Lecture 4

1. What is the supersymmetric phase structure of SCFT at large N?

2. Comments on BH v/s graviton operators in SYM.

How does index capture BH entropy?

Simplest $\frac{1}{4}$-BPS index $(2J = J_1 + J_2)$

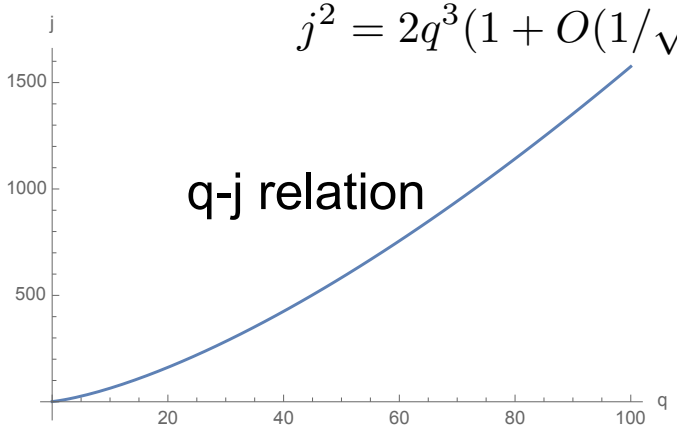
$$\mathcal{I}_N(\tau) \;=\; \mathrm{Tr}\,(-1)^F\, \mathrm{e}^{2\pi \mathrm{i}\tau(2J+Q)} \;=\; \sum_{n\geq 0} d_N(n)\, \mathrm{e}^{2\pi \mathrm{i}\,\tau\, n/3}$$

$\frac{1}{4}$-BPS BH charges

$$(J_1, J_2, Q) = N^2(j_1, j_2, q)$$

obeying

$$j^2 = 2q^3(1 + O(1/\sqrt{q}))$$



Simplest: $j_1 = j_2 = j \neq 0$

$$S_{\mathrm{BH}} \;=\; N^2\sqrt{3q^2 - 4j} \;\equiv\; N^2\, s(n/N^2)$$

cf $d_N(n)$

What is the meaning of BH/exponential growth of states in the index?

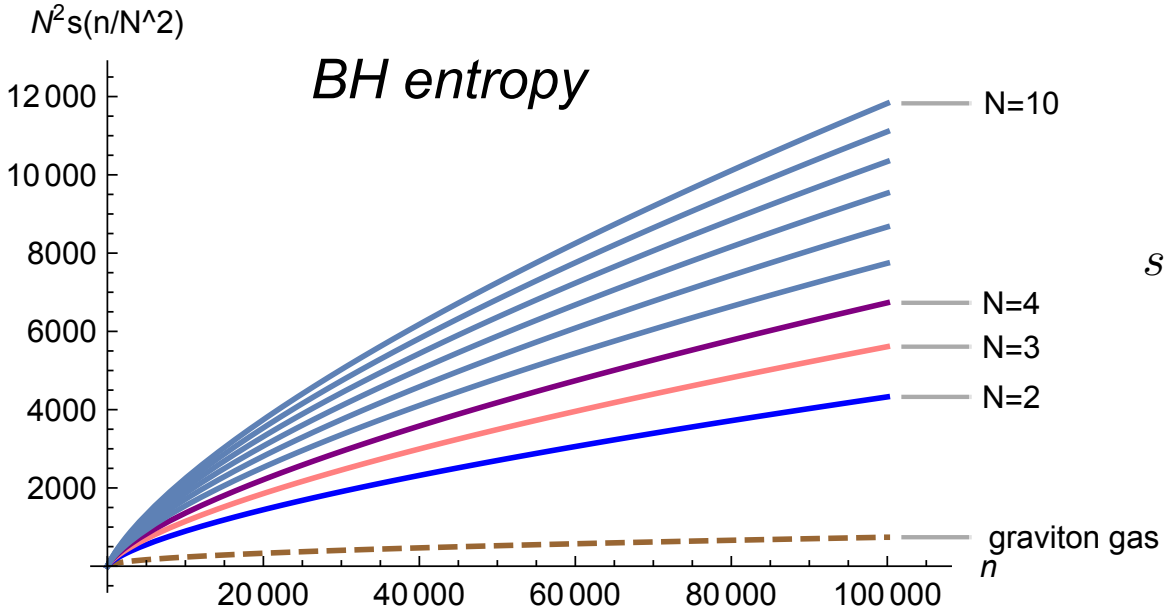
$$\frac{1}{N^2}\log|d_N(n)| \overset{?}{\longrightarrow} s(n/N^2)$$

