# Pion and kaon GPDs

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FÍSICA MATEMÁTICAS COMPUTACIÓN

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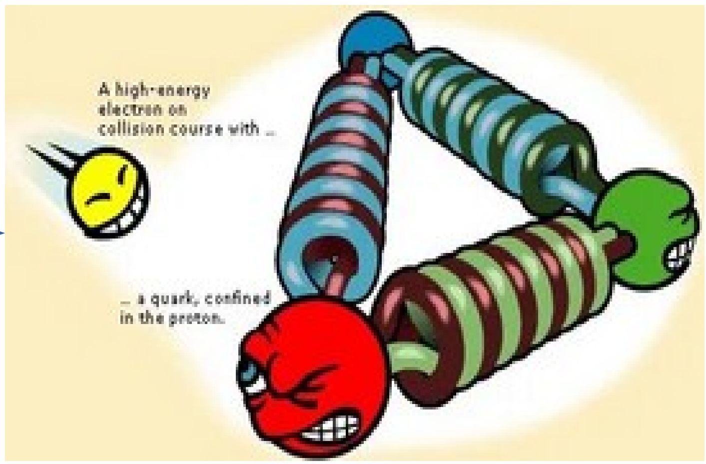
# QCD and hadron physics

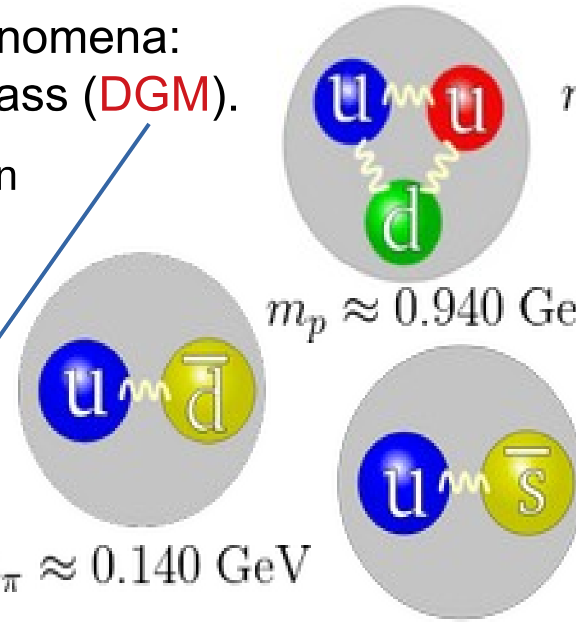
- **QCD** is characterized by two **emergent** phenomena: confinement and dynamical generation of mass (DGM).

Gluons and quarks have never been seen isolated in nature; only colorless bound states (hadrons) have.




**'Higgs' masses**

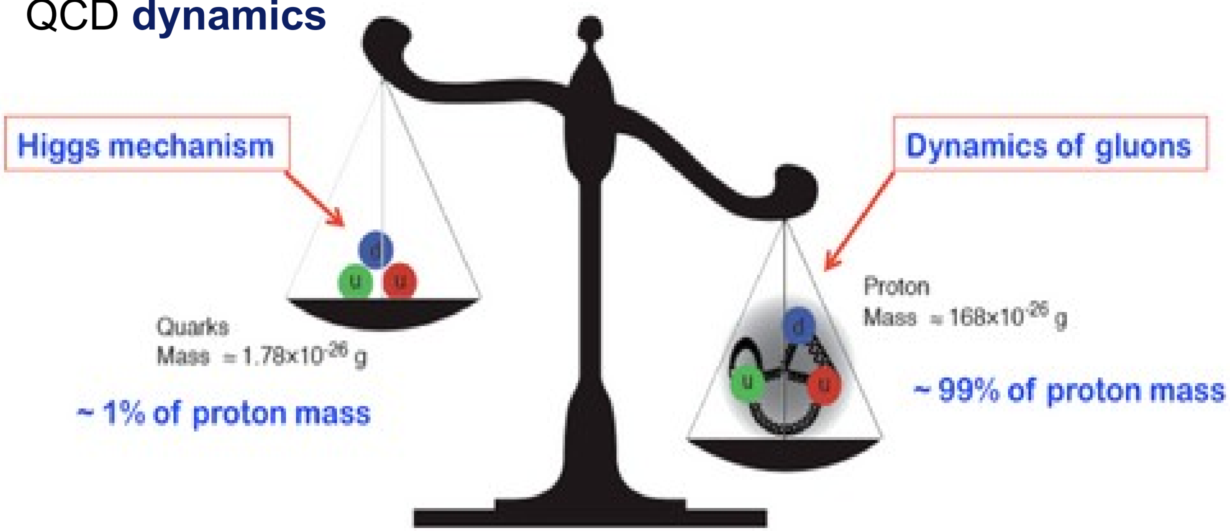
$$m_{u/d} \approx 0.004 \text{ GeV}$$
$$m_s \approx 0.095 \text{ GeV}$$



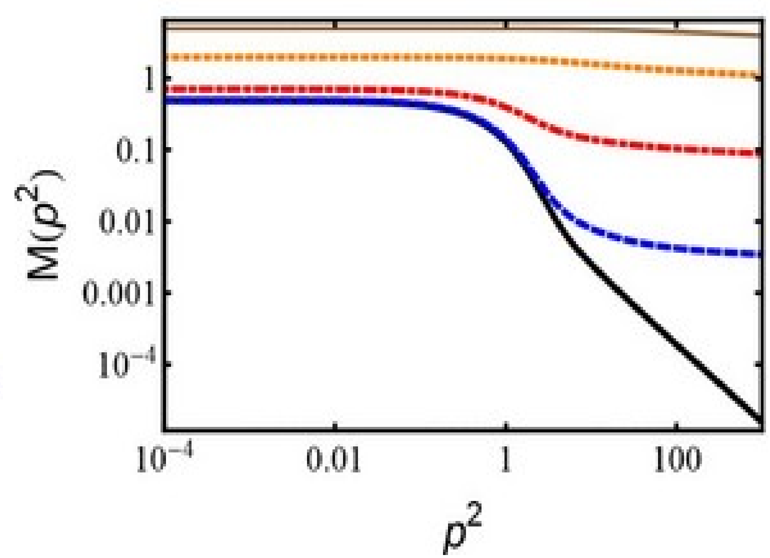

$$m_p \approx 0.940 \text{ GeV}$$
$$m_\pi \approx 0.140 \text{ GeV}$$
$$m_K \approx 0.490 \text{ GeV}$$

