Low-energy analyses of kaon Primakoff reactions

J. Ruiz de Elvira

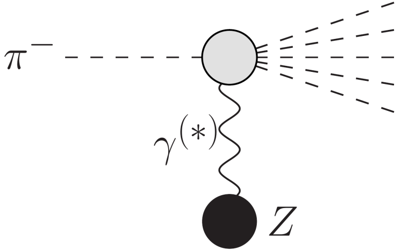
Institute for Theoretical Physics, University of Bern

4th Workshop on Perceiving the Emergence of Hadron Mass through AMBER@CERN,
December 1st, 2020

Primakoff reactions at Compass: $\gamma\pi^- \to \dots$

[B. Kubis contribution to the AMBER/COMPASS++ kickoff workshop]

- Pion beam at small momentum transfer:



- $\gamma\pi^- \to \gamma\pi^-$: Compton scattering
  - $\triangleright$ pion polarizabilities
  - $\hookrightarrow$ fundamental information on pion structure

[COMPASS (2015)]

- $\gamma\pi^- \to \pi^-\pi^0$
  - $\triangleright$ test of the QCD chiral anomaly

[COMPASS (???)]

- $\gamma\pi^- \to (3\pi)^-$
  - $\triangleright$ test of chiral dynamics

[COMPASS (2012)]

Pion polarizabilities

Charged pion polarizabilities: Compton scattering at $\nu = s - u = t = 0$

- ChPT prediction: difference of the electric and magnetic polarizabilities

$$\alpha_E^{\pi^+} - \beta_M^{\pi^+} = \frac{8\alpha_{\rm em}}{F_\pi^2 M_\pi}\left(L_9^r + L_{10}^r\right) + \mathcal{O}(M_\pi)$$

- Low-energy theorem: $h_A/h_V \propto L_9^r + L_{10}$ from $\pi^+ \to e^+ \nu_e \gamma$
- two loop result: $\alpha_E^{\pi^+} - \beta_M^{\pi^+} = 5.7\,(1.0) \times 10^{-4}\,\text{fm}^3$ [Gasser et al. (2006)]

  vs $\alpha_E^{\pi^+} - \beta_M^{\pi^+} = 4.0\,(1.2)(1.4) \times 10^{-4}\,\text{fm}^3$ [COMPASS (2015)]
- Convergence of the chiral series:

$$\alpha_E^{\pi^+} = \underbrace{2.68}_{\mathcal{O}(p^4)} + \underbrace{0.08 + 0.33 - 0.70}_{\mathcal{O}(p^6)} = 2.39$$

$$\beta_M^{\pi^+} = \underbrace{-2.68}_{\mathcal{O}(p^4)} + \underbrace{0.07 - 0.01 + 0.75}_{\mathcal{O}(p^6)} = -1.87$$

$\hookrightarrow$ large corrections for $\alpha_E^{\pi^+}/\beta_M^{\pi^+}$: NLO $\xrightarrow[24\%]{11\%}$ NNLO

- Dispersive results: $\alpha_E^{\pi^+} - \beta_M^{\pi^+}\Big|_{\pi-\text{pole}} = 5.8\,(4) \times 10^{-4}\,\text{fm}^3$ [Hoferichter et al. (2017)]

$\gamma\pi^- \to \pi^-\pi^0$

Manifestation of the Wess-Zumino-Witten anomaly [Adler, Lee, Treiman, Zee (1971)]

$$F_{3\pi} = \frac{eN_c}{12\pi^2 F_\pi^3} = 9.76(3)\,\mathrm{GeV}^{-3}$$

- Experimental determination: $F_{3\pi} = 12.9\,(9)\,(5)\,\mathrm{GeV}^{-3}$ [Serpukhov (1987)]
- EM + chiral order corrections: $F_{3\pi} = 10.7 \pm 1.2\,\mathrm{GeV}^{-3}$ [Ametller, Knecht, Talavera (2001)]
- Determination from $\pi^- e^- \to \pi^- e^- \pi^0$: $F_{3\pi} = 9.6 \pm 1.1\,\mathrm{GeV}^{-3}$ [Giller et al. (2005)]

- Dispersion relations:

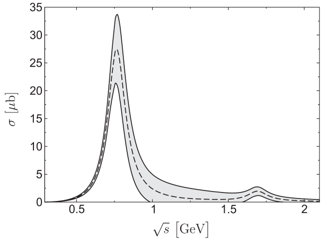
  energy dependence controlled by analyticity and unitarity

  ▷ $\pi\pi$ phase shift as input

  ▷ model independent extraction

  of $\rho \to \pi\gamma$ radiative coupling

  $\hookrightarrow$ extract the chiral anomaly using the $\rho(770)$ spectrum

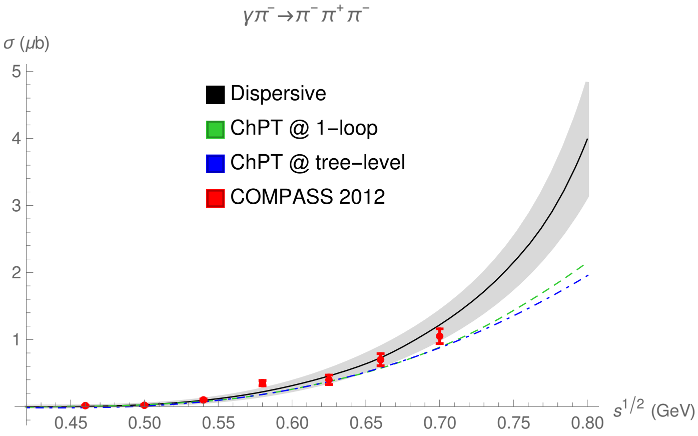


[Hoferichter, Kubis, Sakkas, Zanke (2012,2017)]

$\gamma\pi^- \to (3\pi)^-$

Test of chiral dynamics

- One-loop ChPT calculation [Kaiser (2010)]
- Experimental results [COMPASS (2012)]
- Dispersive result for the pion pole + resonances [Colangelo, Monnard, JRE (in progress)]



Charged kaon polarizabilities

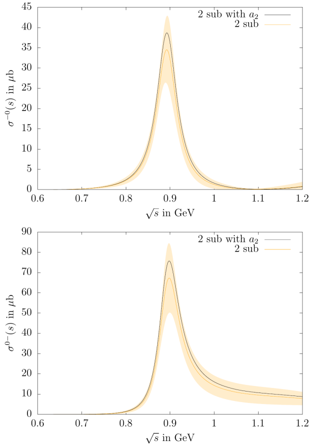
- One loop result: analogous to pion case [Guerrero, Prades (1997)]

$$\alpha_E^{K^+} - \beta_M^{K^+} = \frac{8\alpha_{\rm em}}{F_K^2 M_K}\left(L_9^r + L_{10}^r\right) + \mathcal{O}(M_K) \approx 1.2 \times 10^{-4}\,\mathrm{fm}^{-3}$$

- two-loop calculations not available: size of corrections?
- dispersive results only for a $\gamma\gamma \to \pi\pi$ cross check
  [Garcia-Martin, Moussallam (2010)]

Pion production: $\gamma K^- \to (\pi K)^-$

- Partial one-loop results [Ebertshäuser (2001), Hacker (2008)]
- Ongoing dispersive analysis: [Dax, Stamen, Kubis (to appear)]
  - ▷ Khuri-Treiman equations: single variable amplitudes
  - ▷ $\pi\pi \to \bar{K}K$ and πK phase shifts as input
  - ▷ subtraction constants extracted from $K^* \to K\gamma$ or/and chiral anomaly
  - ↪ combined with future data framework to extract the anomaly and $K^*(892)$ radiative couplings



[Dax, Stamen, Kubis (to appear)]

Dispersion relations in a nutshell

- Effective field theories $\Rightarrow$ systematically improvable but

  $\triangleright$ number of LECs increase rapidly

  $\triangleright$ **convergence** problems: low-lying resonances, strong rescattering effects

- Dispersion relations: **analyticity**, **crossing**, **unitarity**

  $\triangleright$ analyticity constrains the energy dependence of scattering amplitude

  $\triangleright$ crossing symmetry connects different physical regions

  $\triangleright$ unitarity constrains imaginary part

- **Roy(-Steiner) eqs.** =Partial-Wave (Hyberbolic) Dispersion Relations coupled by unitarity and crossing symmetry

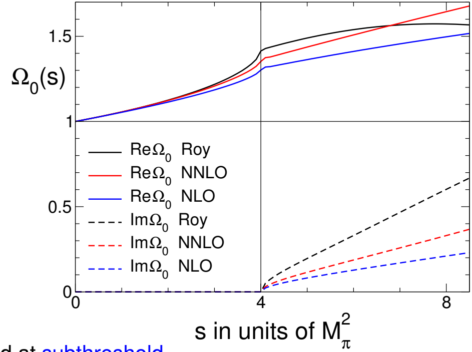
  $\hookrightarrow$ **model independent** approach

  $\hookrightarrow$ **analytic continuation** for the complex plane $\Rightarrow$ resonances, unphysical regions

ChPT vs dispersion theory

- How large are rescattering effects?

$\hookrightarrow$ look at $F^s_\pi(s) = P(s)\Omega_0(s)$, $\quad \Omega_0(s) = \exp \dfrac{s}{\pi}\displaystyle\int_{s_\pi}^{\infty} \dfrac{ds'}{s'} \dfrac{\delta^0_0(s')}{s' - s - i\epsilon}$



[courtesy of Heiri Leutwyler 2015]

- ChPT converges rapid at subthreshold

- Slow convergence already at threshold

$\hookrightarrow$ dispersion theory for the energy dependence, ChPT for subraction constants

From dispersion relations to Roy-equations

- $\pi\pi \to \pi\pi \Rightarrow$ fully crossing symmetric in Mandelstam variables s, t, and $u = 4M_\pi - s - t$
- Start from twice-subtracted **fixed-t** DRs

$$T^I(s,t) = c(t) + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ds'}{s'^2}\left[\frac{s^2}{(s'-s)} - \frac{u^2}{(s'-u)}\right] \mathrm{Im}\; T^I(s',t)$$

- Subtraction functions c(t) are determined via crossing symmetry

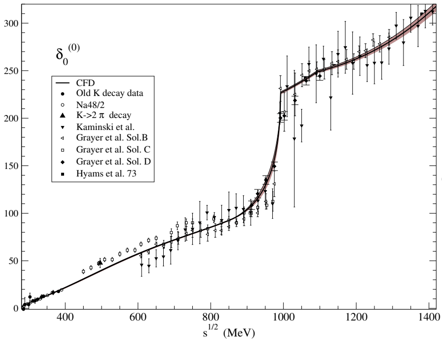
  $\hookrightarrow$ functions of scattering lengths: a_0^0 and a_0^2
- PW-projection and expansion yields **Roy-equations** [Roy (1971)]

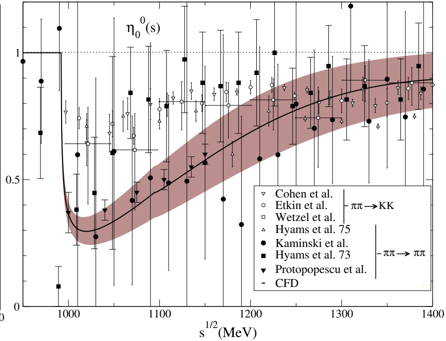
$$t_J^I(s) = ST_J^I(s) + \sum_{J',I'}(2J'+1)\int_{4m_\pi^2}^{\infty} ds' K_{JJ'}^{II'}(s',s)\, \mathrm{Im}\, t_{J'}^{I'}(s')$$

- $K_{JJ'}^{II'}(s',s) \Rightarrow$ analytically known
- elastic unitarity: $t(s) = \dfrac{\sin\delta(s) e^{i\,\delta(s)}}{\sigma(s)}$

  $\hookrightarrow$ self-consistent equation for phase shifts

- Solution for the $\pi\pi$ S0-wave





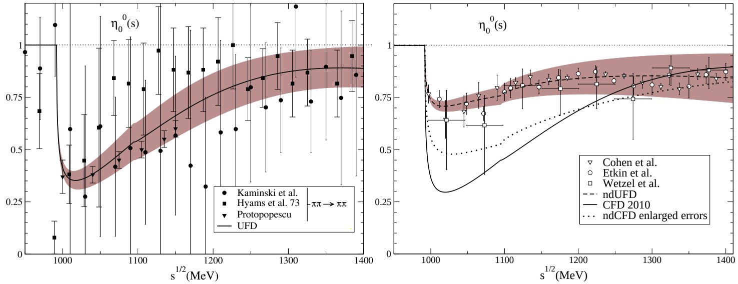
[Garcia-Martin, Kaminski, Pelaez, JRE, Yndurain (2011)]

Beyond the elastic region: dip vs no-dip solutions

- The dip vs no-dip ⇒ long-standing controversy [Pennington, Bugg, Zou, Achasov, · · ·]

  ↪ no clear preference for any of the two scenarios in previous works

- Is it possible to satisfy Roy Equations in a non-dip scenario?