Large-N limit

$$n, N \to \infty \quad \text{fixed } \nu = n/N^2$$

Cardy-like limit

$$n/N^2 \to \infty \quad \text{fixed } N$$



$$s(\nu) \underset{\nu\to\infty}{\sim} \frac{\pi}{2\cdot 3^{1/6}}\nu^{2/3} + \dots$$

AdS/CFT prediction: free energy of matrix model saddle = on-shell action of BH

$$\mathcal{I}_N(\tau) \Longleftrightarrow d_N(n)$$

Change of ensemble

$$\mathcal{I}_N(\tau) = \sum_{n\geq 0} d_N(n)\, \mathrm{e}^{2\pi \mathrm{i}\, \tau\, n/3}$$

$(\tau \to \tau + 1$ from now on$)$

$$\log|d_N(n)| \stackrel{?}{\longrightarrow} N^2\, s(n/N^2)$$

$$\Updownarrow$$

$$\log \mathcal{I}_N(\tau) \stackrel{?}{\longrightarrow} \frac{\pi \mathrm{i}\, N^2}{27} \frac{(2\tau+1)^3}{\tau^2}$$

Not $SL_2(\mathbb{Z})$ modular form!

$n \to \infty$ fixed N

$n, N \to \infty$ fixed $j = \frac{n}{N^2}$

$\tau \to 0$ fixed N

Cardy-like limit

$N \to \infty$ fixed τ

Large-N limit

The N=4 superconformal index is an integral over unitary matrices

$$\mathcal{I}_N(\tau) = \int [DU] \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i_{\rm s}(e^{2\pi i n(\tau+1)/3}) \,{\rm tr}\, U^n \,{\rm tr}\,(U^\dagger)^n \right)$$

$U(N)$ matrix

$$i_{\rm s}({\rm x}) = \frac{3\,{\rm x}^2 - 3\,{\rm x}^4 - 2\,{\rm x}^3 + 2\,{\rm x}^6}{(1-{\rm x}^3)^2}$$

$$\mathcal{I}_N(\tau) = \int [D\underline{u}] \exp\left(-S(\underline{u})\right) \qquad S(\underline{u}) = \sum_{i,j=1}^{N} V(u_{ij})$$

$\underline{u} \equiv \{u_i\}_{i=1,\ldots,N}$

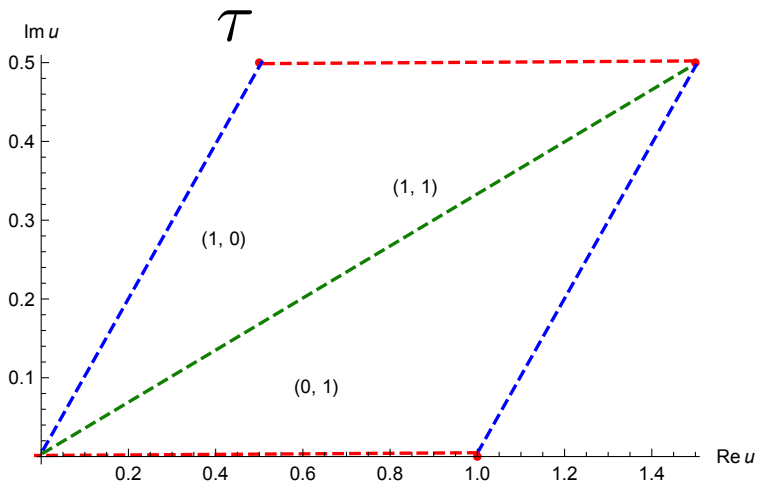
Pure two-particle interaction

Complex coupling.