**Pions** and **Kaons** emerge as QCD's (pseudo)-**Goldstone** bosons.

Emergence of hadron masses (EHM) from QCD **dynamics**

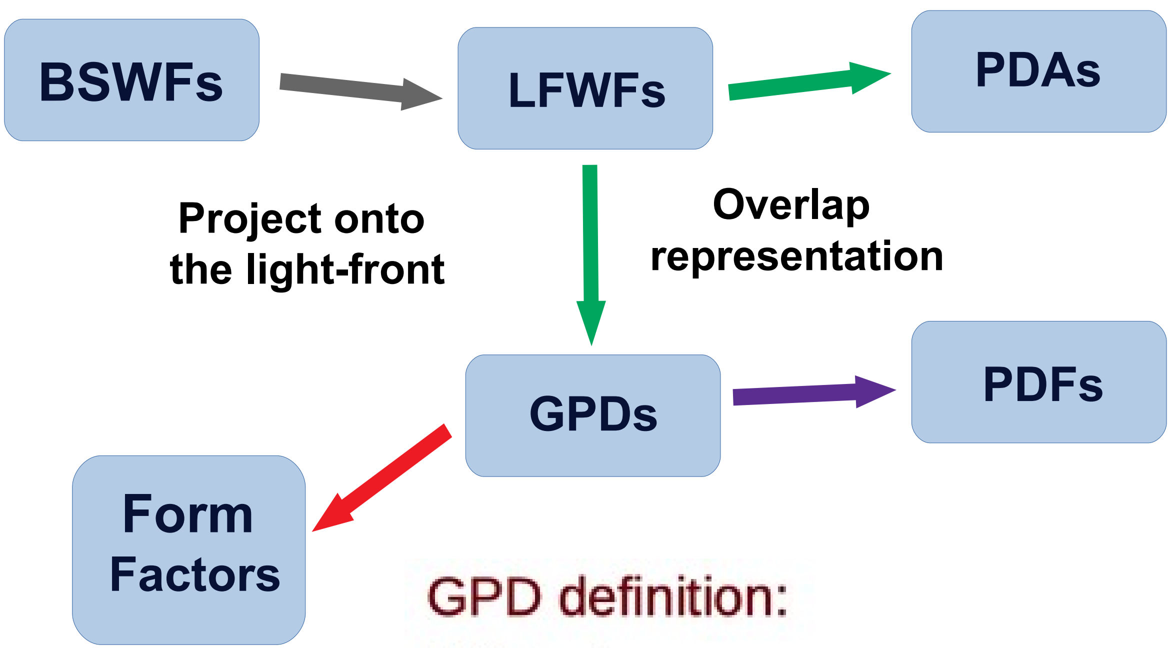



Dynamical Chiral Symmetry Breaking (**DCSB**)



# LFWFs, PDFs and PDAs from DSEs

- **Goal:** get a **broad picture** of the pion/Kaon structure.




**The approach:**
Compute ***everything*** from the **LFWF,** obtained as ***solutions*** from quark **DSE** and meson **BSE**.

- Already on the market:
  PDAs, PDFs, Form factors...
  K. Raya et al., arXiv: 1911.12941 [nucl-th]
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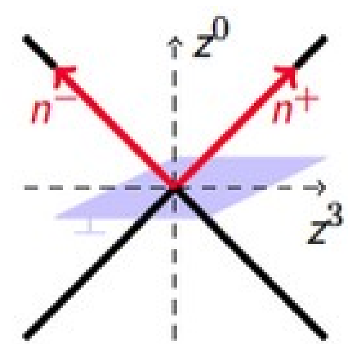
**GPD definition:**

$$H_\pi^q(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi, P+\frac{\Delta}{2}\middle|\, \bar{q}\left(-\frac{z}{2}\right)\gamma^+ q\left(\frac{z}{2}\right) \middle|\, \pi, P-\frac{\Delta}{2}\right\rangle_{\substack{z^+=0\\ z_\perp=0}}$$

with $t=\Delta^2$ and $\xi = -\Delta^+/(2P^+)$.

- green: $\int dk_\perp$
- red: $\int dx$
- purple: $t=0, \xi=0$

Muller et al., Fortchr. Phys. 42 (1994) 101
Radyushkin, Phys. Lett. B380 (1996) 417
Ji, Phys. Rev. Lett. 78 (1997) 610



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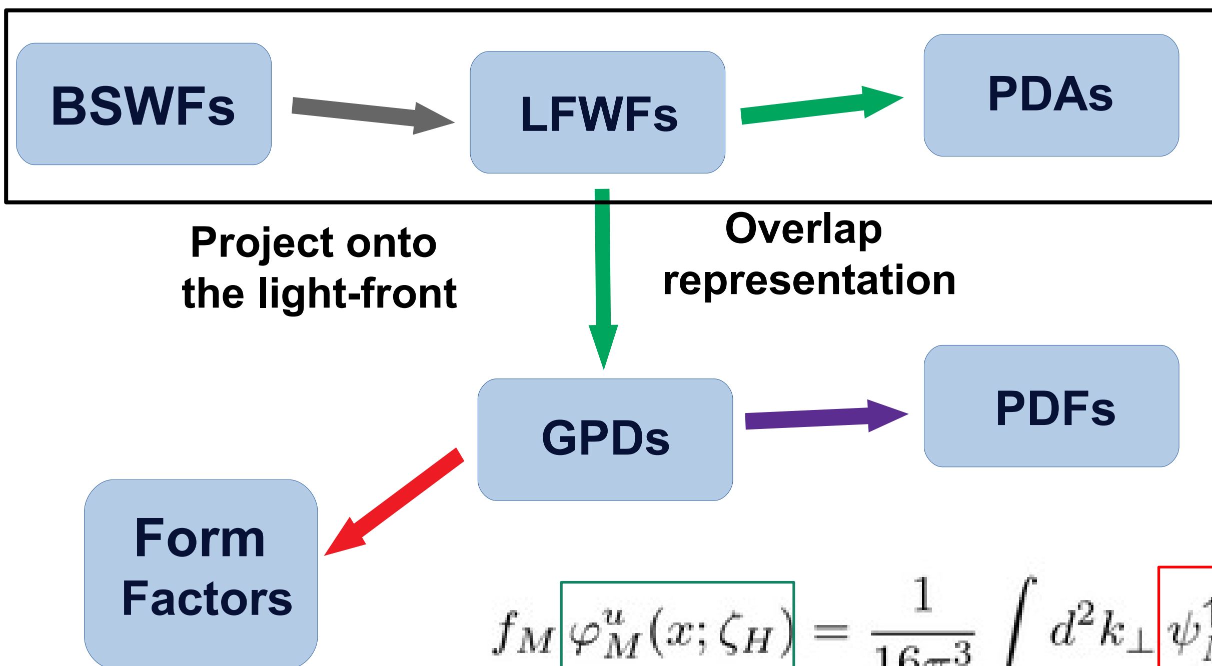