- How do the dip vs non-dip solutions satisfy Roy equations

	dip	non-dip	enlarged errors
χ^2-like	1.0	**3.5**	1.7

[Garcia-Martin, Kaminski, Pelaez, JRE, Yndurain (2011)]

↪ the non-dip scenario is **rejected** by DR

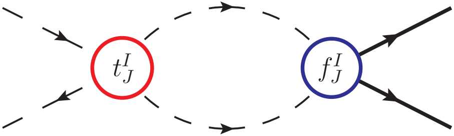
Key differences compared to $\pi\pi$ Roy equations

- Crossing: coupling between $\pi K \to \pi K$ (s-channel) and $\pi\pi \to \bar{K}K$ (t -channel)

  $\Rightarrow$ need a different kind of dispersion relations [Hite, Steiner 1973, Büttiker et al. 2004]

- Unitarity in t-channel, e.g. single-channel approximation

$$\mathrm{Im} f_\pm^J(t) = \sigma_t^\pi f_\pm^J(t) t_J^I(t)^*$$

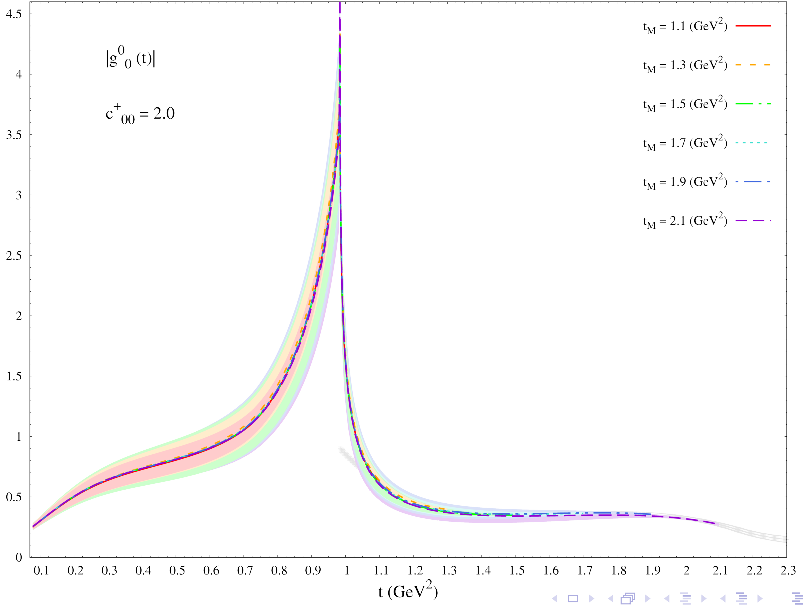


$\Rightarrow$ **Watson's theorem**: phase of $f_\pm^J(t)$ equals δ_{IJ} [Watson 1954]

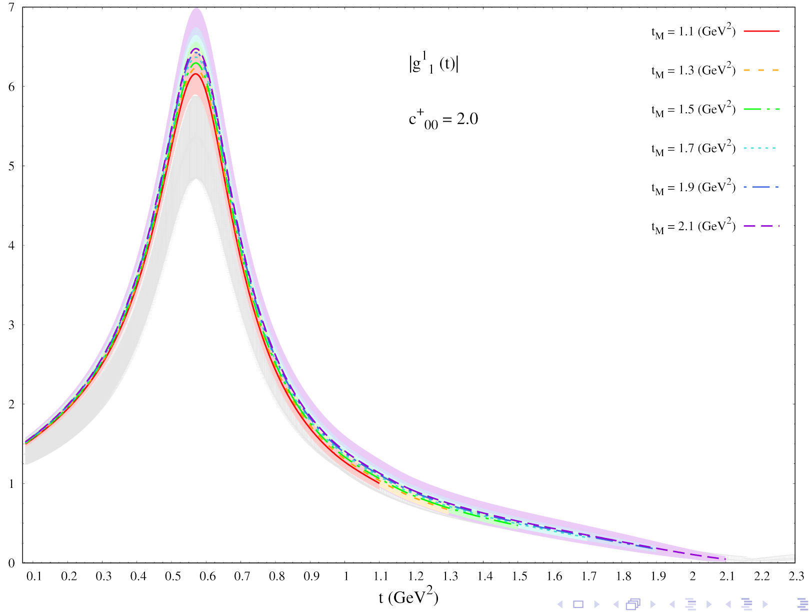
$\triangleright$ solution in terms of Omnès function [Muskhelishvili 1953, Omnès 1958]

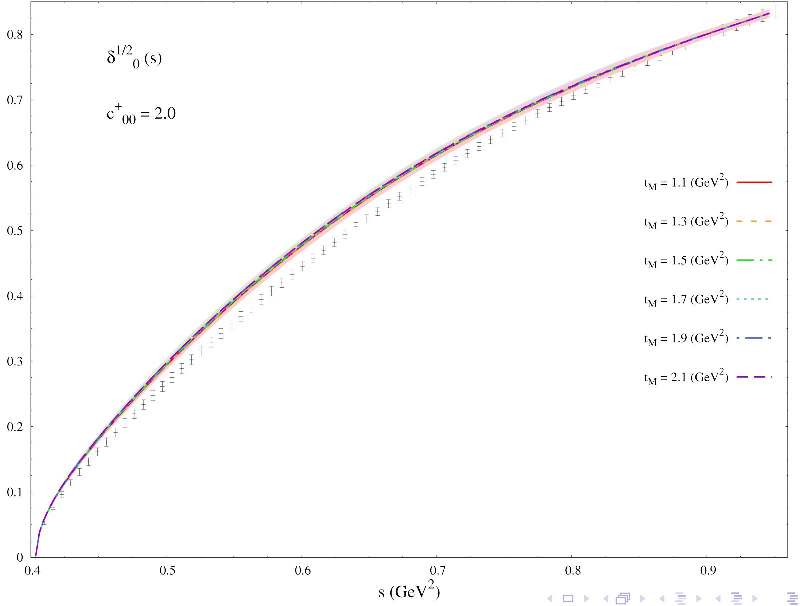
$\hookrightarrow$ $\pi\pi$ scattering δ_{IJ} need as input

$\pi\pi \to \bar{K}K$ scattering: preliminary results



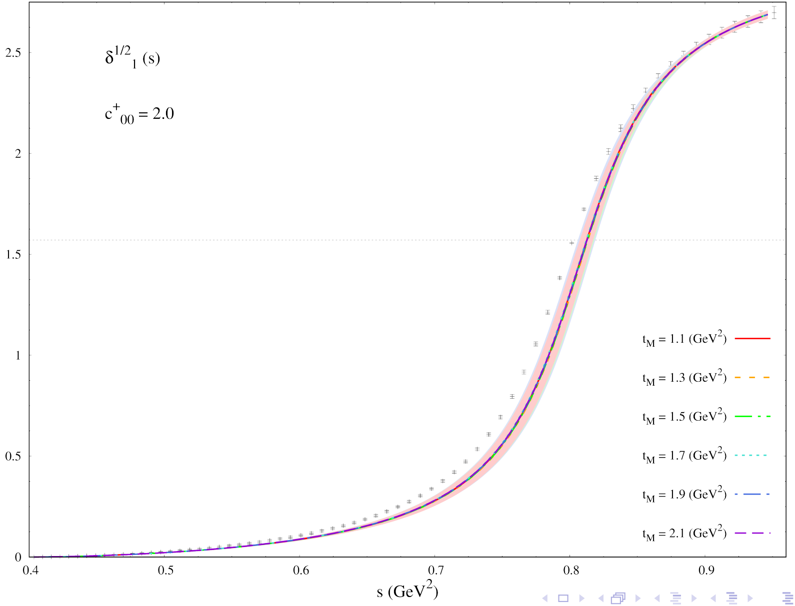
$\pi\pi \to \bar{K}K$ scattering: preliminary results



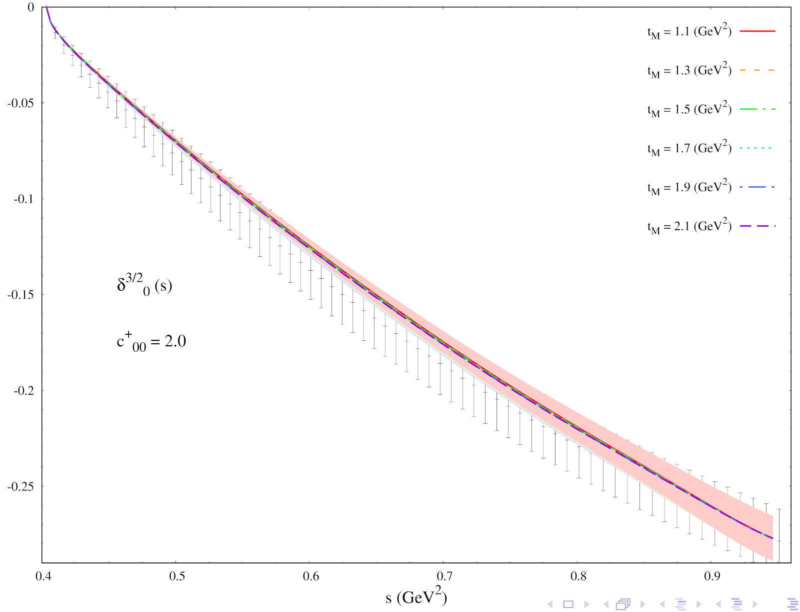


δ1/2 0 (s)
c+ 00 = 2.0
tM = 1.1 (GeV2)
tM = 1.3 (GeV2)
tM = 1.5 (GeV2)
tM = 1.7 (GeV2)
tM = 1.9 (GeV2)
tM = 2.1 (GeV2)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.4
0.5
0.6
0.7
0.8
0.9
s (GeV2)

$\pi\pi \to \bar{K}K$ scattering: preliminary results



$\pi\pi \to \bar{K}K$ scattering: preliminary results



Roy equations and resonance pole parameters

- $t_{IJ}(s)$ known in the I Riemann sheet

- **Resonances**

  $\hookrightarrow$ poles on unphysical Riemann sheets

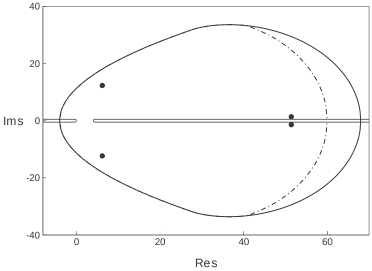
- First and second Riemann sheets continuous along the cut

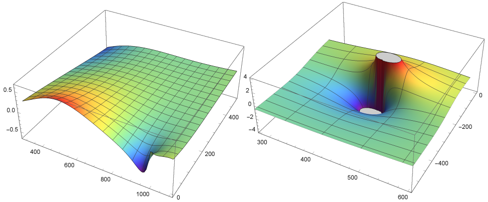
$$S^{II}(s - i\epsilon, t) = S^{I}(s + i\epsilon, t)$$

$$t^{II}_{IJ}(s) = \frac{t_{IJ}(s)}{1 + 2i\,\sigma(s) t_{IJ}(s)}$$

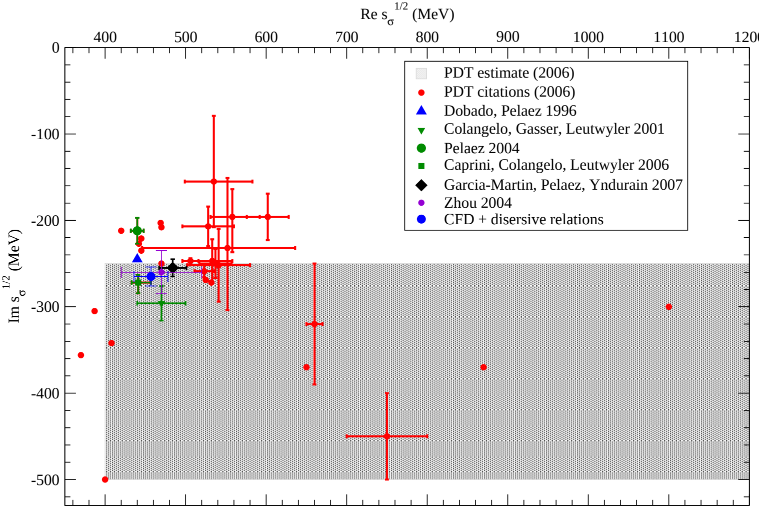
- Elastic scattering:

  $\hookrightarrow$ **II** RS is **known** exactly



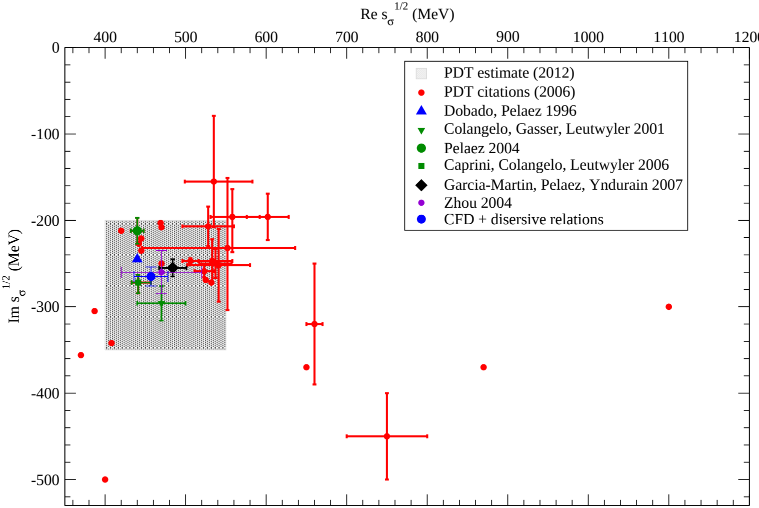


$f_0(500)$ pole 2010



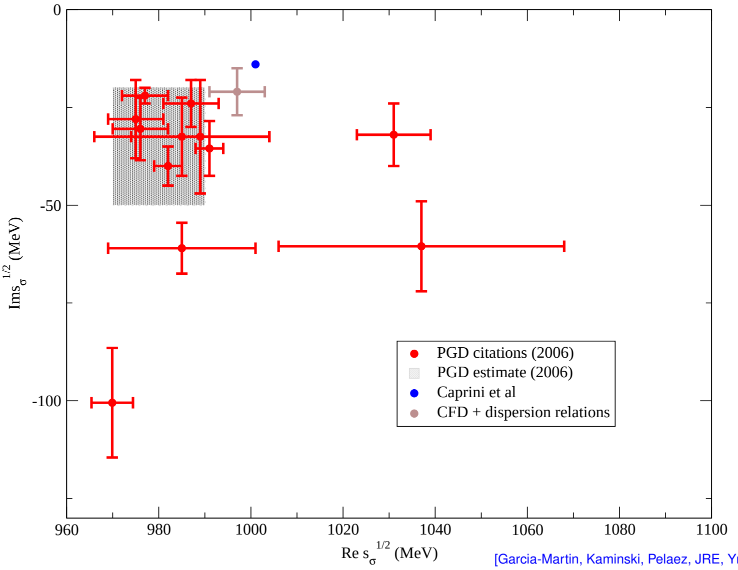
[Garcia-Martin, Kaminski, Pelaez, JRE, Yndurain (2011)]