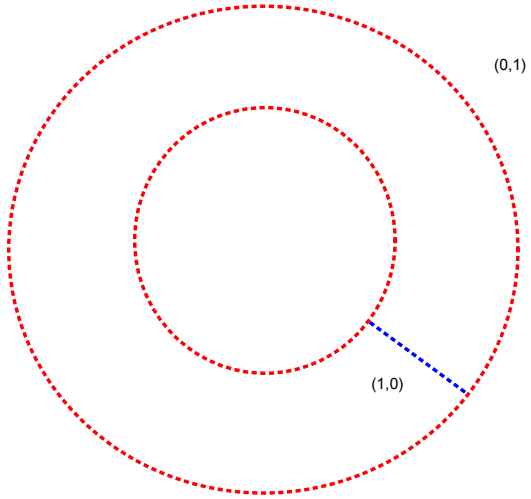
[cf. Gross, Witten '80; Wadia '80]

Matrix integral has complex saddles

[Cabo-Bizet, S.M. 1909.]



$$u \mapsto e^{2\pi i u}$$

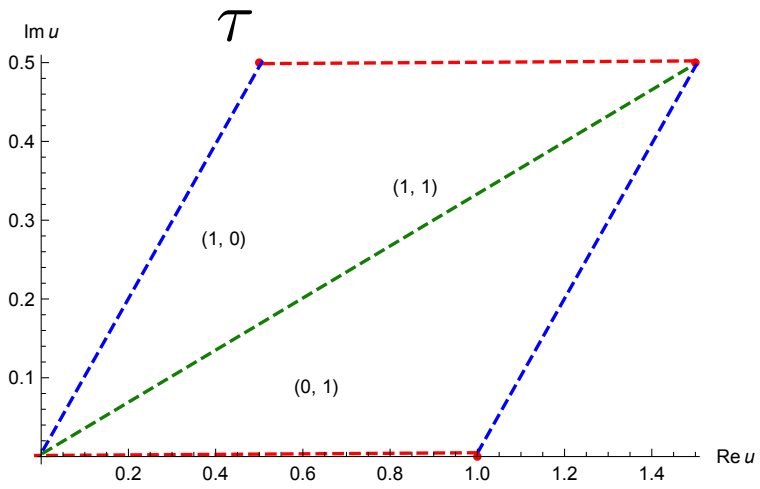


Family of saddles labelled by lattice points (m, n)

$$m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \qquad \gcd(m, n) = 1$$

Matrix integral has complex saddles with macroscopic entropy

[Cabo-Bizet, S.M. 1909.]



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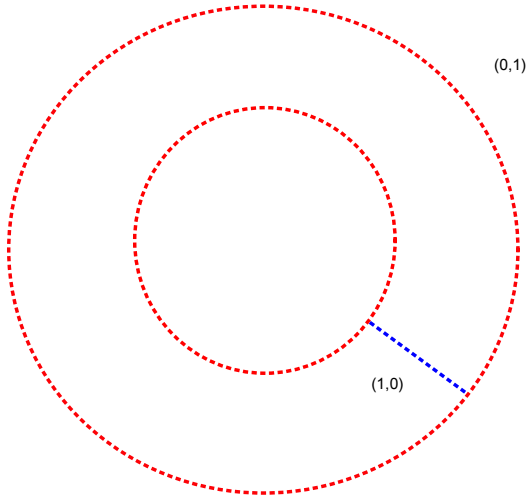


Family of saddles labelled by lattice points (m, n)

$$m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \qquad \gcd(m, n) = 1$$

$$\mathcal{S}_{(0,1)} = \text{entropy of pure } \mathrm{AdS}_5 = O(1)$$

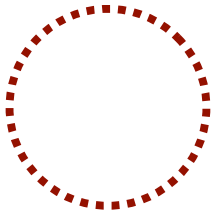
$$\mathcal{S}_{(1,0)} = \text{entropy of } \mathrm{AdS}_5 \text{ BH} = O(N^2)$$

Lemma: Periodic eigenvalue distributions are saddle points of periodic potentials

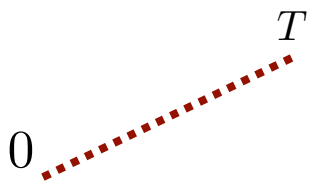
$$\partial_{u_i} S(\underline{u}) = \partial_{\overline{u}_i} S(\underline{u}) = 0 \ , \qquad S(\underline{u}) = \sum_{i,j=1}^{N} V(u_{ij})$$

$$V(-z) = V(z) \qquad V(z+T) = V(z)$$

1. Uniform distribution along [0,T] $u_i = \frac{i}{N} T + u_0$ is a solution of the variational equations.



2. Action of this distribution = average value of V



$$S_{\text{eff}}(T) = \sum_{i,j=1}^{N} V\left(\frac{i}{N} T\right) = N^2 \int_0^1 dx\, V(xT)$$