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$\zeta_H$ : hadron scale

$$f_M \boxed{\varphi_M^u(x;\zeta_H)} = \frac{1}{16\pi^3}\int d^2k_\perp \boxed{\psi_{M_u}^{\uparrow\downarrow}(x,k_\perp^2;\zeta_H)}$$
$$= N_c \mathrm{tr} Z_2(\zeta_H,\Lambda)\int_{dk}^{\Lambda} \delta_n^x(k_n)\gamma_5\gamma\cdot n \boxed{\chi_M(k_{\eta\bar{\eta}};P;\zeta_H)}$$

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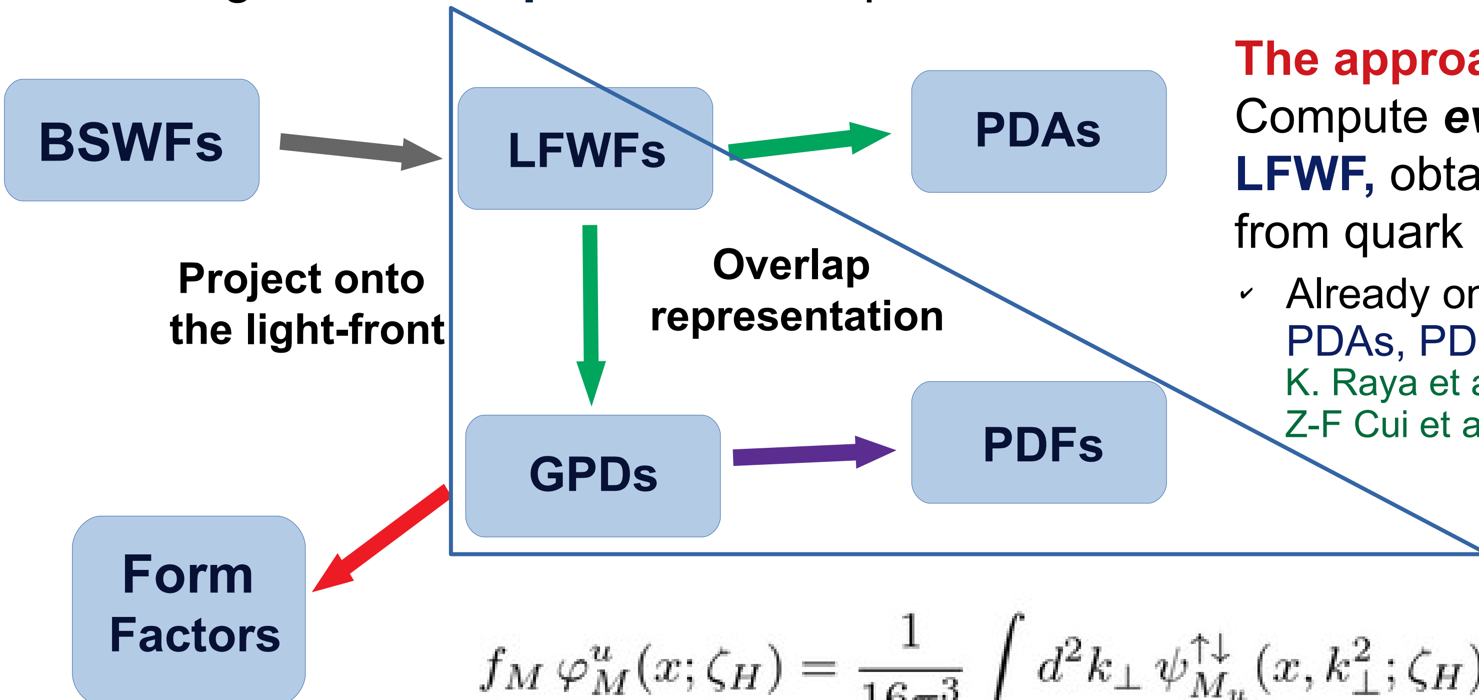


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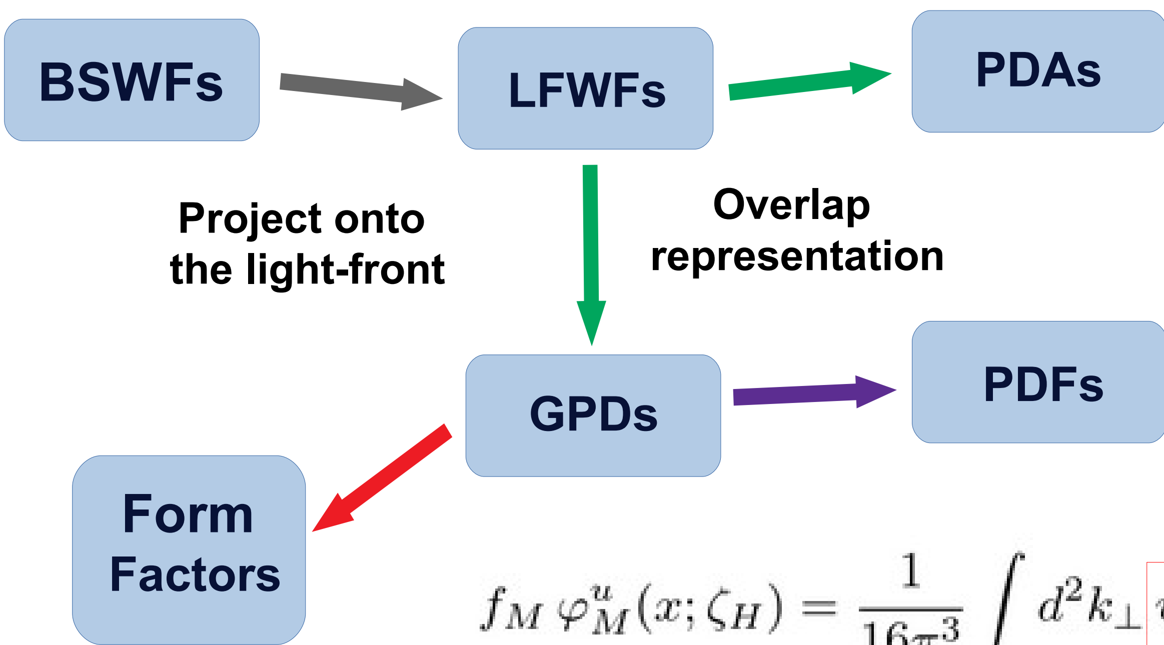


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S.-S. Xu et al,. Phys.Rev.D97094014(2018)