$f_0(500)$ pole 2012



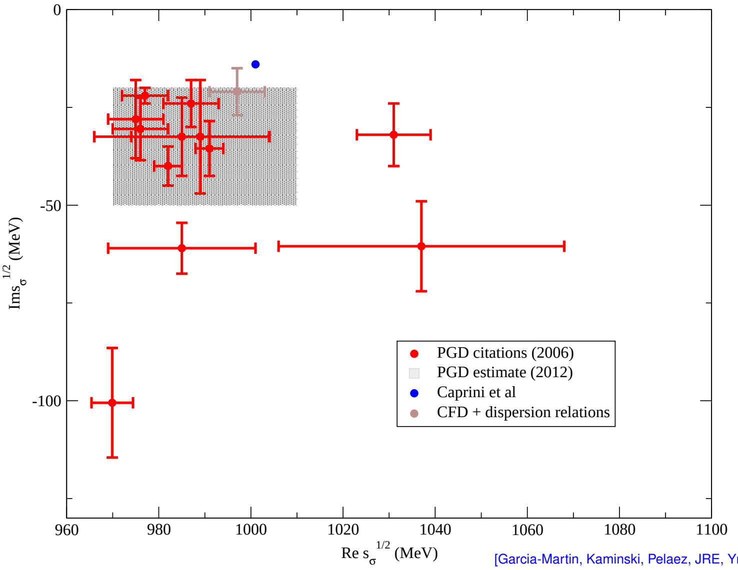
[Garcia-Martin, Kaminski, Pelaez, JRE, Yndurain (2011)]

$f_0(980)$ pole 2010



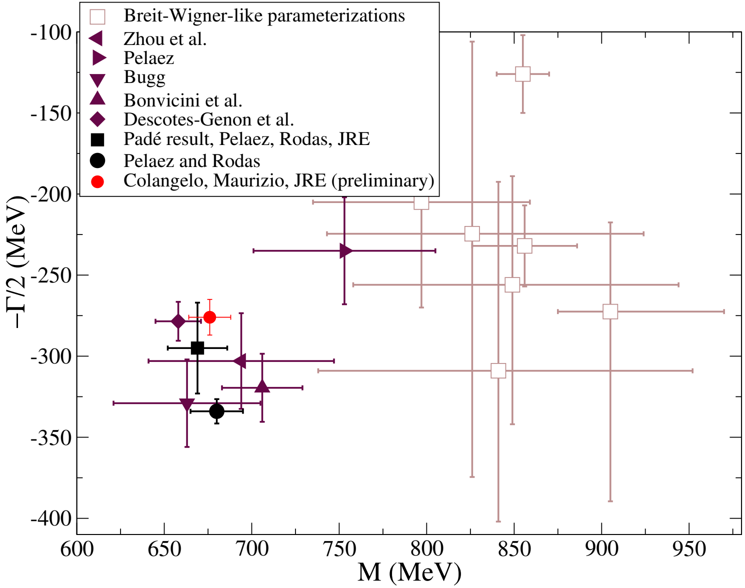
[Garcia-Martin, Kaminski, Pelaez, JRE, Yndurain (2011)]

$f_0(980)$ pole 2012



[Garcia-Martin, Kaminski, Pelaez, JRE, Yndurain (2011)]

Preliminary results: kappa pole



Inelastic resonance pole parameters

- Assume two-coupled channels

$$T(s) = \begin{pmatrix} t^{(11)}(s) & t^{(12)}(s) \\ t^{(12)}(s) & t^{(22)}(s) \end{pmatrix} \qquad \Sigma(s) = \begin{pmatrix} \sigma_1(s) & 0 \\ 0 & \sigma_2(s) \end{pmatrix}$$

- First and higher Riemann sheets continuous above a given threshold

$$S^{II}(s - i\epsilon, t) = S^{I}(s + i\epsilon, t)$$

$$T^{II}(s) = T(s) \cdot (\mathbb{1} + 2i\,\Sigma(s)T(s))^{-1}$$

- Amplitude for the first channel in the III Riemann sheet:

$$t^{(11)}(s)^{III} = t^{(11)}(s) - \frac{2i\sigma_2(s)t^{(12)}(s)^2}{1 + 2i\sigma_2(s)t^{(22)}(s)}$$

$\hookrightarrow$ III and IV RS require crossed channels

- Can we still get any information about inelastic resonances?

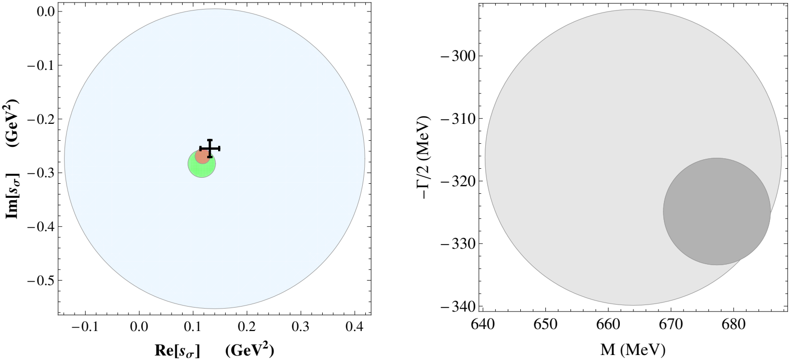
$\hookrightarrow$ analytic expansion: Laurent series, Padé approximants, continuous fractions ...

Padé approximants and resonance pole parameters

- Analytical continuation given by Padé approximants:

$$t(s) = P^N_M(s, s0) + \mathcal{O}\left(s - s_0\right)^{N+M+1}, \qquad P^N_M(s, s_0) = Q^N(s, s_0)/R^M(s, s_0)$$

- Check with elastic resonances: $f_0(500)$ and $\kappa(700)$

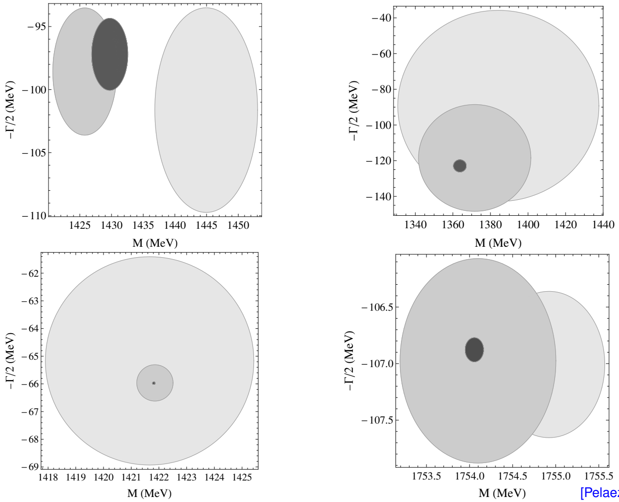


[Masjuan, Cillero, JRE (2014)] [Pelaez, Rodas, JRE (2016)]

- Consistent agreement but larger uncertainties

Padé approximants and inelastic resonances

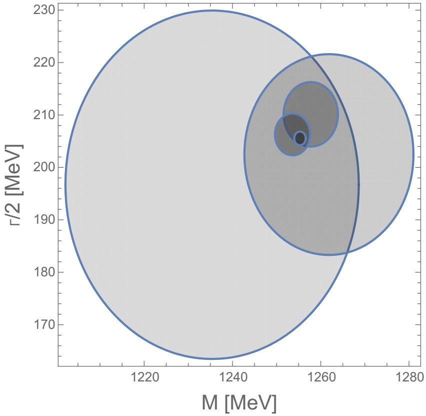
- Analytic continuation above the highest inelastic threshold
- Determination of strange resonances: $K_0^*(1430)$, $K_1^*(1410)$, $K_2^*(1430)$, $K_3^*(1780)$



[Pelaez, Rodas, JRE (2016)]

Padé approximants and the $f_0(1370)$

- $f_0(1370)$ still controversial resonance
  - ▷ PDG estimate $\sqrt{s_{f_0(1370)}} = (1200 - 1500) - i\,(150 - 250)$ MeV
  - ▷ Key for the identification of the lightest glueball
- Analytic continuation from $\pi\pi$ and $\pi\pi \to \bar{K}K$ dispersion relation



[Rodas, Pelaez, JRE (preliminary)]

Thank you

Spare slides

Motivation: Why πK scattering?

- **Low energies:** test chiral dynamics in the strange-quark sector

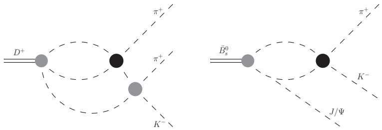
  $\triangleright$ Scattering lengths lowest energy observables

  $\hookrightarrow$ Spontaneous and explicit chiral symmetry breaking

- **Higher energies:** resonances, hadron spectrum

  $\hookrightarrow$ $\kappa(700)$ non-ordinary meson, PDG "needs confirmation"

- **Input for Heavy-meson decays**: CP-violation and New Physics searches



- **Crossed channel $\pi\pi \to \bar{K}K$**: first inelastic contribution to $\pi\pi$ scattering

  $\hookrightarrow$ $\Gamma(f_0(500) \to \bar{K}K)$ nature of the $f_0(500)/\sigma$ meson

  $\hookrightarrow$ Nucleon form factors, $g-2\ldots$

Pion-kaon scattering lengths: ChPT

- Simplest scattering process involving strangeness

  $\hookrightarrow$ test chiral dynamics in the strange-quark sector

- Two independent amplitudes: $I_s = \{1/2, 3/2\}$ or $I_\pm = \{+, -\}$,
  $T^{1/2} = T^+ + 2T^-, \quad T^{3/2} = T^+ - T^-.$

- LO prediction: $a^- = \frac{m_\pi m_K}{8\pi(m_\pi + m_K)F_\pi^2} + \mathcal{O}(m_i^4), \qquad a^+ = \mathcal{O}(m_i^4)$ [Weinberg 66]

- NLO: $a^-_{LECs} = \frac{2 m_K m_\pi^3}{\pi(m_\pi + m_K)F_\pi^4} L_5 + \mathcal{O}(m_i^6)$ [Bernard, Kaiser, Meißner 91]

$$a^+_{LECs} = \frac{2 m_K^2 m_\pi^2}{\pi(m_\pi + m_K)F_\pi^4}\left(4(L_1 + L_2 - L_4) + 2L_3 - L_5 + 2(2L_6 + L_8)\right) + \mathcal{O}(m_i^6)$$

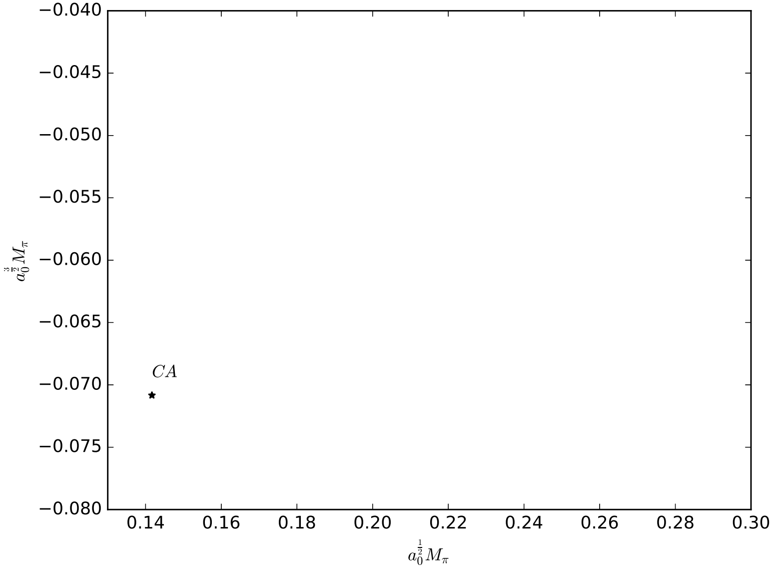
- Size of higher order corrections?

