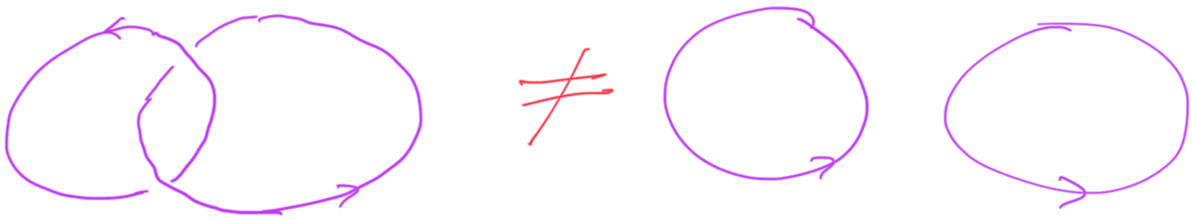
Because CS theory is topological $\int_{\partial D} A \sim \int_D F \sim 0$

we can deform $\gamma \to \gamma'$ without changing correlators

$$\langle \ldots W_\gamma \ldots \rangle = \langle \ldots W_{\gamma'} \ldots \rangle$$

we cannot cross W's through each other

e.g.



Famously, Witten showed that correlation fns of Wilson lines give quantum knot invariants

$\therefore$ W_γ topological in γ, Reidemeister moves easy to understand

no crossing of lines.

4d CS also has Wilson line operators but these are more subtle:

1) There is no $A_{\bar z}$ component

$\Longrightarrow$ cannot parallel-transport on C

restrict to γ along $\Sigma_{(w,\bar w)}$ at $z = z_0$

2) Open Wilson lines stretching to ∞ are gauge invariant

4d CS is IR-free so $A \to 0$ at large distance $w, \bar w \to \infty$

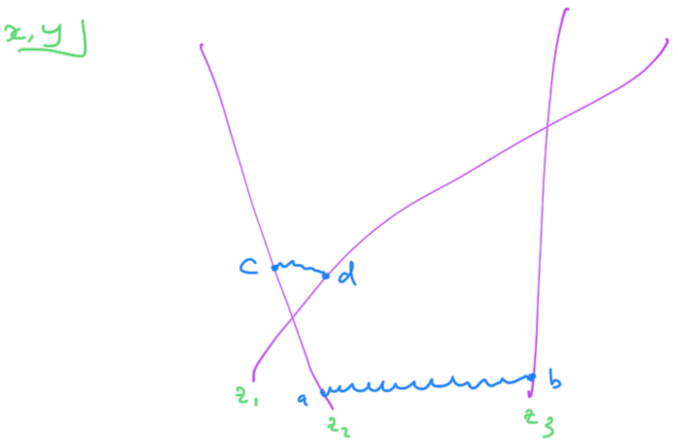
3) Wilson lines couple to $\partial^n A$ and not just A

$$\int A^a t^a[[z]]$$

return to this below

With these observables we can do std pert. theory

For $C = \mathbb{C}$ we have a unique vacuum $A = 0$ and we can compute Feynman diagrams e.g.



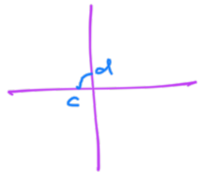
a b : diffeo invariance

can scale metric on Σ

s.t. a, b very far apart

$A \to 0$ in IR free th.

$c \sim d$: Scaling straightens lines and localizes interaction near crossing



This localization of interactions is how the R-matrix picture emerges

Away from crossings  no interactions

is free excitation in rep V_i

Near crossing localized interactions generate R matrix

Let's compute the 1^{st} one using our propagator

$$= h \; t^a \otimes t^b \int dx_1 dy_2 \; P^{ab}(x_1 - x_2,\, y_1 - y_2,\, z_1 - z_2,\, \bar{z}_1 - \bar{z}_2)$$

$$= h \; t^a \otimes t^a \int dx\, dy \; \frac{\bar{z}_1 - \bar{z}_2}{\left(x^2 + y^2 + |z_1 - z_1|^2\right)^2}$$