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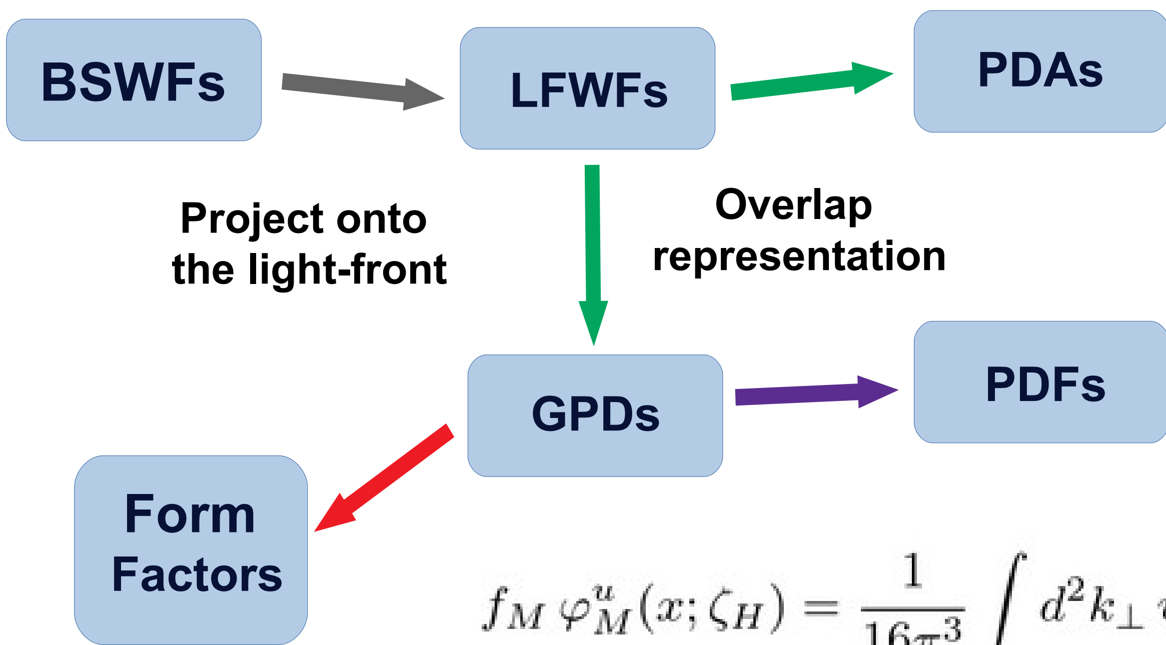


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$$u^M(x;\zeta_H) = H_M^u(x,0,0;\zeta_H) = |\varphi_M^u(x;\zeta_H)|^2\int\frac{d^2\mathbf{k}_\perp}{16\pi^3}\left|\psi_{M_u}^{\uparrow\downarrow}(\mathbf{k}_\perp^2;\zeta_\mathbf{H})\right|$$

$$\bar{h}^M(x;\zeta_H) = u^M(1-x;\zeta_H) \qquad (h=d\,,s)$$

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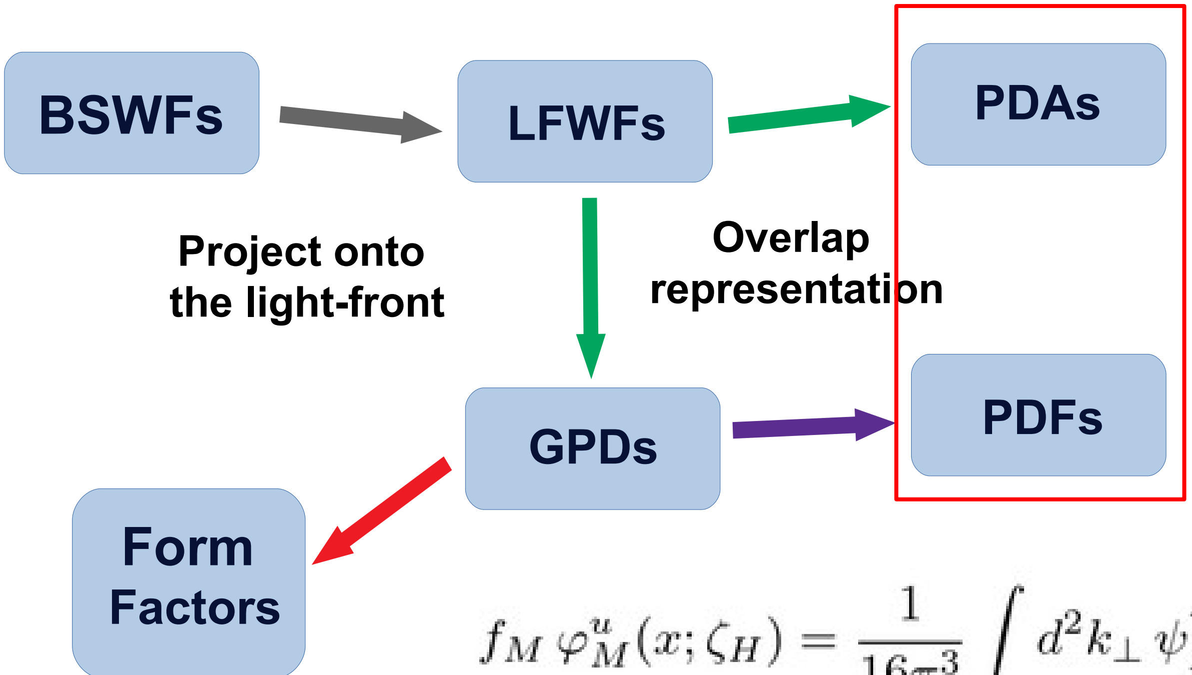


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$\int dk_\perp$

$\int dx$

$t=0, \xi=0$

Direct connection between meson PDAs and PDFs at the hadronic scale, $\zeta_H$, grounded on the *factorization approximation*, only valid for integrated quantities and not beyond $\zeta_H$ due to parton splitting effects.

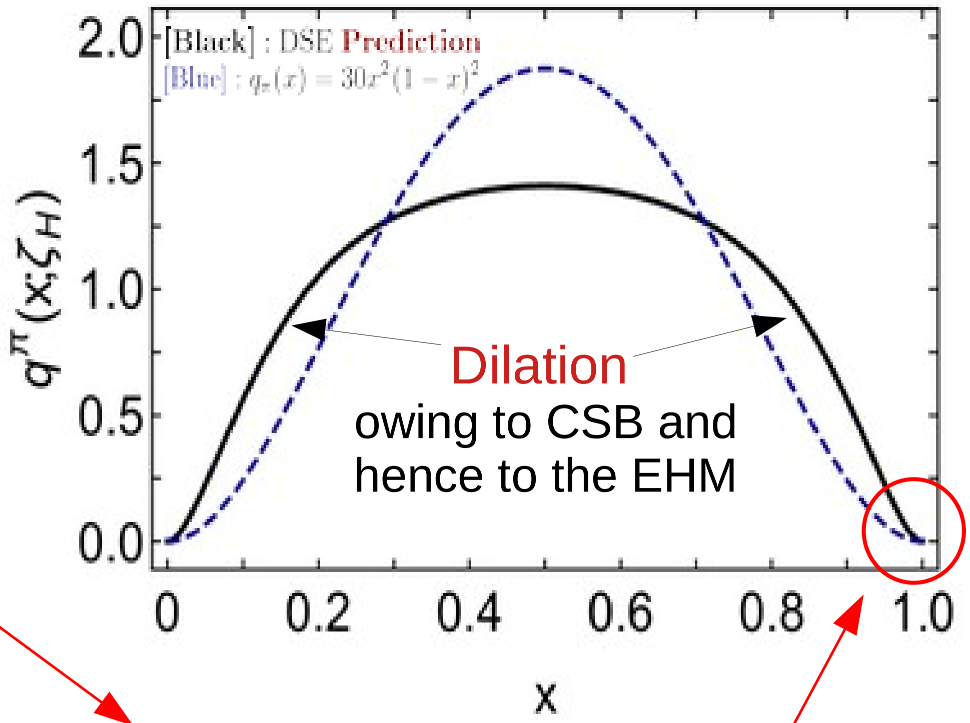
# LFWFs, PDFs and PDAs from DSEs

Symmetry-preserving DSE computation of the valence-quark PDF:

[L. Chang et al., Phys.Lett.B737(2014)23]
[M. Ding et al., Phys.Rev.D101(2020)054014

$$q^{\pi}(x;\zeta) = N_c \mathrm{tr} \int_{dk} \delta_n^x(k_\eta) \Gamma_\pi^P(k_{\bar{\eta}\eta};\zeta)\, S(k_{\bar{\eta}};\zeta) \times \{n \cdot \frac{\partial}{\partial k_\eta} [\Gamma_\pi^{-P}(k_{\eta\bar{\eta}};\zeta) S(k_\eta;\zeta)]\}$$

$$q_{\text{\O}}^{\pi}(x;\zeta_H) = 213.32\, x^2(1-x)^2 \times [1 - 2.9342\sqrt{x(1-x)} + 2.2911\, x(1-x)]$$




$$q^M(x;\zeta_H) \overset{x\simeq 1}{=} c(\zeta_H)\,(1-x)^{\beta(\zeta_H)}$$

$$\beta(\zeta_H) = 2$$

Farrar, Jackson, Phys.Rev.Lett 35(1975)1416
Berger, Brodsky, Phys.Rev.Lett 42(1979)940

- The EHM-triggered broadening shortens the extent of the domain of convexity lying on the neighborhood of the endpoints, induced too by the QCD dynamics
- It cannot however spoil the asymptotic QCD behaviour at large-x (and, owing to isospin symmetry, at low-x)

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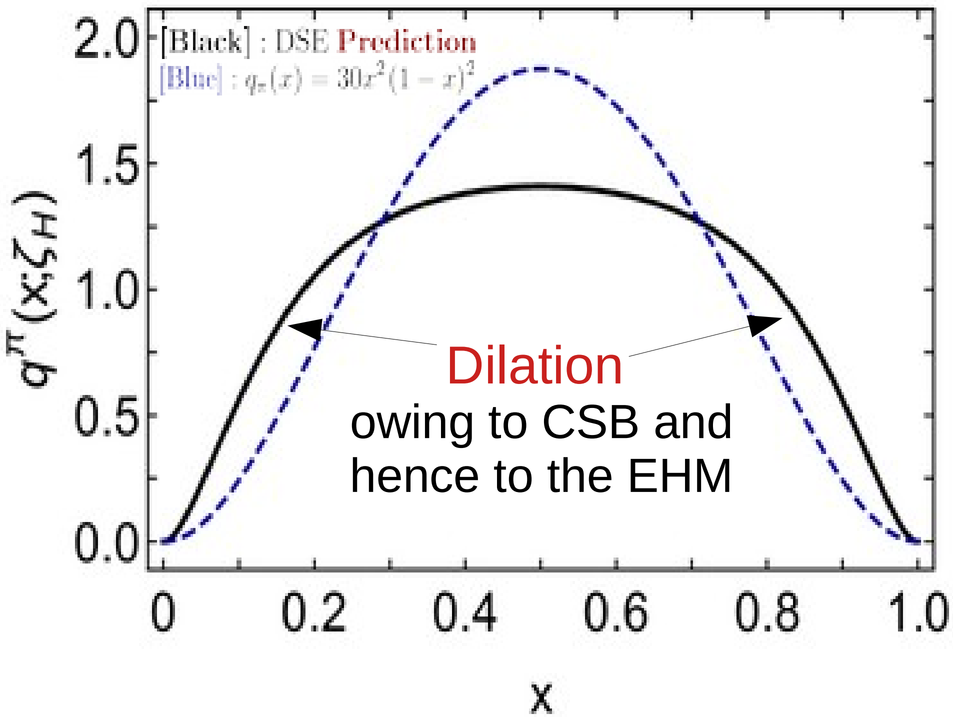


PDA computation using the BSA obtained with the DB kernel:

[L. Chang et al., Phys.Rev.Lett.110(2013)132001]

$$f_M\, \varphi_M^u(x;\zeta_H) = N_c \mathrm{tr} Z_2(\zeta_H,\Lambda) \int_{dk}^{\Lambda} \delta_n^x(k_\eta) \gamma_5 \gamma \cdot n \chi_M(k_{\eta\bar\eta};P;\zeta_H)$$

$$\varphi_\pi^{\mathrm{DB}}(x;\zeta_H) = 20.227\, x(1-x) \times \left[1 - 2.5088\sqrt{x(1-x)} + 2.0250\, x(1-x)\right]$$

Both computations scale-independent, albeit describing properly the hadronic degrees of freedom only at the hadronic scale

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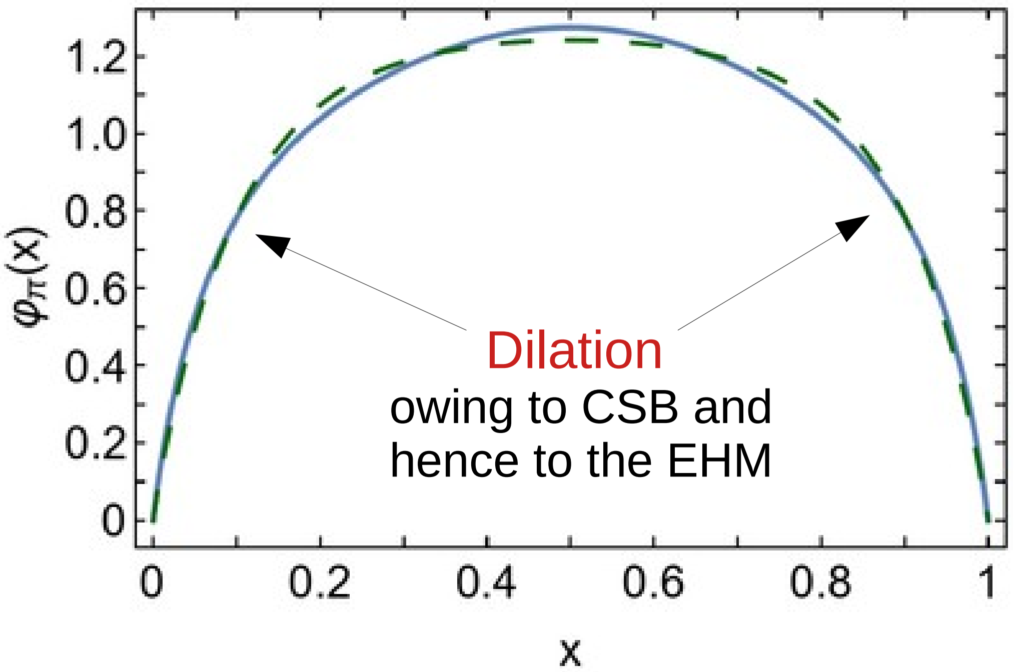