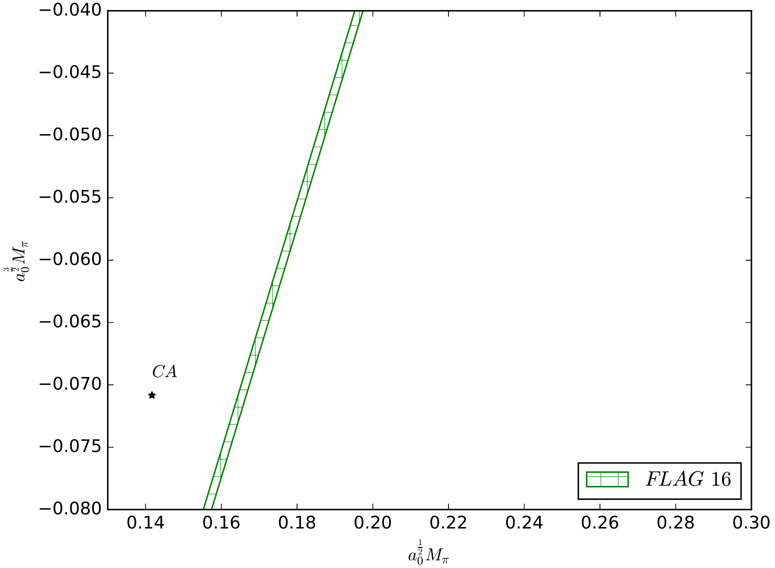
  $\hookrightarrow$ Low Energy Theorem

$$a^-\big|_{N^nLO} \propto a^-\big|_{LO}\left(\frac{M_\pi}{4\pi F_\pi}\right)^2\left(\frac{M_K}{4\pi F_\pi}\right)^{2n}, \quad n \geq 2$$ [Weinberg 66]

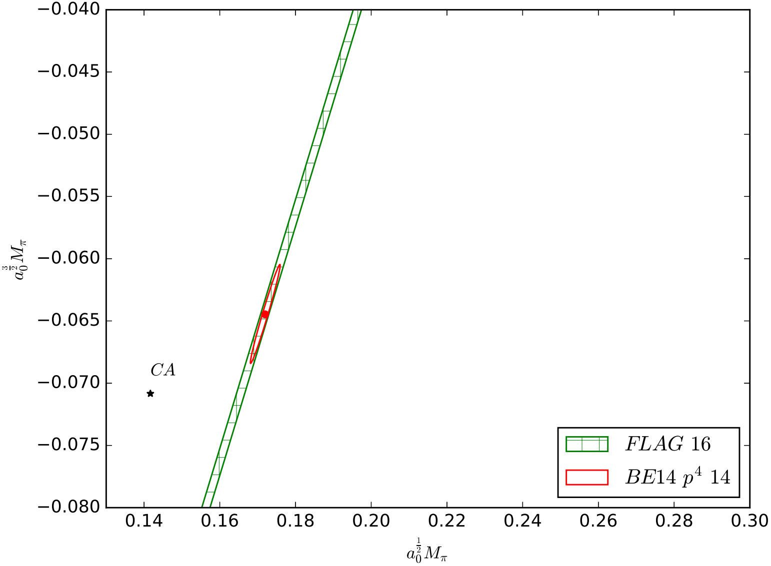
- NNLO: C_{1-32}, 10 for a^- and 23 for a^+ [Bijnens,Dhonte,Talavera 2004], [Bijnens, Ecker 2014]

πK scattering lengths

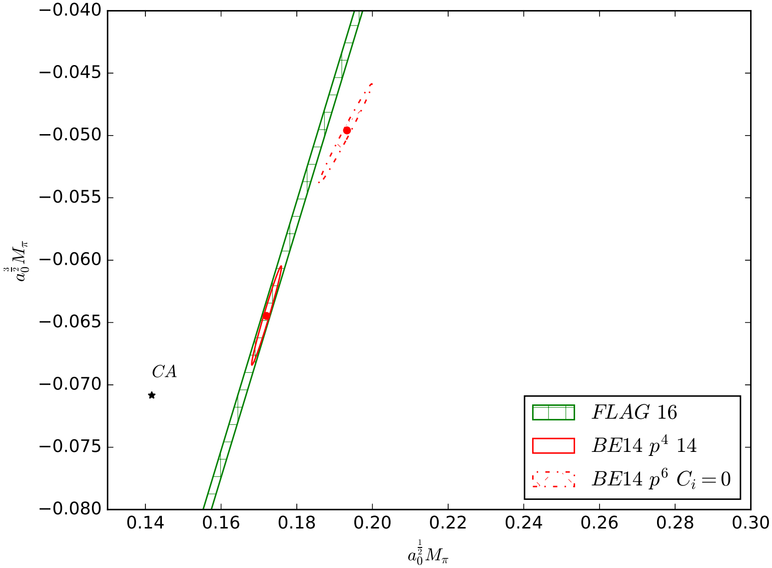



−0.040
−0.045
−0.050
−0.055
−0.060
−0.065
−0.070
−0.075
−0.080
$a_0^{\frac{3}{2}} M_\pi$
0.14
0.16
0.18
0.20
0.22
0.24
0.26
0.28
0.30
$a_0^{\frac{1}{2}} M_\pi$
CA
$FLAG\ 16$

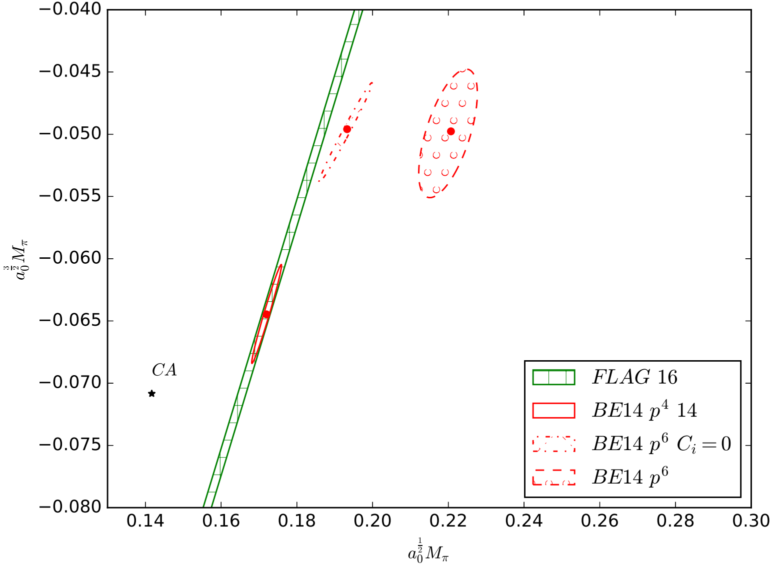
πK scattering lengths



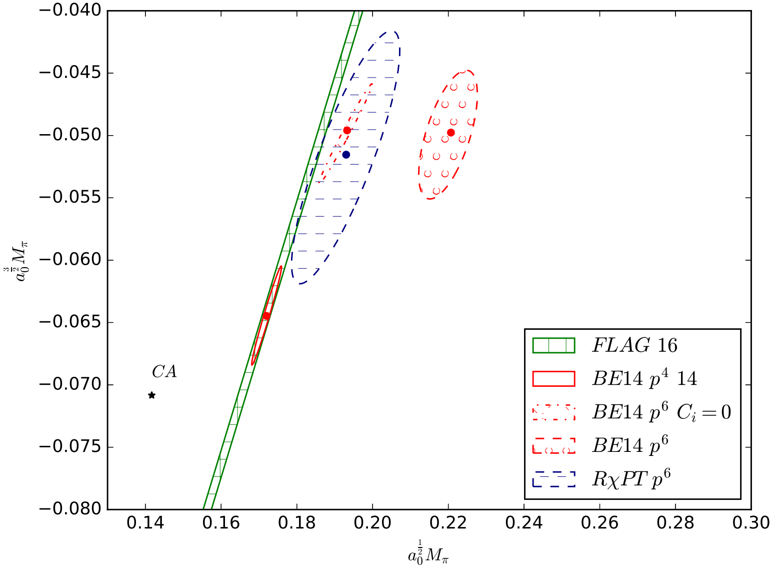
πK scattering lengths



πK scattering lengths



πK scattering lengths



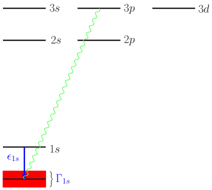
Current status

- **Experimental values**: DIRAC collaboration

  ▷ lifetime of πK atoms at CERN $\Rightarrow$ isovector scattering length

$$\Gamma_{1S} \propto \left| T_{(\pi^+ K^- \to \pi^0 K^0)} \right|^2 \propto |a^-|^2$$

[Deser, Goldberger, Baumann, Thirring 1954]



▷ Current result:

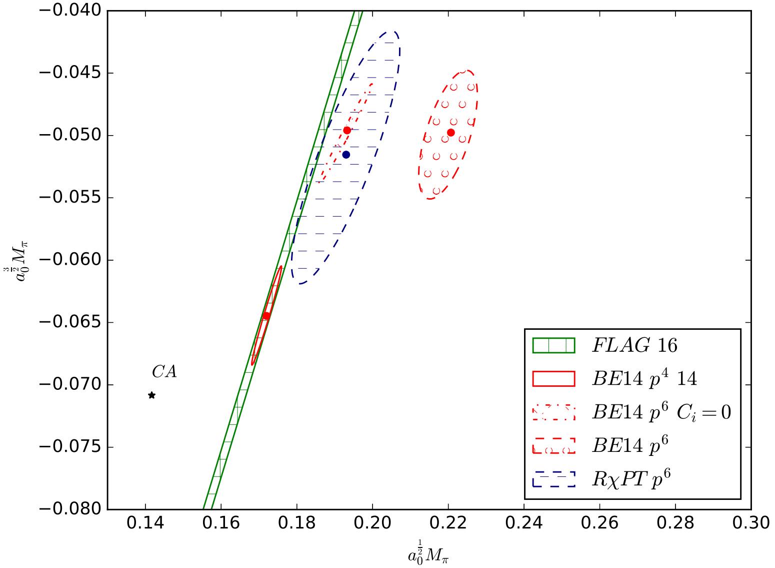
$$a^- = (-0.072^{+0.031}_{-0.020})\, m_\pi^{-1}$$

[DIRAC 2017]

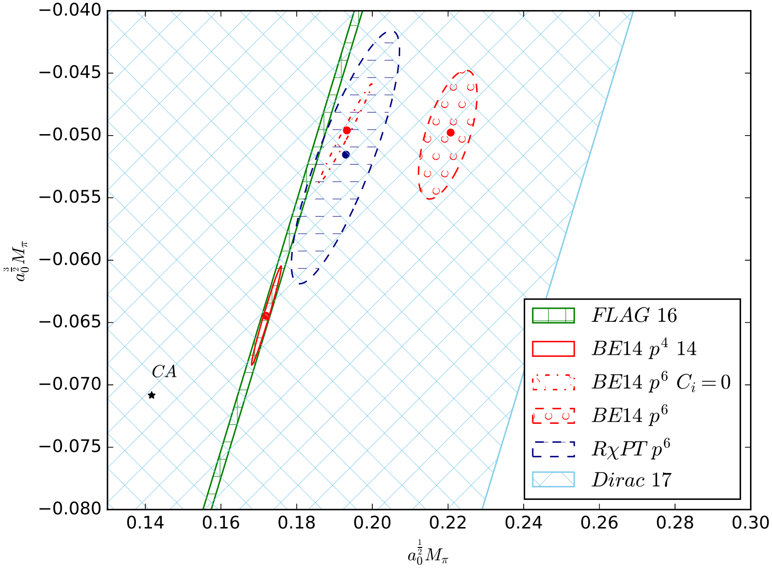
$\hookrightarrow$ **huge uncertainties**