3. Need separate argument for contour deformation.

The potential of our index is quasi-periodic under lattice translations

$$\mathcal{I}_N(\tau) \;=\; \eta(\tau)^{2N} \int [D\underline{u}] \exp\bigl(-S(\underline{u})\bigr)$$

$$\exp\bigl(-S(\underline{u})\bigr) \;=\; \prod_{i\neq j=1}^{N} \Gamma_{\rm e}(u_{ij}+2\tau;\tau,\tau)\,\Gamma_{\rm e}\bigl(u_{ij}+\frac{1}{3}(2\tau+1);\tau,\tau\bigr)^3$$

[Dolan-Osborn '08]

$$\Gamma_{\rm e}(z;\sigma,\tau) \;=\; \prod_{j,k=0}^{\infty} \frac{1-{\rm e}^{-2\pi{\rm i}z}\,{\rm e}^{2\pi{\rm i}\sigma({\rm j}+1)}\,{\rm e}^{2\pi{\rm i}\tau({\rm k}+1)}}{1-{\rm e}^{2\pi{\rm i}z}\,{\rm e}^{2\pi{\rm i}\sigma j}\,{\rm e}^{2\pi{\rm i}\tau k}}$$

$$\Gamma_{\rm e}(z;\tau,\tau) \;=\; \Gamma_{\rm e}(z+1;\tau,\tau) \;=\; \theta_0(z;\tau)^{-1}\,\Gamma_{\rm e}(z+\tau;\tau,\tau)$$

[Felder-Varchenko '99; Spiridonov '10; Spiridonov-Vartanov '12]

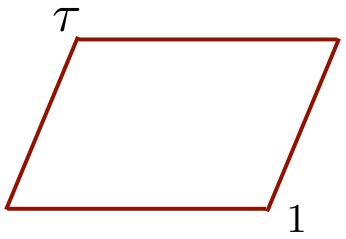
From quasi-elliptic to elliptic functions

$$\theta_0(z;\tau) = (1-\zeta)\prod_{n=1}^{\infty}(1-q^n\zeta)\,(1-q^n\zeta^{-1})$$

[Jacobi]

$$\theta_0(z) = \theta_0(z+1) = -e^{2\pi i z}\,\theta_0(z+\tau)$$

Lattice $\mathbb{Z}\tau + \mathbb{Z}$



$$q = e^{2\pi i \tau}$$
$$\zeta = e^{2\pi i z}$$

$$z = z_1 + \tau z_2$$

Construct

$$P(z;\tau) := q^{\frac{1}{2}B_2(z_2)}\,\theta_0(z;\tau) \qquad B_2(x) = x^2 - x + \frac{1}{6}$$

1. $q^{-\frac{1}{12}}P(z;\tau) = \theta_0(z;\tau)$ when $z_2 = 0$

2. $|P(z)|$ is elliptic

3. $|P(z)|$ is modular!

$$-\log|P(z;\tau)| = \lim_{s\to 1}\frac{\tau_2^s}{2\pi}\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}}\frac{e^{2\pi i(nz_2 - mz_1)}}{|m\tau+n|^{2s}}$$

[Kronecker's second limit formula]

The elliptic gamma function can be deformed to a doubly periodic function

Construct $Q(z;\tau) = q^{\frac{1}{3}B_3(z_2)} \dfrac{\theta_0(z;\tau)^{z_2}}{\Gamma_{\rm e}(z+\tau;\tau,\tau)}$ $(z = z_1 + \tau z_2)$

Kronecker-Eisenstein series

Period functions

1. $Q(z;\tau) = \Gamma_{\rm e}(z+\tau;\tau,\tau)^{-1}$ when $z_2 = 0$

2. $|Q(z)|$ is doubly periodic under $z \mapsto z+1$, $z \mapsto z+\tau$

3. $$\log\left|Q\left(\frac{z}{\tau};-\frac{1}{\tau}\right)\right| - \tau\log|Q(z;\tau)| = F(z;\tau)$$

Bloch formula

$$F(z;\tau) = \frac{\tau_2^2}{2\pi^2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{e^{2\pi {\rm i}(nz_2 - mz_1)}}{(m\tau+n)(m\overline{\tau}+n)^2}$$

Elliptic dilogarithm [S.Bloch-D.Wigner '78]

[Ramakrishnan '86] [Zagier '90] [Duke-Imamoglu '07] [Pasol-Zudilin '18]

We can thus extend the SYM action to a doubly-periodic function (periods 1 and τ)

- The new action S(u) reduces to the old action at Im(u)=0 $\Longrightarrow$ does not change the partition function!