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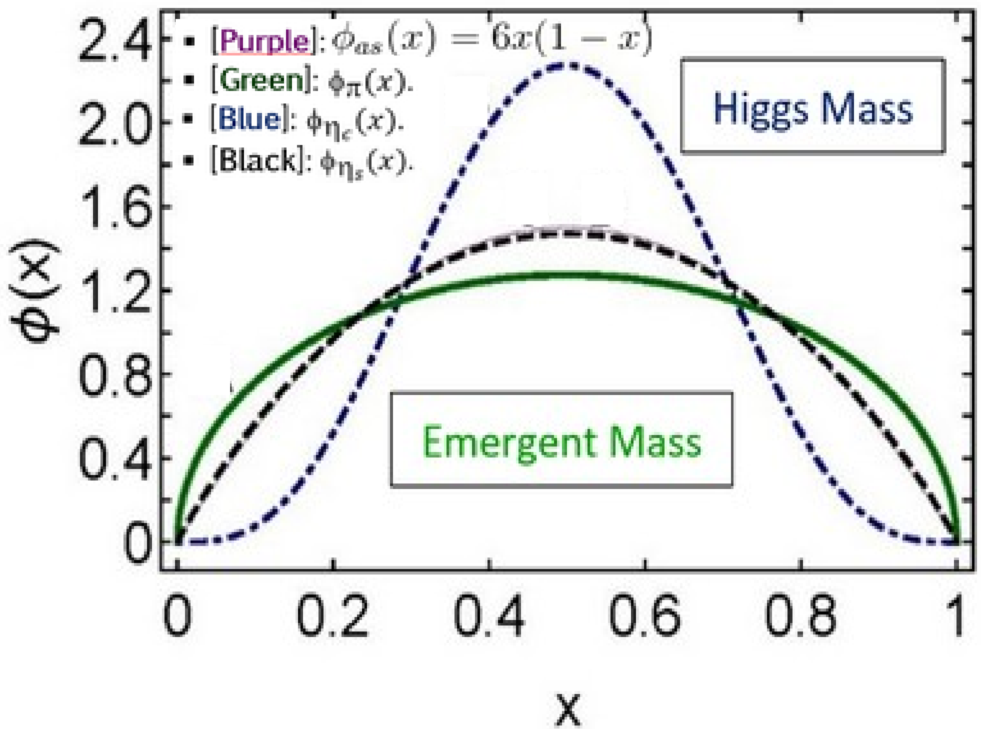
A. Aguilar et al. Eur.Phys.J. A55 (2019) no.10, 190




- Dominance of QCD dynamics (EHM) expressed by **broad and concave** PDAs (light sector)
- Dominance of Higgs mass generation (explicit CSB) reflected by **narrow** PDAs (heavy sector)
- **s-quark mass** lies on the boundary where both effects appear **balanced**

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# Off-forward extension of PDFs

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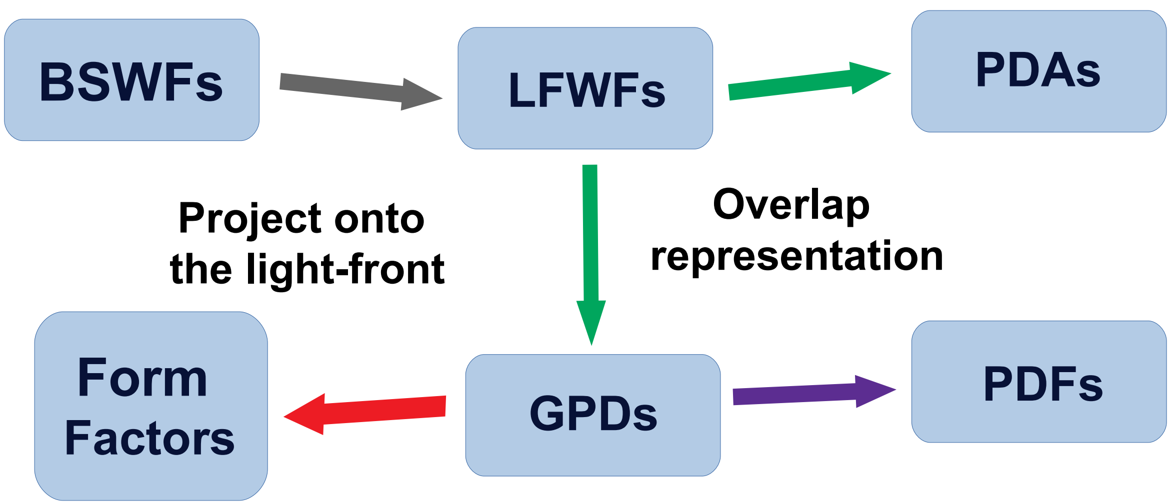


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$$x-\xi \geq 0\,;\,\xi \geq 0$$

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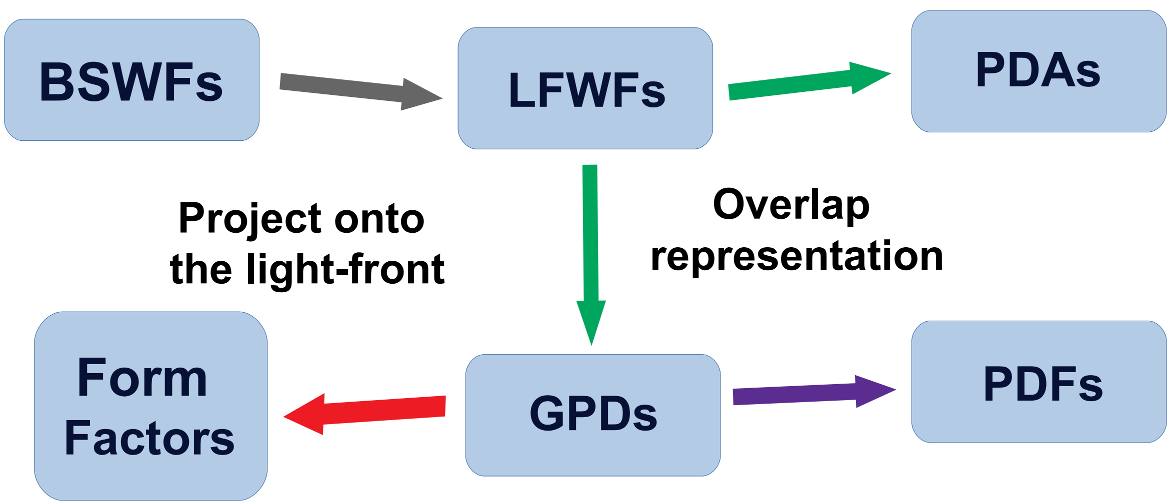