▷ Room for improvement: near future increase statistics by 10

πK scattering lengths



πK scattering lengths



Current status

- **Lattice analysis**: unquenched results only

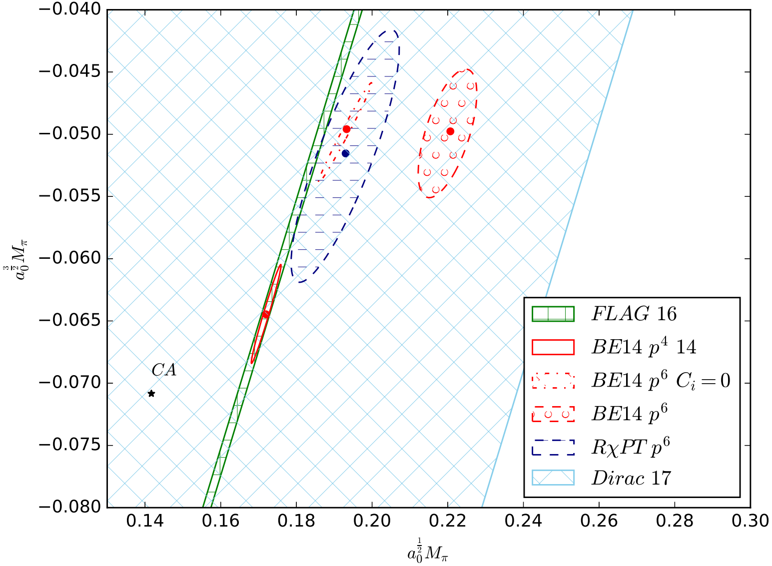
Current status

- **Lattice analysis**: unquenched results only

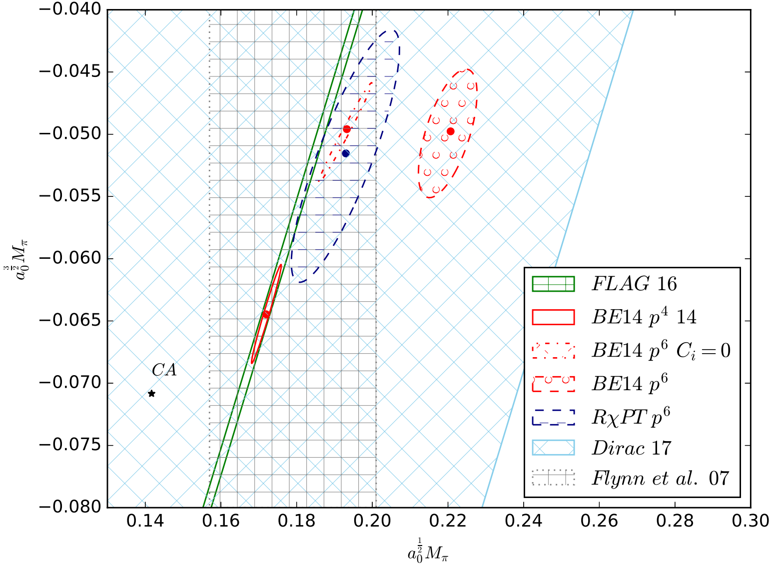
  ▷ Constraint from semileptonic K_{l3} decays [Flynn, Nieves 2007]

$$a^{1/2} = 0.179(17)(14) m_\pi^{-1}$$

Current status



Current status



Current status

- **Lattice analysis**: unquenched results only

  ▷ Constraint from semileptonic K_{l3} decays [Flynn, Nieves 2007]

$$a^{1/2} = 0.179(17)(14) m_\pi^{-1}$$

Current status

- **Lattice analysis**: unquenched results only

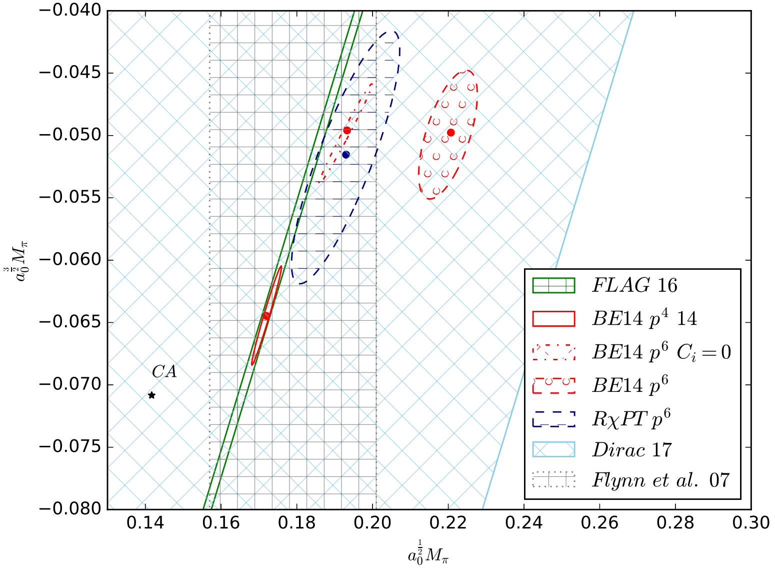
  ▷ Constraint from semileptonic K_{l3} decays [Flynn, Nieves 2007]

$$a^{1/2} = 0.179(17)(14)m_\pi^{-1}$$

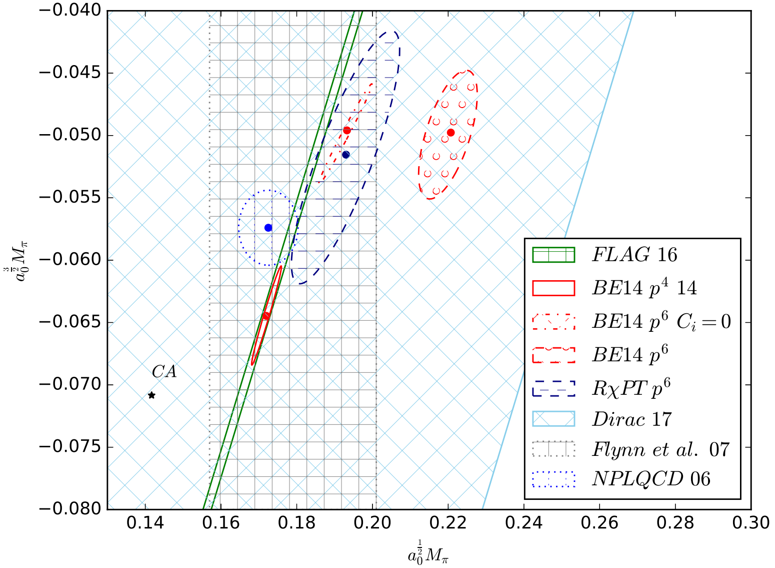
  ▷ NPLQCD: dynamical $N_f = 2 + 1$ calculation, $m_\pi = 290 - 600$ MeV [NPLQCD 2006]

$$a^{1/2} = (0.173^{+0.003}_{-0.016})m_\pi^{-1}, \quad a^{3/2} = (-0.057^{+0.003}_{-0.006})m_\pi^{-1}$$

Current status



Current status



Current status

- **Lattice analysis**: unquenched results only

  ▷ Constraint from semileptonic K_{l3} decays [Flynn, Nieves 2007]

$$a^{1/2} = 0.179(17)(14)m_\pi^{-1}$$

  ▷ NPLQCD: dynamical $N_f = 2 + 1$ calculation, $m_\pi = 290 - 600$ MeV [NPLQCD 2006]

$$a^{1/2} = (0.173^{+0.003}_{-0.016})m_\pi^{-1}, \quad a^{3/2} = (-0.057^{+0.003}_{-0.006})m_\pi^{-1}$$

Current status

- **Lattice analysis**: unquenched results only
  - ▷ Constraint from semileptonic K_{l3} decays [Flynn, Nieves 2007]

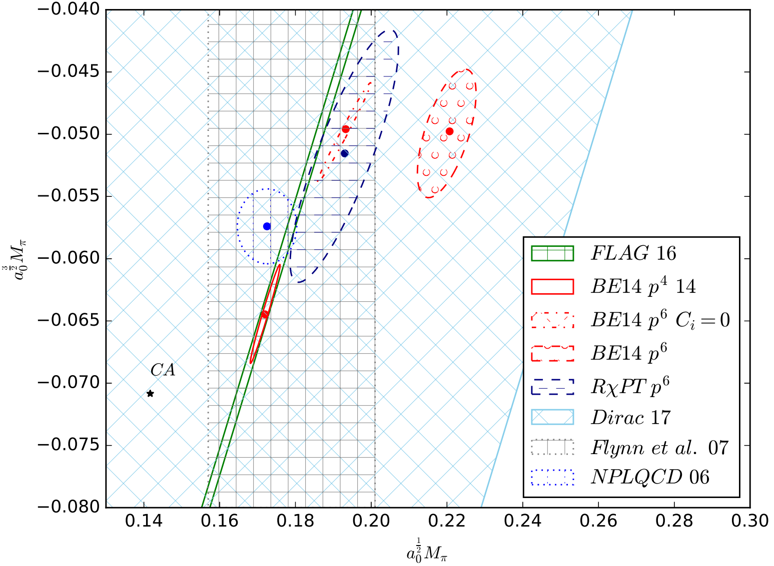
$$a^{1/2} = 0.179(17)(14)m_\pi^{-1}$$

  - ▷ NPLQCD: dynamical $N_f = 2+1$ calculation, $m_\pi = 290 - 600$ MeV [NPLQCD 2006]

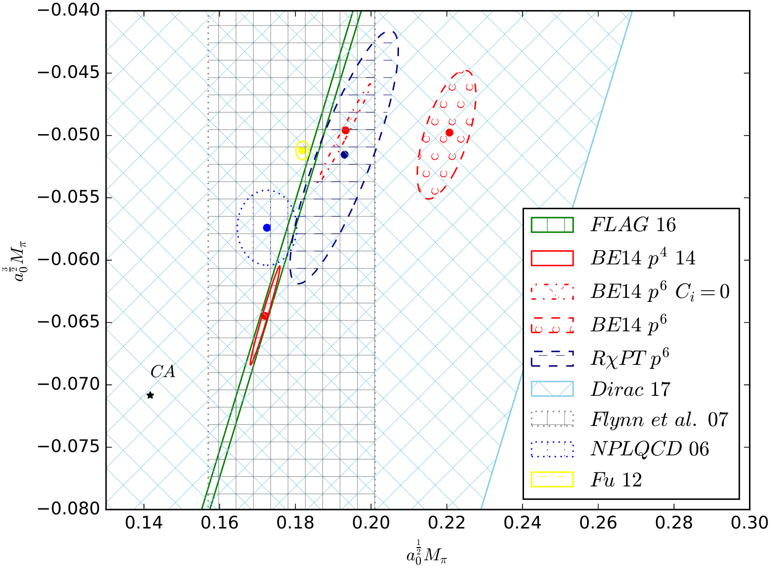
$$a^{1/2} = (0.173^{+0.003}_{-0.016})m_\pi^{-1}, \quad a^{3/2} = (-0.057^{+0.003}_{-0.006})m_\pi^{-1}$$

  - ▷ Z. Fu: dynamical $N_f = 2+1$ fermions, $m_\pi = 330 - 466$ MeV [Fu 2012]

$$a^{1/2} = 0.182(4)m_\pi^{-1}, \quad a^{3/2} = -0.051(2)m_\pi^{-1}$$


−0.040
−0.045
−0.050
−0.055
−0.060
−0.065
−0.070
−0.075
−0.080
0.14
0.16
0.18
0.20
0.22
0.24
0.26
0.28
0.30
$a_0^{\frac{3}{2}} M_\pi$
$a_0^{\frac{1}{2}} M_\pi$
CA
$FLAG$ 16
$BE14\ p^4$ 14
$BE14\ p^6\ C_i = 0$
$BE14\ p^6$
$R\chi PT\ p^6$
$Dirac$ 17
$Flynn\ et\ al.$ 07
$NPLQCD$ 06

Current status



Current status

- **Lattice analysis**: unquenched results only

▷ Constraint from semileptonic K_{l3} decays [Flynn, Nieves 2007]

$$a^{1/2} = 0.179(17)(14)m_\pi^{-1}$$

▷ NPLQCD: dynamical $N_f = 2 + 1$ calculation, $m_\pi = 290 - 600$ MeV [NPLQCD 2006]

$$a^{1/2} = (0.173^{+0.003}_{-0.016})m_\pi^{-1}, \quad a^{3/2} = (-0.057^{+0.003}_{-0.006})m_\pi^{-1}$$

▷ Z. Fu: dynamical $N_f = 2 + 1$ fermions, $m_\pi = 330 - 466$ MeV [Fu 2012]

$$a^{1/2} = 0.182(4)m_\pi^{-1}, \quad a^{3/2} = -0.051(2)m_\pi^{-1}$$

Current status

- **Lattice analysis**: unquenched results only

▷ Constraint from semileptonic K_{l3} decays [Flynn, Nieves 2007]

$$a^{1/2} = 0.179(17)(14)m_\pi^{-1}$$

▷ NPLQCD: dynamical $N_f = 2 + 1$ calculation, $m_\pi = 290 - 600$ MeV [NPLQCD 2006]

$$a^{1/2} = (0.173^{+0.003}_{-0.016})m_\pi^{-1}, \quad a^{3/2} = (-0.057^{+0.003}_{-0.006})m_\pi^{-1}$$