- Doubly-periodic extension to complex plane (torus) allows us to easily find saddle-point solutions.

  Family of saddles labelled by

  $$(m,n) \quad m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \quad \gcd(m,n) = 1$$

- Action of saddle calculated as the integral along a given direction is easy — use Fourier series (Bloch formula)!

The action of the (m,n) saddle is a simple rational function of τ

$$S_{\rm eff}(m,n;\tau) \;=\; \frac{N^2\pi{\rm i}}{27\,m}\,\frac{\big(2(m\tau+n)+\chi_1(m+n)\big)^3}{(m\tau+n)^2} \qquad m\neq 0$$

$$S_{\rm eff}(0,1;\tau) \;=\; 0$$

ℓ	0	1	2
χ_1	0	1	-1

Dirichlet character mod 3.

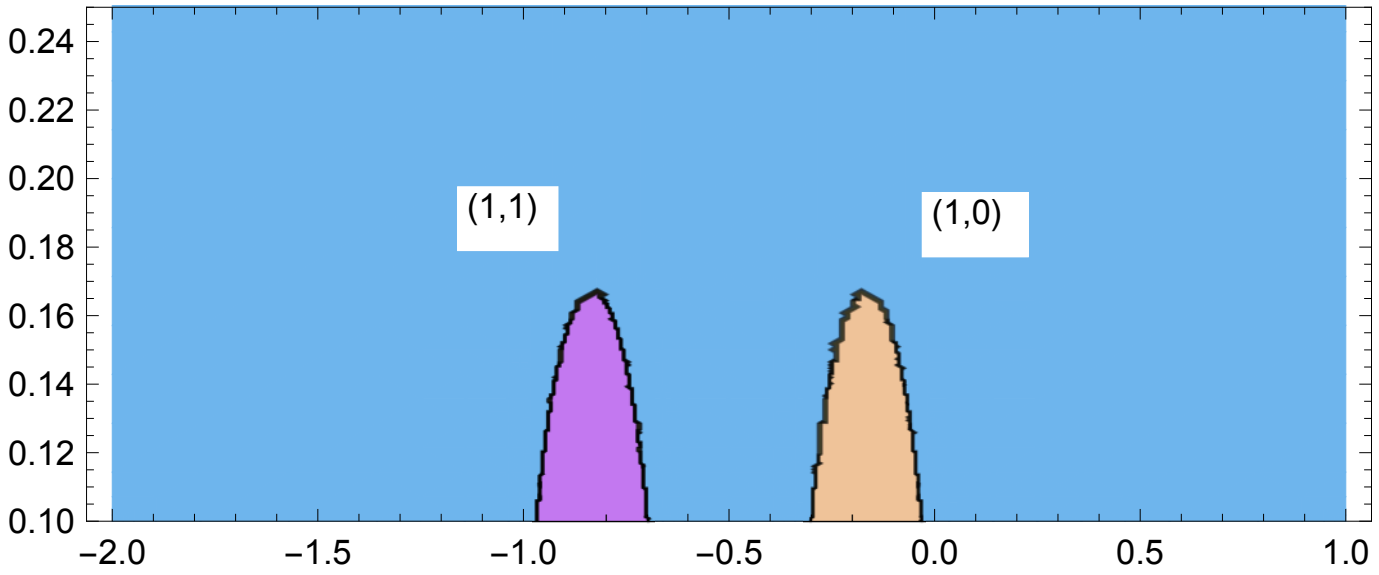
In particular,

$$S_{\rm eff}(1,0;\tau) \;=\; \frac{N^2\pi{\rm i}}{27}\,\frac{(2\tau+1)^3}{\tau^2}$$