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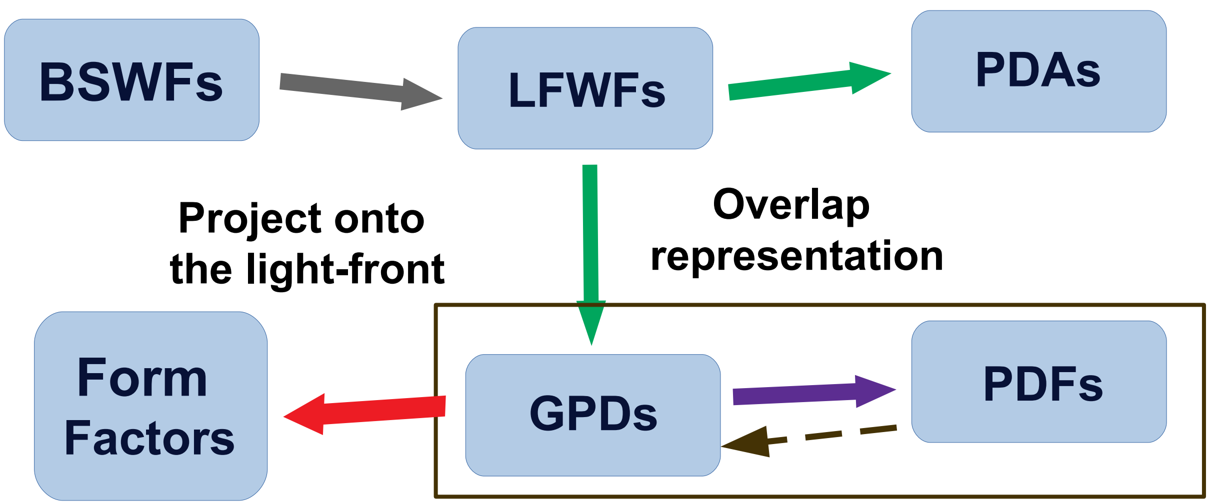


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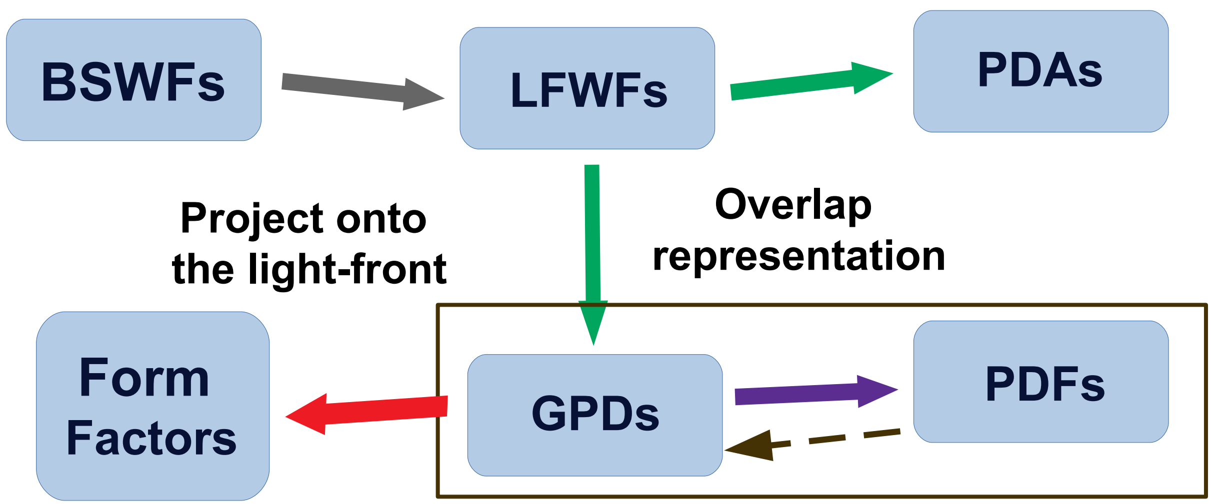


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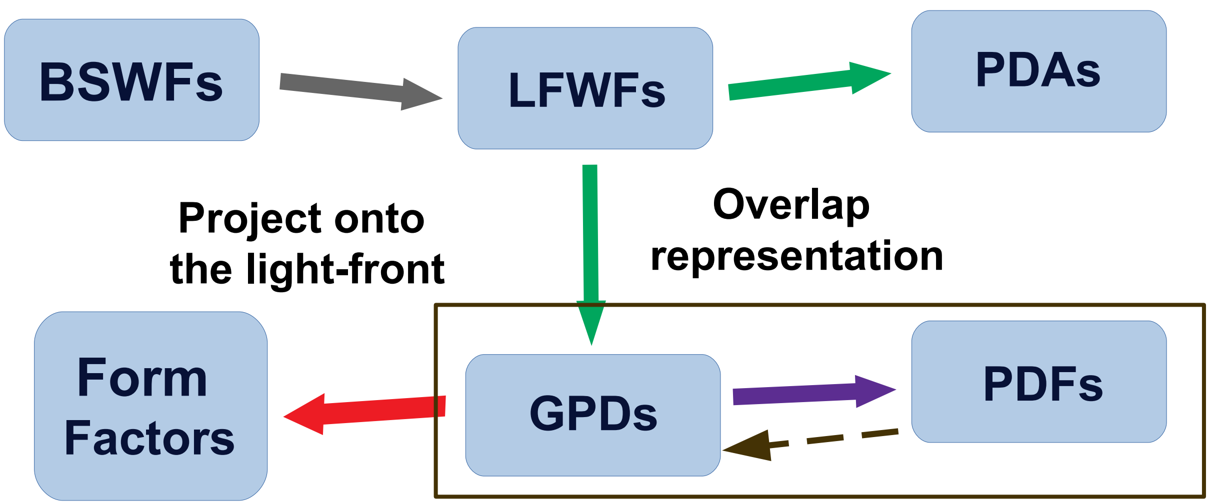
# Off-forward extension of PDFs

- **Goal:** get a **broad picture** of the pion/Kaon structure.

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Compute ***everything*** from the **LFWF,** obtained as ***solutions*** from quark **DSE** and meson **BSE**.