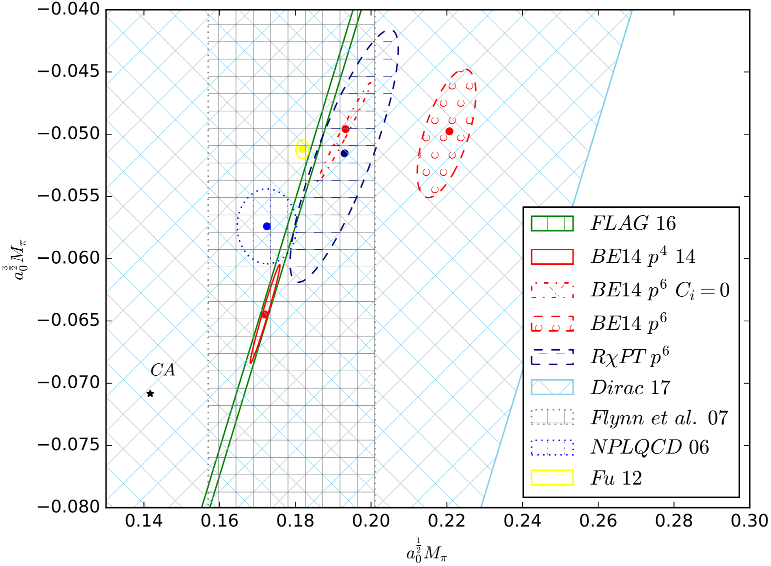
▷ Z. Fu: dynamical $N_f = 2 + 1$ fermions, $m_\pi = 330 - 466$ MeV [Fu 2012]

$$a^{1/2} = 0.182(4)m_\pi^{-1}, \quad a^{3/2} = -0.051(2)m_\pi^{-1}$$

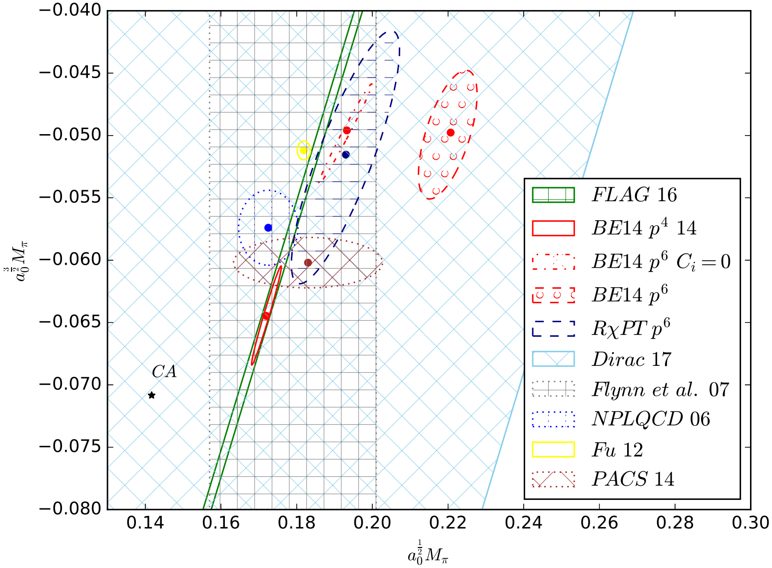
▷ PACs: dynamical $N_f = 2 + 1$ fermions, $m_\pi = 170 - 710$ MeV [PACs-Cs 2014]

$$a^{1/2} = 0.142(14)(27)m_\pi^{-1}, \quad a^{3/2} = -0.047(2)(2)m_\pi^{-1}$$

Current status



Current status



Current status

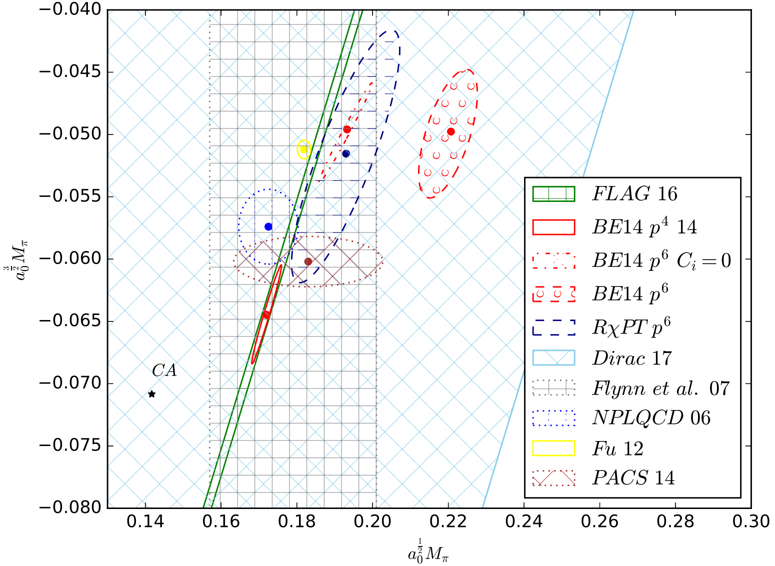
- **Dispersive determination**: Roy-Steiner equations analysis

  ▷ Most precise results up to date

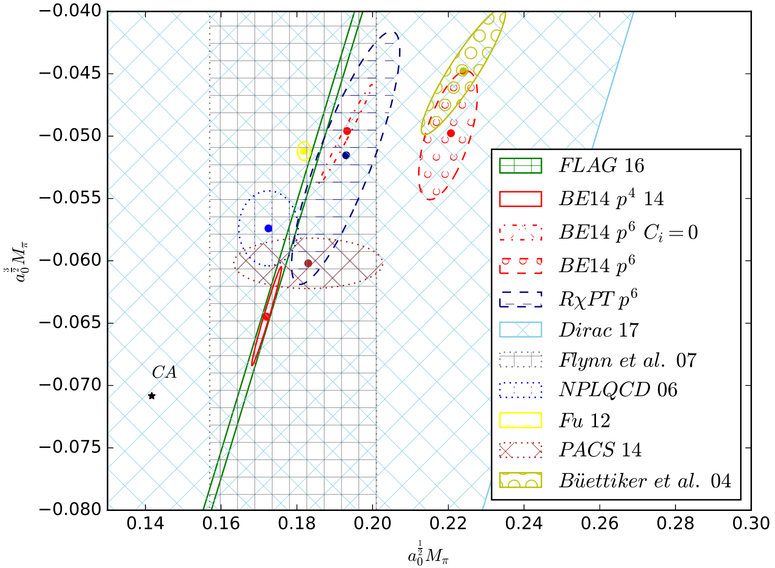
$$a^{1/2} = 0.224(22)m_\pi^{-1}, \quad a^{3/2} = -0.045(8)m_\pi^{-1}$$

[Büttiker, Descontes-Genon, Moussallam 2003]

Current status



Current status



Current status

- **Dispersive determination**: Roy-Steiner equations analysis

  ▷ Most precise results up to date

$$a^{1/2} = 0.224(22) m_\pi^{-1}, \quad a^{3/2} = -0.045(8) m_\pi^{-1}$$

[Büttiker, Descontes-Genon, Moussallam 2003]

  ↪ More than 3.5σ from $\mathcal{O}(p^4)$ ChPT results

Current status

- **Dispersive determination**: Roy-Steiner equations analysis

  ▷ Most precise results up to date

$$a^{1/2} = 0.224(22)m_\pi^{-1}, \quad a^{3/2} = -0.045(8)m_\pi^{-1}$$

[Büttiker, Descontes-Genon, Moussallam 2003]

  ↪ More than 3.5σ from $\mathcal{O}(p^4)$ ChPT results

- **This talk**: where does this discrepancy come from?

Current status

- **Dispersive determination**: Roy-Steiner equations analysis

  ▷ Most precise results up to date

$$a^{1/2} = 0.224(22) m_\pi^{-1}, \quad a^{3/2} = -0.045(8) m_\pi^{-1}$$

[Büttiker, Descontes-Genon, Moussallam 2003]

  ↪ More than 3.5σ from $\mathcal{O}(p^4)$ ChPT results

- **This talk**: where does this discrepancy come from?

  ↪ Overview of Roy-Steiner equation for πK scattering

Why Roy-Steiner equations?

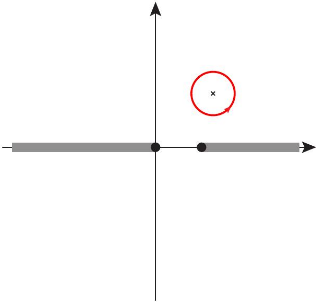
Roy(-Steiner) eqs. = Partial-Wave (Hyberbolic) Dispersion Relations coupled by unitarity and crossing symmetry

- **Respect all symmetries**: analyticity, unitarity, crossing
- **Model independent** $\Rightarrow$ the actual parameterization of the data is irrelevant once it is used in the integral.
- Framework allows for systematic improvements (subtractions, higher partial waves, ...)
- **PW(H)DRs** help to study processes with high precision:
  - $\pi\pi$-scattering: [Ananthanarayan et al. (2001), García-Martín et al. (2011)]
  - πK-scattering: [Büttiker et al. (2004)]
  - $\gamma\gamma \to \pi\pi$ scattering: [Hoferichter et al. (2011, 2018)]
  - πN scattering: [Hoferichter et al. (2015)]

From Cauchy's theorem to dispersion relations

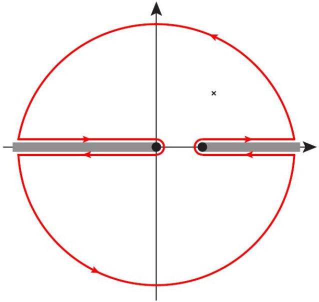
- Cauchy's Theorem

$$t(s) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{ds'\, t(s')}{s' - s}$$



- Cauchy's Theorem

$$t(s) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{ds'\, t(s')}{s'-s}$$

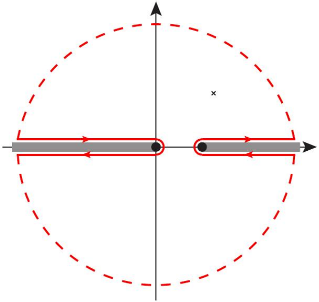


From Cauchy's theorem to dispersion relations

- Dispersion relation

$$t(s) = \frac{1}{\pi}\int_{\text{cuts}} \frac{ds'\,\mathrm{Im}\,t(s')}{s'-s}$$

$\hookrightarrow$ **analyticity**



From Cauchy's theorem to dispersion relations

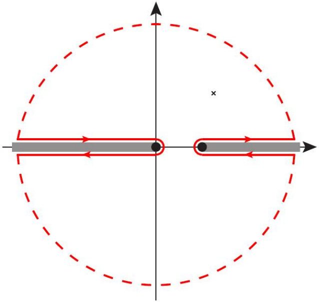
- Dispersion relation

$$t(s) = \frac{1}{\pi}\int_{\text{cuts}} \frac{ds'\,\mathrm{Im}\,t(s')}{s'-s}$$

$\hookrightarrow$ **analyticity**

- Subtractions

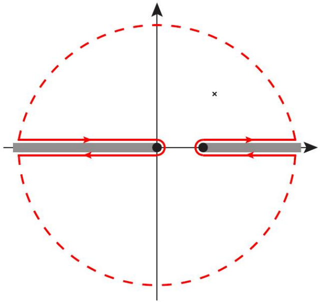
$$t(s) = t(0) + \frac{s}{\pi}\int_{\text{cuts}} \frac{ds'\,\mathrm{Im}\,t(s')}{s'(s'-s)}$$



From Cauchy's theorem to dispersion relations

- Dispersion relation

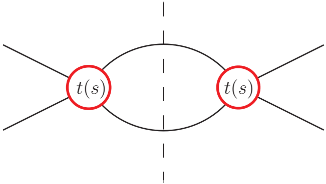
$$t(s) = \frac{1}{\pi}\int_{\text{cuts}} \frac{ds'\,\mathrm{Im}\,t(s')}{s'-s}$$



$\hookrightarrow$ **analyticity**

- Subtractions

$$t(s) = t(0) + \frac{s}{\pi}\int_{\text{cuts}} \frac{ds'\,\mathrm{Im}\,t(s')}{s'(s'-s)}$$



- Imaginary part from **Cutkosky rules**
  $\hookrightarrow$ forward direction: optical theorem

- Unitarity for partial waves

$$\mathrm{Im}\;t(s) = \sigma(s)|t(s)|^2 + \cdots\,, \quad t(s) = \frac{\eta(s)e^{2\,i\,\delta_{IJ}(s)}-1}{2i\sigma(s)}, \quad \sigma(s) = \sqrt{1-\frac{4m^2}{s}}$$

Solving Roy-equations: flow information

- **Roy-equations** rigorously valid for a finite energy range

  $\Rightarrow$ introduce a matching point s_m

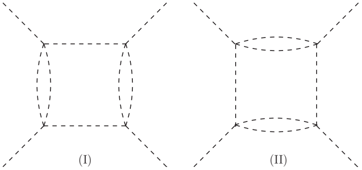
- only partial waves with $J \leq J_{\max}$ are solved
- Assume **isospin limit**
- **Input**
  - High-energy region: $\mathrm{Im}\, t_J^I(s)$ for $s \geq s_m$ and for all J
  - Higher partial waves: $\mathrm{Im}\, t_J^I(s)$ for $J > J_{\max}$ and for all s
  - Inelasticities $\eta(s)$
- **Output**
  - Self-consistent solution for $\delta_{IJ}(s)$ for $J \leq J_{\max}$ and $s_{\mathrm{th}} \leq s \leq s_m$
  - Subtraction constants: scattering lengths/subtraction parameters

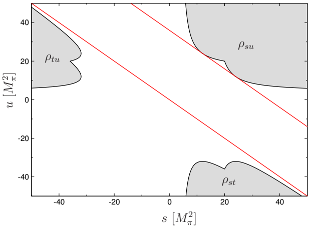
- Convergence for $T^I(s,t)$ guaranteed for $t < 4m^2$
- Where does the partial wave expansion converge?
- Assumption: Mandelstam analyticity [Mandelstam (1958,1959)]

$$T(s,t) = \frac{1}{\pi^2}\iint \mathrm{d}s'\mathrm{d}t'\,\frac{\rho_{st}(s',t')}{(s'-s)(t'-t)} + (t\leftrightarrow u) + (s\leftrightarrow u)$$

$\hookrightarrow$ integration on the support of the double spectral densities ρ

- Boundaries of ρ





- Lehmann ellipses

$\hookrightarrow$ largest ellipses, which do not enter any ρ [Lehmann (1958)]

Roy-equations: existence and uniqueness

- Solution characterized by subtraction constants and high-energy input (a, A)
- **Existence** and **uniqueness** depends on δ_i at s_m

$$m = \sum_i m_i, \qquad m_i = \begin{cases} \left\lfloor \frac{2\delta_i(s_{\rm m})}{\pi} \right\rfloor & \text{if } \delta_i(s_{\rm m}) > 0, \\ -1 & \text{if } \delta_i(s_{\rm m}) < 0, \end{cases}$$

$$\lfloor x \rfloor \Rightarrow \text{largest integer} \leq x.$$

[Gasser, Wanders 1999, Wanders 2000]