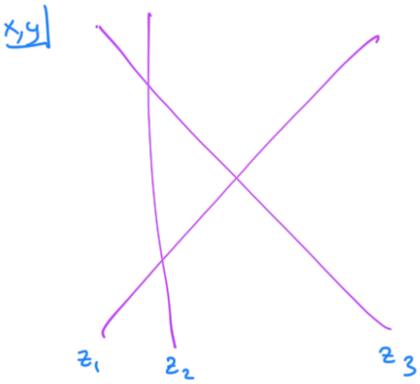
only $\langle A_x A_y \rangle$ contributes

$$= h \; C \; \frac{1}{z_1 - z_2} \equiv r_{12}(z_1 - z_2)$$

$z_1 - z_2$

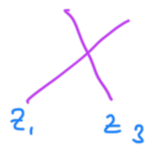
we recover semi-classical r-matrix of Drinfeld!

The Yang-Baxter equation is even simpler



Because the theory is topological in $\Sigma_{(x,y)}$ we are free to move for example

In usual top. th. we cannot move past



But we are in a four-dimensional theory
These Wilson lines are separated in $\mathbb{C}_{(z,\bar{z})}$
So we can move past the "crossing" point

But this is precisely the YBE!

The Wilson line operators $P e^{\int_\gamma A}$ are a generalization of conventional Wilson lines

$$\int A^a T^a \longrightarrow \int A^a(\bar{z}=0)\, T^a[[z]]$$

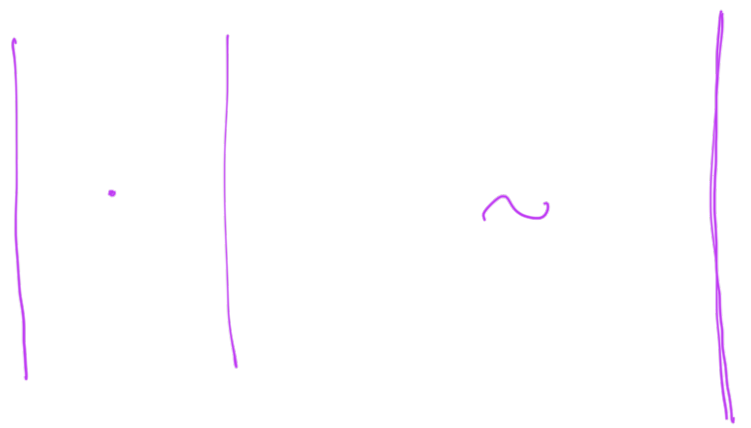
Classically

$$T^a_n = T^a_0 z^n$$

$$[T^a_n, T^b_m] = f^{ab}{}_c T^c_{n+m}$$

Quantum mechanically there are corrections to T^a_n exactly of the type considered by Drinfeld

Why do we need these generalized Wilson lines? The OPE of two ordinary Wilson lines

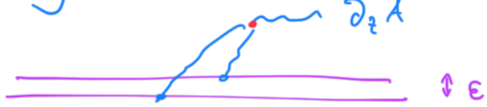


$\mathcal{O}_1 \sim e^{\int A^a T^a_0}$ $\Downarrow$ couples to A

$\mathcal{O}_2 \sim e^{\int A^a T^a_0}$ $\Downarrow$ couples to A

$\mathcal{O}_3 \sim e^{\int A \; 1_0 t}$ $\Downarrow$ couples to $\partial_z A$

explicitly



is non zero as $\varepsilon \to 0$

In other words even if you start with $T^a_{n>0} = 0$ in classical theory, you will generate $T^a_{n>0} \neq 0$ in the quantum thy

<u>Exercise:</u> above diagram involves computing integral

$$h f_{bca} T^b \otimes T^c \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int dx\, dy\, d^2 z \; P(x_1 - x, y, z, \bar{z}) \wedge dz \wedge A^a(x, y, z, \bar{z}) \wedge P(x_2 - x, y - \varepsilon, z, \bar{z})$$

1-form from $dz \wedge A^3$ interaction

Show that in the $\varepsilon \to 0$ limit this reduces to

$$h f_{bca} T^b \otimes T^c \int dx\, dy\, d^2 z \, A^a \, \partial_z \delta^{(3)}(y, z, \bar{z}) = -h f_{bca} T^b \otimes T^c \int dx \, \partial_z A^a$$