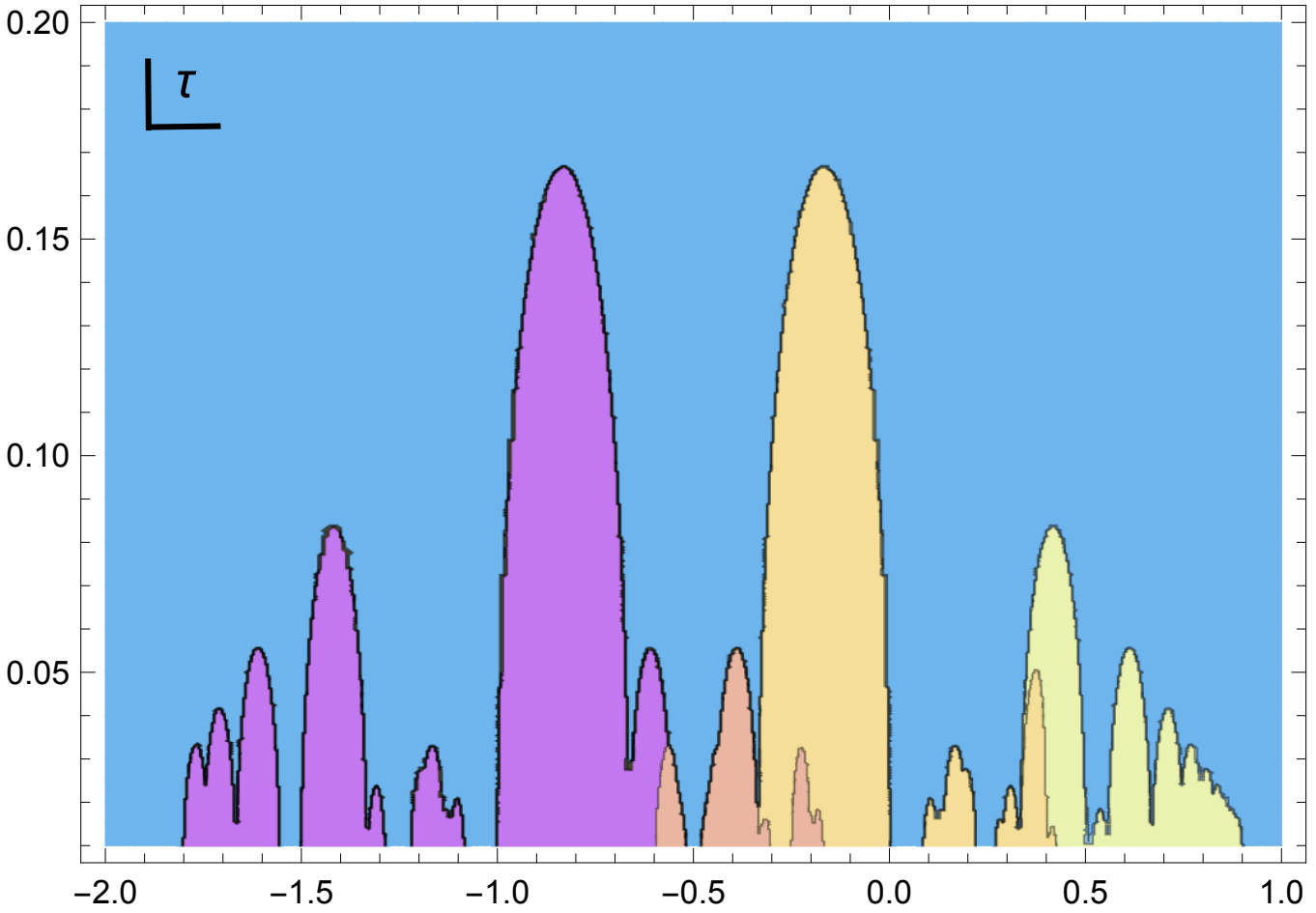
= the on-shell sugra action of AdS5 BH

Phase structure: black hole dominates in a region surrounding $\tau = 0$

[Cabo-Bizet, S.M. 1909.]



Large-N free energy dominated by BH near $\tau = 0$, new phases near rational points



[Cabo-Bizet, S.M. 1909.]

Comments on formula for index

$$\mathcal{I}(\tau) \sim \sum_{m,n} \exp\left(-S_{\text{eff}}(m,n;\tau)\right)$$

[Cabo-Bizet, S.M. 1909.]