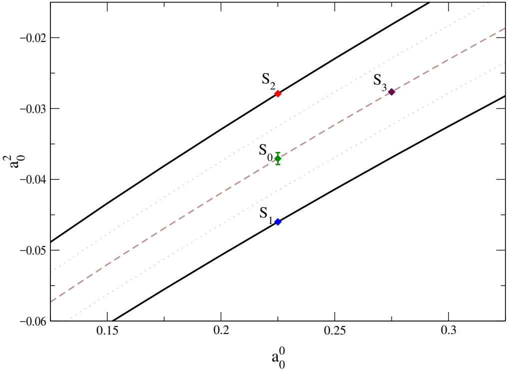
- $m = 0$, a unique solution exists for any (a, A)
- $m > 0$, m-parameter family of solutions for any (a, A)
- $m < 0$, only for a specific choice of the input constrained by $|m|$ conditions
- **Physical solution** characterized by *smooth* matching

$\hookrightarrow$ for $m = 0$, a determined by **non-cusp condition**

Universal band for $\pi\pi$ scattering

- What happens in $\pi\pi$ scattering?
  - $\triangleright$ two-subtraction constants: a_0 and a_2
  - $\triangleright$ $m = 0$, unique solution for any combination of scattering lengths
  - $\triangleright$ non-cusp for $\delta_0^0(s_m)$ condition removes cusps for $\delta_1^1(s_m)$ and $\delta_0^2(s_m)$
  - $\hookrightarrow$ universal band in the a_0^0, a_0^2 plane

[Ananthanarayan et al. (2001)]



Solving Roy-Steiner equations for πK: t-channel

- General solution in terms of Omnes function [Muskhelishvili 1953, Omnes 1958]

$$\Omega(t) = \exp\left(\frac{t}{\pi}\int_{t_\pi}^{tm} \frac{\delta(t')\mathrm{d}t'}{t'(t'-t)}\right), \text{ with } \quad \Omega(t \sim t_m) \sim |t' - t_m|^{\frac{\delta(t_m)}{\pi}}$$

Solving Roy-Steiner equations for πK: t-channel

- General solution in terms of Omnes function [Muskhelishvili 1953, Omnes 1958]

$$\Omega(t) = \exp\left(\frac{t}{\pi}\int_{t_\pi}^{tm}\frac{\delta(t')\mathrm{d}t'}{t'(t'-t)}\right), \text{ with } \quad \Omega(t \sim t_m) \sim |t' - t_m|^{\frac{\delta(t_m)}{\pi}}$$

- $\delta_{IJ}(t_m) > \pi \Rightarrow$ one free parameter α_{IJ}

Solving Roy-Steiner equations for πK: t-channel

- General solution in terms of Omnes function [Muskhelishvili 1953, Omnes 1958]

$$\Omega(t) = \exp\left(\frac{t}{\pi}\int_{t_\pi}^{tm} \frac{\delta(t')dt'}{t'(t'-t)}\right), \text{ with } \quad \Omega(t \sim t_m) \sim |t' - t_m|^{\frac{\delta(t_m)}{\pi}}$$

- $\delta_{IJ}(t_m) > \pi \Rightarrow$ one free parameter α_{IJ}

$$g_{IJ}(t) = \Delta_{IJ}(t) + \frac{t^n \Omega_{IJ}(t)}{t_m - t}\left(\alpha_{IJ} + \frac{t}{\pi}\int_{t_\pi}^{t_m} \frac{dt'}{t'^{\,n+1}} \frac{\Delta_{IJ}(t')\sin\delta_{IJ}(t')}{t'-t} \frac{tm-t}{|\Omega(t')|} + \frac{t}{\pi}\int_{t_m}^{\infty} \frac{dt'}{t'^{\,n+1}} \frac{\operatorname{Im} g_{IJ}(t')}{t'-t} \frac{tm-t}{|\Omega(t')|}\right)$$

$\Delta_{IJ}(t) \equiv$ subtraction constants, s-channel pw, J>1 t-channel pw

Solving Roy-Steiner equations for πK: t-channel

- General solution in terms of Omnes function [Muskhelishvili 1953, Omnes 1958]

$$\Omega(t) = \exp\left(\frac{t}{\pi}\int_{t_\pi}^{tm}\frac{\delta(t')\mathrm{d}t'}{t'(t'-t)}\right), \text{ with } \quad \Omega(t\sim t_m)\sim |t'-t_m|^{\frac{\delta(t_m)}{\pi}}$$

- $\delta_{IJ}(t_m) > \pi \Rightarrow$ one free parameter α_{IJ}

$$g_{IJ}(t) = \Delta_{IJ}(t) + \frac{t^n\Omega_{IJ}(t)}{t_m-t}\left(\alpha_{IJ} + \frac{t}{\pi}\int_{t_\pi}^{t_m}\frac{\mathrm{d}t'}{t'^{\,n+1}}\frac{\Delta_{IJ}(t')\sin\delta_{IJ}(t')}{t'-t}\frac{tm-t}{|\Omega(t')|} + \frac{t}{\pi}\int_{t_m}^{\infty}\frac{\mathrm{d}t'}{t'^{\,n+1}}\frac{\mathrm{Im}\,g_{IJ}(t')}{t'-t}\frac{tm-t}{|\Omega(t')|}\right)$$

$\Delta_{IJ}(t) \equiv$ subtraction constants, s-channel pw, J>1 t-channel pw

- $g_{IJ}(t_m)$ is continuous but $g_{IJ'}(t_m)$ has a cusp (divergent)

$\hookrightarrow$ α_{IJ} analytically determined from a non-cusp condition

Solving Roy-Steiner equations for πK: t-channel

- General solution in terms of Omnes function [Muskhelishvili 1953, Omnes 1958]

$$\Omega(t) = \exp\left(\frac{t}{\pi}\int_{t_\pi}^{tm}\frac{\delta(t')\mathrm{d}t'}{t'(t'-t)}\right), \text{ with } \quad \Omega(t\sim t_m)\sim |t'-t_m|^{\frac{\delta(t_m)}{\pi}}$$

- $\delta_{IJ}(t_m) > \pi \Rightarrow$ one free parameter α_{IJ}

$$g_{IJ}(t) = \Delta_{IJ}(t) + \frac{t^n\Omega_{IJ}(t)}{t_m-t}\left(\alpha_{IJ} + \frac{t}{\pi}\int_{t_\pi}^{t_m}\frac{\mathrm{d}t'}{t'^{\,n+1}}\frac{\Delta_{IJ}(t')\sin\delta_{IJ}(t')}{t'-t}\frac{tm-t}{|\Omega(t')|} + \frac{t}{\pi}\int_{t_m}^{\infty}\frac{\mathrm{d}t'}{t'^{\,n+1}}\frac{\mathrm{Im}\, g_{IJ}(t')}{t'-t}\frac{tm-t}{|\Omega(t')|}\right)$$

$\Delta_{IJ}(t) \equiv$ subtraction constants, s-channel pw, J>1 t-channel pw

- $g_{IJ}(t_m)$ is continuous but $g_{IJ'}(t_m)$ has a cusp (divergent)

$\hookrightarrow$ α_{IJ} analytically determined from a non-cusp condition

- $\Omega(t')^{-1}$ obeys a dispersion relation $\Rightarrow$ $\Omega_{IJ}^{-1}(t) = P_n(t) - \frac{1}{\pi}\int_{t_\pi}^{t_m}\frac{\mathrm{d}t'}{t'^{\,n}}\frac{\sin\delta_{IJ}(t')}{|\Omega_{IJ}(t')|(t'-t)}$

$\hookrightarrow$ $\Delta_{IJ}(t)|_{ST}$ polynomial in t $\Rightarrow$ $g_{IJ}(t_m)|_{ST}$ can be computed analytically

$\hookrightarrow$ $\Delta_{IJ}(t)|_{\mathrm{s-channel}}$ expanded as polynomial using splines

Solving Roy-Steiner equations for πK: t-channel

- General solution in terms of Omnes function [Muskhelishvili 1953, Omnes 1958]

$$\Omega(t) = \exp\left(\frac{t}{\pi}\int_{t_\pi}^{tm}\frac{\delta(t')dt'}{t'(t'-t)}\right), \text{ with } \quad \Omega(t\sim t_m) \sim |t'-t_m|^{\frac{\delta(t_m)}{\pi}}$$

- $\delta_{IJ}(t_m) > \pi \Rightarrow$ one free parameter α_{IJ}

$$g_{IJ}(t) = \Delta_{IJ}(t) + \frac{t^n\Omega_{IJ}(t)}{t_m - t}\left(\alpha_{IJ} + \frac{t}{\pi}\int_{t_\pi}^{t_m}\frac{dt'}{t'^{\,n+1}}\frac{\Delta_{IJ}(t')\sin\delta_{IJ}(t')}{t'-t}\frac{tm-t}{|\Omega(t')|} + \frac{t}{\pi}\int_{t_m}^{\infty}\frac{dt'}{t'^{\,n+1}}\frac{\mathrm{Im}\, g_{IJ}(t')}{t'-t}\frac{tm-t}{|\Omega(t')|}\right)$$

  $\Delta_{IJ}(t) \equiv$ subtraction constants, s-channel pw, J>1 t-channel pw

- $g_{IJ}(t_m)$ is continuous but $g_{IJ}{}'(t_m)$ has a cusp (divergent)

  $\hookrightarrow \alpha_{IJ}$ analytically determined from a non-cusp condition

- $\Omega(t')^{-1}$ obeys a dispersion relation $\Rightarrow \Omega_{IJ}^{-1}(t) = P_n(t) - \frac{1}{\pi}\int_{t_\pi}^{t_m}\frac{dt'}{t'^{\,n}}\frac{\sin\delta_{IJ}(t')}{|\Omega_{IJ}(t')|(t'-t)}$

  $\hookrightarrow \Delta_{IJ}(t)|_{ST}$ polynomial in $t \Rightarrow g_{IJ}(t_m)|_{ST}$ can be computed analytically

  $\hookrightarrow \Delta_{IJ}(t)|_{\mathrm{s-channel}}$ expanded as polynomial using splines

Exact solution of the MO problem while solving the s-channel RS equations

Solving s-channel: strategy

- Parameterize S and P waves up to $s < s_m$

  $\triangleright$ Imposing a continuous and differentiable matching point

- Introduce as many subtractions as necessary to match d.o.f [Gasser, Wanders 1999]

  $\triangleright$ $m = 0$, three non-cusp conditions: $S^{1/2}$, $S^{3/2}$ and $P^{1/2}$

  $\hookrightarrow$ three free parameters: c_{00}^{+}, c_{01}^{+}, c_{00}^{-}

- Minimize difference between LHS and the RHS on a grid of points W_j

$$\chi^2_{\text{phys}} = \sum_{l,I_s} \sum_{j=1}^{N} \frac{\left(\text{Re}\, f_{l\pm}^{I_s}(W_j) - F[c_{nm}^{I_s}, f_{l\pm}^{I_s}](W_j)\right)^2}{\text{Re}\, f_{l\pm}^{I_s}(W_j)^2}$$

  $\triangleright$ $F[c_{nm}^{I_s}, f_{l\pm}^{I_s}](W_j) \equiv$ right hand side of RS-equations

- Parameterization and subthreshold parameters are the fitting parameters

Roy–Steiner equations for πK: flow of information

Limited range of validity: hyperbolic dispersion relations

$$\sqrt{s} \leq \sqrt{s_m} = 0.985\,\mathrm{GeV} \qquad\qquad t \leq t_m = 2\,\mathrm{GeV}^2$$

Input/Constraints

- S- and P-waves above matching point $s > s_m$ $(t > t_m)$
- Inelasticities
- Higher waves (D-, F-, $\cdots$)
- $\pi\pi$ phase shifts below the $\bar{K}K$ threshold

Output

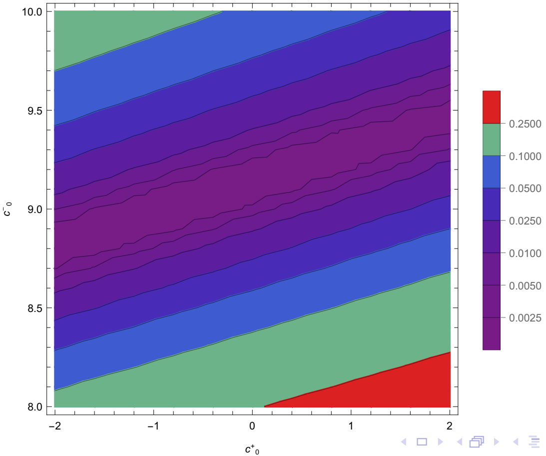
- S- and P-wave phase-shifts at low energies $s < s_m$ $(t < t_m)$
- Subtraction constants
- Subthreshold parameters: c_{00}^{+}, c_{01}^{+}, c_{00}^{-}