In other words we see that

In other words

$$0 \neq T_1^a \sim \hbar f^a{}_{bc} T_0^b \otimes T_0^c$$

in the quantum theory

This is the 4d CS version of Drinfeld's Yangian where level 1 generators have have a level 0 tail when acting on multi-excitation states

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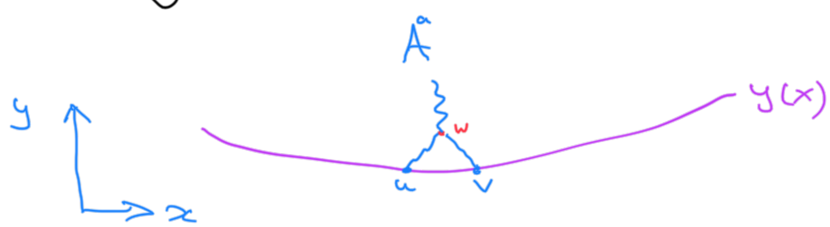
Like many theories, 4d CS can have gauge anomalies. Correlators should be invariant under gauge transformations

$$A \to A + D\epsilon$$

but quantum effects can mess this up

In 4d CS the Wilson lines have a framing anomaly. This comes from the diagram



When the Wilson line bends in $\Sigma_{(x,y)}$

sending $A^a \to A^a + D\varepsilon^a$ modifies the correlator

by a non-zero term

$$h\, f^{abc} T^b T^c \int_{u,v,w} P(x-v)\, D\varepsilon^a\, P(x-u)$$

$$\overset{IBP}{\sim} -h\, \underbrace{f^{abc} f^{bcd}}_{\text{dual coxeter \#}} T^d \int_{u,v,w} \underbrace{D\left(P\ P\right)}_{\text{singular when } u,v,w \text{ coincide}} \varepsilon^a$$

$$\sim -h\, h^\vee \int dx\, \partial_z \varepsilon\, \frac{d^2 y}{d x^2}$$

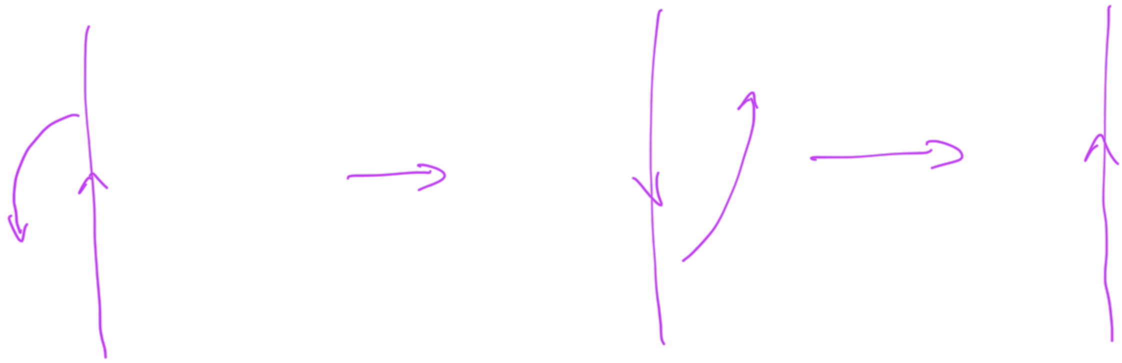
Luckily, it is possible to cancel this anomaly for general Wilson lines on $y(x)$ by not placing the Wilson lines at fixed point $z = z_0$ but rather at

$$z = z_0 - h\,(h^\vee)\,\frac{dy}{dx}$$

For example, rotating a Wilson line

... by 2π

in $x-y$ plane by



corresponds to Lorentz Boosting it (by 2π)

Framing anomaly $\Rightarrow$ need to shift spectral parameter

$$z_0 \longrightarrow z_0 - h\, h^\vee$$

up to factors of 2π

In the evaluation rep

$$Q^{(1)} \sim z\, Q^{(0)}$$

so the shift of z means under a Boost

$$Q^{(1)} \longrightarrow Q^{(1)} - h\, h^\vee\, Q^{(0)}$$

This is exactly how B acts in Drinfeld

$$[B\ Q^{(1)}] \sim h\ Q^{(0)}$$