- Complete asymptotics (perturbation exp) around each saddle
- Note: not exact formula [see Cabo-Bizet 2012.]
- cf. SCFT2 elliptic genus [Dijgraaf-Maldacena-Moore-Verlinde '00]
- Family of gravitational solns? [cf AdS3 Maldacena-Strominger '98]

(m,n) saddles are interpreted as partially deconfined phases

- Action:

$$S_{\rm eff}(m,n;\tau) \;=\; \frac{N^2\pi{\rm i}}{27\,m}\,\frac{\big(2(m\tau+n)+\chi_1(m+n)\big)^3}{(m\tau+n)^2} \qquad m\neq 0$$

- Entropy: $S_{m,n}(Q,J) = \dfrac{1}{m}\,S_{1,0}(\widetilde{Q},\widetilde{J}/m)$

$$\widetilde{Q} \;=\; Q \;-\; 2\,\frac{n}{m}\,(2J+Q)\,, \qquad 2\widetilde{J}+\widetilde{Q} \;=\; \frac{2J+Q}{m}$$

- New Cardy-like limits $\qquad \tau \to -\dfrac{n}{m}$

Cardy-like limit can be derived easily from Fourier expansion of the elliptic functions

$$\log Q(z;\tau) \underset{\tau\to 0}{\longrightarrow} \frac{1}{4\pi^2\tau^2}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{\mathbf{e}(-mz_1)}{m^3}\,,$$

$$= \frac{\pi\mathrm{i}}{3}\frac{1}{\tau^2}B_3\big(\{z_1\}\big)\,.$$

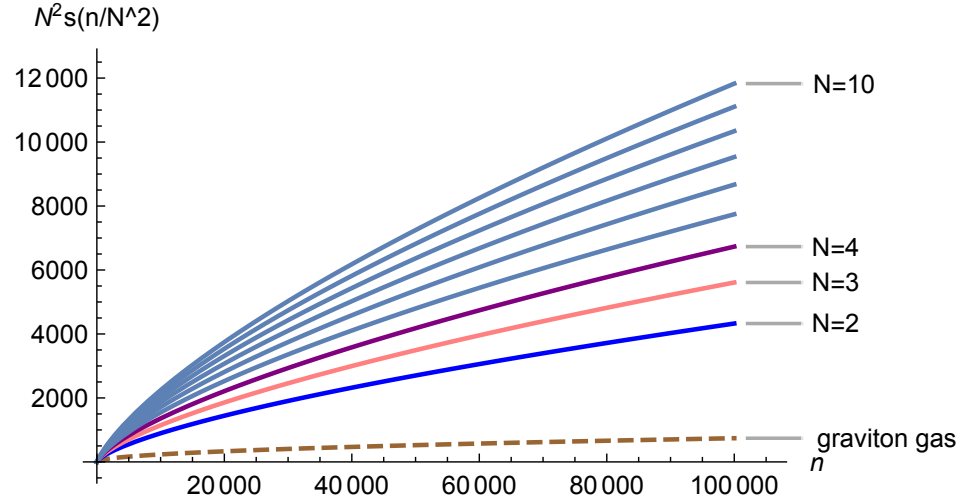
$$\log P(z;\tau) \underset{\tau\to 0}{\longrightarrow} -\frac{\pi\mathrm{i}}{\tau}B_2\big(\{z_1\}\big)$$

$$S(u) \underset{\tau\to 0}{\longrightarrow}$$

$$\pi\mathrm{i}\sum_{i,j}\left(\frac{1}{3\tau^2}\big(B_3\big(\{u_{1ij}\}\big)+3B_3\big(\{u_{1ij}+\tfrac{1}{3}\}\big)\big)+\frac{1}{\tau}\big(B_2\big(\{u_{1ij}\}\big)-B_2\big(\{u_{1ij}+\tfrac{1}{3}\}\big)\big)\right)$$

Are SYM operators BHs or gravitons?

- At $N = \infty$ single-trace operators = single gravitons.
  At $N = \infty$ multi-trace BPS operators = multi-gravitons.
  Statement sharpened, conjectured to hold at finite N.
  Checked to first few orders. [Chang, Yin '13]
  [cf Janik, Trzetrzelewski, '07; Grant, Grassi, Kim, Minwalla '08]