  $\hookrightarrow$ πK **scattering lengths**

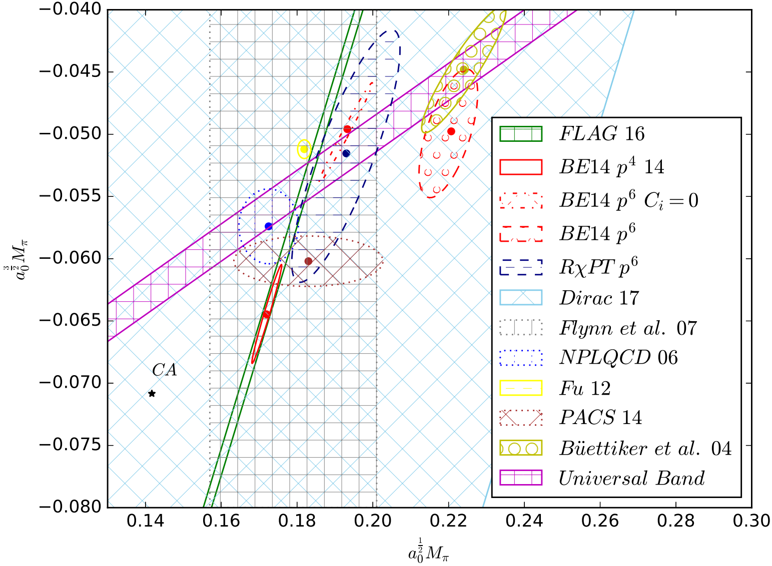
- Uniqueness: $m = 0$
  - ▷ unique solution for any subtraction constant and input
  - ▷ three subtraction constants but only two non-cusp conditions required

    $\hookrightarrow$ πK universal band solution

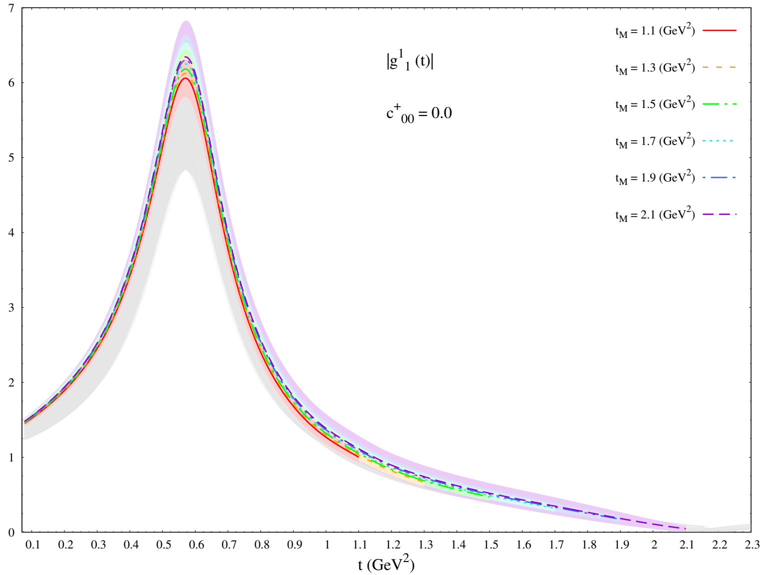
[Colangelo, Maurizio, JRE (preliminary)]



πK scattering lengths: preliminary results

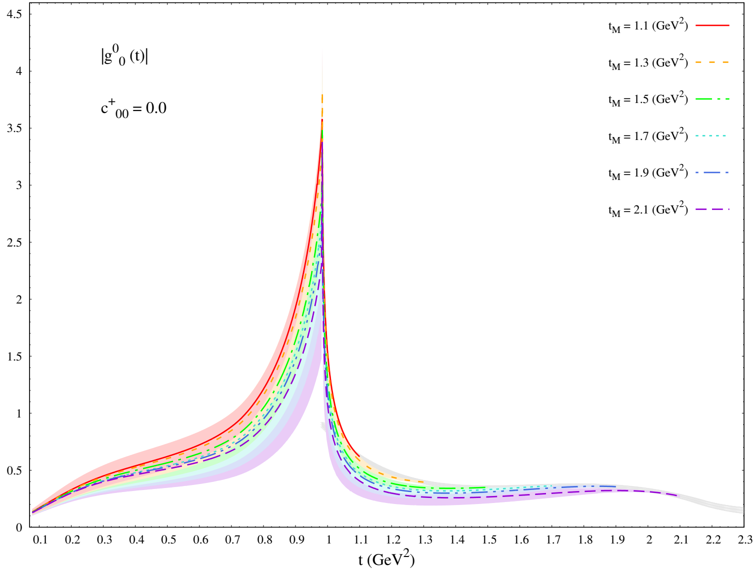


πK universal band: dependence on t_M

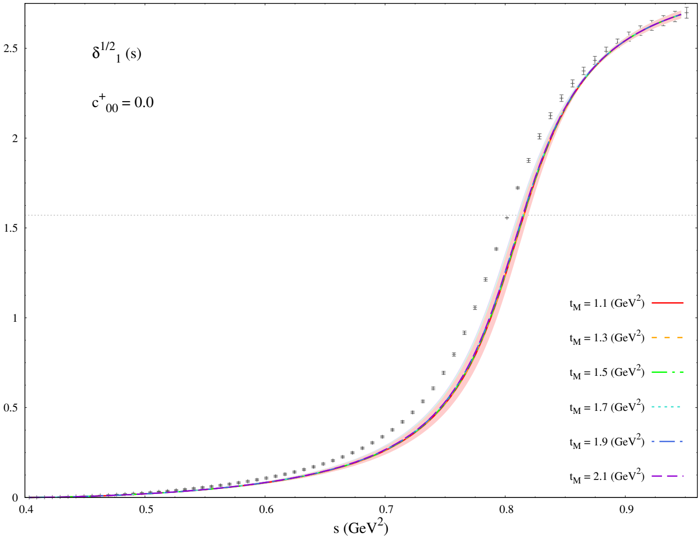


[Colangelo, Maurizio, JRE (preliminary)]

πK universal band: dependence on t_M

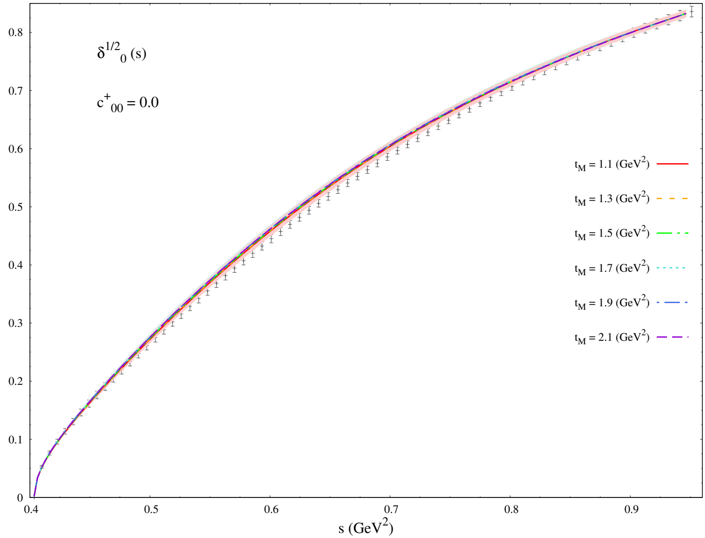


[Colangelo, Maurizio, JRE (preliminary)]



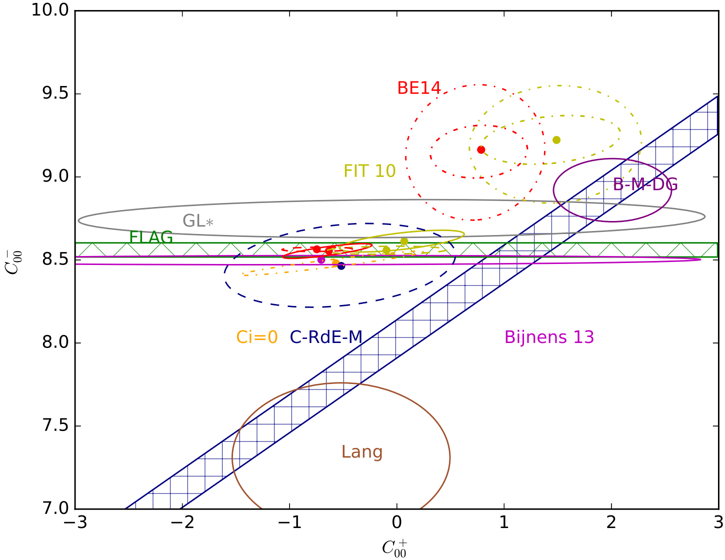
[Colangelo, Maurizio, JRE (preliminary)]

πK universal band: dependence on t_M



[Colangelo, Maurizio, JRE (preliminary)]

πK subthreshold parameter plane



πK Scattering length puzzle: BDM solution

- **Fixed-t** dispersion relation

▷ $c^I(t)$ extracted from a hyperbolic dispersion relation at threshold

$$s \cdot u = m_K^2 - m_\pi^2$$

- Twice/once subtracted version for $I = \{+, -\}$

$$c^+(t) = 8\pi(m_\pi + m_K)a^+ + b^+ t$$

$$c^-(t) = \frac{8\pi(m_\pi + m_K)}{m_\pi m_K} a^- p_\pi(t) q_K(t)$$

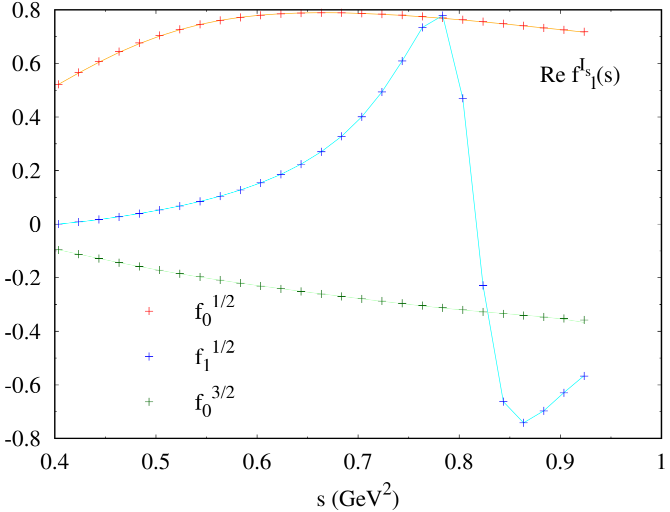
- b^+ extracted from a sum rule involving a^-

- a^+ and a^- extracted from RS self-consistent solution

$$\chi^2_{\text{phys}} = \sum_{l,I_s} \sum_{j=1}^{N} \frac{\left(\text{Re}\, f_l^{I_s}(s_j) - F[f_l^{I_s}](s_j)\right)^2}{\text{Re}\, f_l^{I_s}(s_j)^2}$$

▷ $F[f_l^{I_s}](s_j) \equiv$ right hand side of RS-equations

Roy-Steiner solution for πK



Looking for a unique solution

- **Unique solution**: as many subtractions as necessary to match d.o.f [Gasser, Wanders 1999]

  ↪ two non-cusp conditions ⇒ two free scattering lengths [Büttiker, Descontes-Genon, Moussallam 2003]

- Are the two scattering lengths independent?

$$\frac{8\pi(m_\pi + m_K)a^-_{SR}}{m_\pi m_K} = \frac{1}{\pi}\int_{s_{\rm th}}^{\infty}\frac{ds'}{\lambda(s')}\,\mathrm{Im}\,F^-(s',t') + \frac{1}{\pi}\int_{t_\pi}^{\infty}\frac{dt'}{t'}\,\mathrm{Im}\,\frac{G^1(t',s')}{8p_\pi(t')p_K(t')}$$

	K_s	K_t	D_s	D_t	A_s	A_t	Total
$a^-_{i,SR}$ $(m_\pi 10^2)$	0.86	**6.10**	0.60	0.13	0.00	0.37	**8.05**
a^- $(m_\pi 10^2)$	–	–	–	–	–	–	8.96

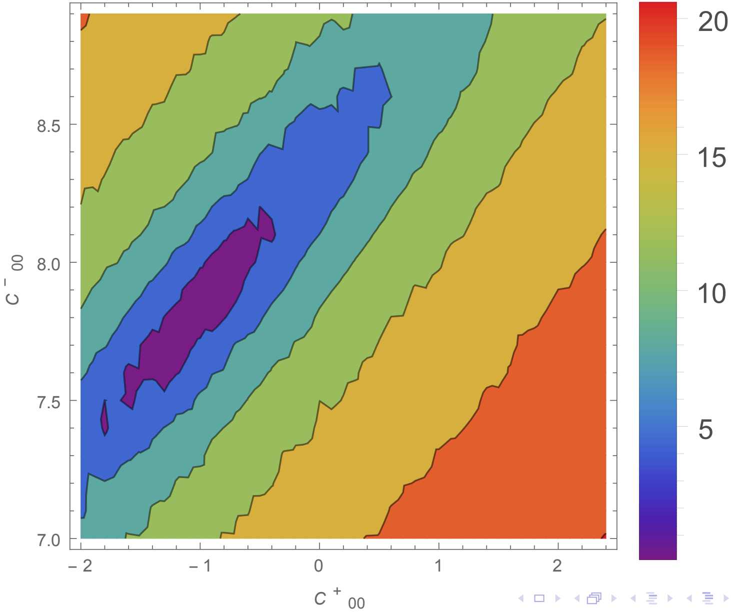
↪ once a^+ is fixed, a^- is obtained from a convergent sum rule

- New approach: **HDR** subtracted at **subthreshold**

  ▷ three non-cusp conditions ⇒ three subthreshold parameters **constrained** by SR

$$\chi^2_{\rm tot} = \chi^2_{\rm phys} + \frac{\left(c^-_{00} - c^-_{00,SR}\right)^2}{\Delta {c^-_{00}}^2} + \frac{\left({c_{10}}^+ - c^+_{10,SR}\right)^2}{\Delta {c^+_{10}}^2}$$

Unique solution for πK scattering



πK scattering without $\chi^2_{c^-_{00}}$