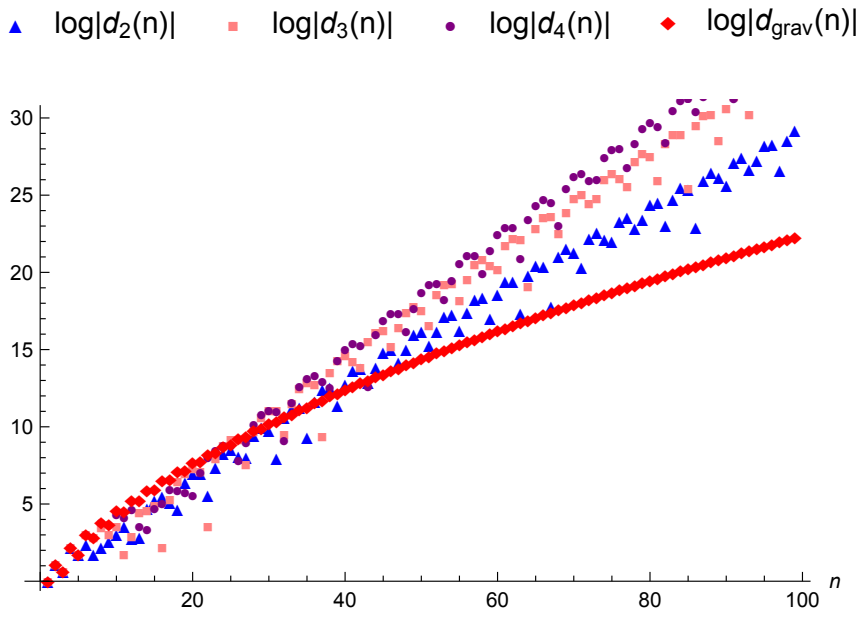
$$\log d_N(n) \underset{n\to\infty}{\sim} \frac{\pi}{2\cdot 3^{1/6}} N^{2/3} n^{2/3} + \cdots$$

$$\log d_{\rm grav}(n) \underset{n\to\infty}{\sim} \frac{\pi}{3}\sqrt{5n} + \cdots$$

- On the other hand, we clearly see much larger growth of SYM operators controlled by N^2, as consistent with BH.

Both! The finite-N index *interpolates* between multi-gravitons and BHs [S.M. 2005]

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
d_2	3	-2	9	-6	11	-6	9	14	-21	36	-17	-18	114	-194	258	-168
d_3	3	-2	9	-6	21	-18	33	-22	36	6	-19	90	-99	138	-9	-210
d_4	3	-2	9	-6	21	-18	48	-42	78	-66	107	-36	30	114	-165	390
d_5	3	-2	9	-6	21	-18	48	-42	99	-96	172	-156	252	-160	195	48
d_6	3	-2	9	-6	21	-18	48	-42	99	-96	200	-198	345	-340	540	-426
d_{grav}	3	-2	9	-6	21	-18	48	-42	99	-96	200	-198	381	-396	711	-750



This can be proved using ideas from representation theory

[S.M. 2005] [cf Dolan '07; Dutta, Gopakumar '07]

- Expand exponential in terms of U(N) characters.
- Frobenius: $\prod_{j=1}^{m} \left(\mathrm{Tr}\, U^j\right)^{k_j} = \sum_{\ell(\lambda)\leq N} \widetilde{\chi}_\lambda(U)\, \chi^\lambda(P) \qquad \left(P \equiv \prod_{j=1}^{m} (j)^{k_j}\right)$

$$\mathcal{I}_N(\mathrm{x}) = \sum_P i_{\mathrm{s}}(\mathrm{x})_P \frac{1}{z_P} \sum_{\ell(\lambda)\leq N} \chi^\lambda(P)^2$$

$$z_P = \prod_{j=1}^{m} k_j!\, j^{k_j}$$

$$f(\mathrm{x})_P := \prod_{j=1}^{m} f(\mathrm{x}^j)^{k_j}$$

- At $N = \infty$, we sum over all partitions, and =1

$$\Rightarrow \quad \mathcal{I}_{N=\infty}(\mathrm{x}) = \prod_{k=1}^{\infty} \frac{1}{1 - i_{\mathrm{s}}(\mathrm{x}^k)} = \prod_{n=1}^{\infty} \frac{(1-\mathrm{x}^{3n})^2}{(1-\mathrm{x}^{2n})^3} = \sum_n d_{\mathrm{grav}}(n)\, \mathrm{x}^n$$

Outlook and Questions

Faith in AdS/CFT reconfirmed (in its simplest form)

- Complex saddles of matrix model important.
  BHs, many other phases (what are they in gravity?)
- Importance of lattice $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$ and torus $\mathbb{C}/\Lambda_\tau$

  Role of modular symmetry: τ^{-1} vs τ^{-2} period functions
- Elliptic deformation of SYM action — Q-exact term?
- Sum over random partitions (cf 2d gauge theory).
- Deeper role of number theory? Relation to L-functions/ Mahler measure/ Algebraic K-theory.

Thank you very much for the attention and all the questions!