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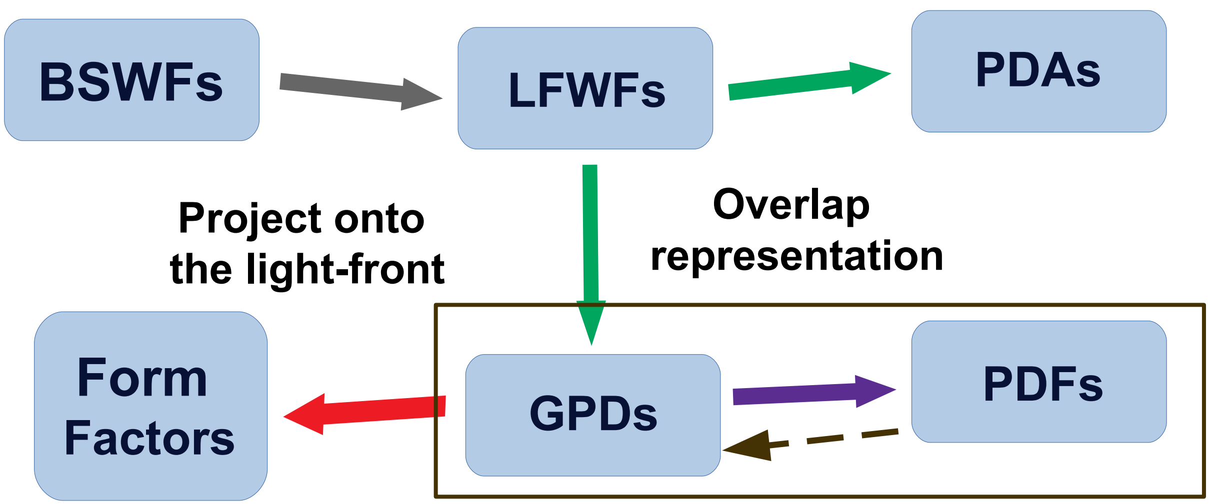


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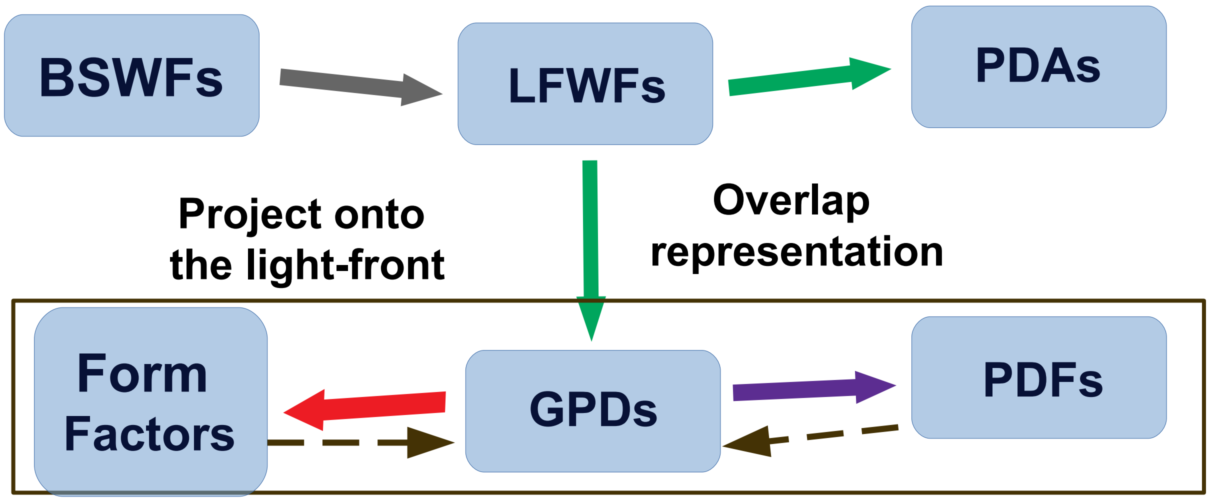


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Realistic (DSE) PDF
\+
Realistic (DSE) FF

→ Realistic GPD prediction

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- The impact-parameter GPD

$$u^M(x,b_\perp^2;\zeta_H) = \int_0^\infty \frac{d\Delta_\perp}{2\pi}\Delta_\perp J_0(b_\perp\Delta_\perp)\; H_M^u(x,\xi,t;\zeta_H)|_{\xi=0}$$

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Pion's case

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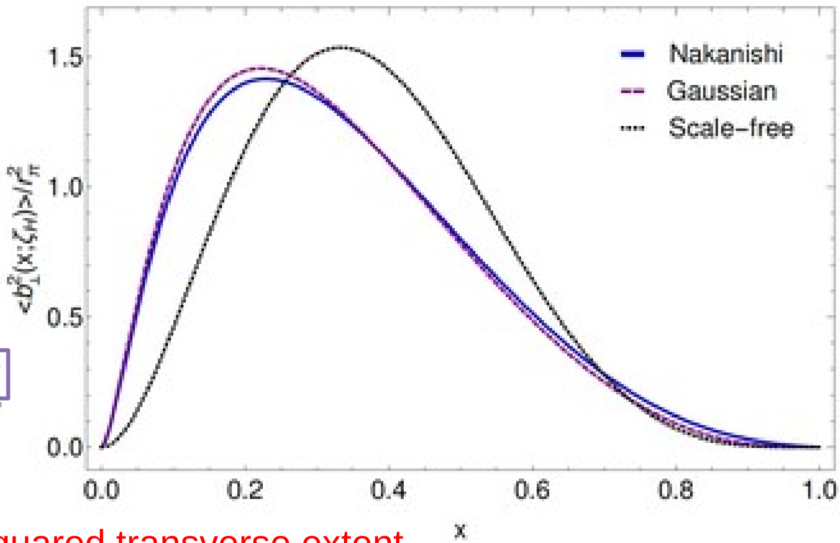



Compact expressions in terms of the PDF and $\Phi_M^u$

Mean-squared transverse extent

# Beyond factorization: Nakanishi representation

## Modeling the LFWF:

S-S Xu et al., PRD 97 (2018) no.9, 094014.

- Considering the Kaon as a example, we employ a Nakanishi-like representation:

$$n_K \chi_K^{(2)}(k_-^K; P_K) = \underbrace{\mathcal{M}(k; P_K)}_{1} \int_{-1}^{1} d\omega \, \underbrace{\rho_K(\omega)}_{2} \underbrace{\mathcal{D}(k; P_K)}_{3} \,,$$

**1: Matrix structure (leading BSA):**

$$\mathcal{M}(k; P_K) = -\gamma_5[\gamma \cdot P_K M_u + \gamma \cdot k(M_u - M_s) + \sigma_{\mu\nu} k_\mu P_{K\nu}] \,,$$

**2: Sprectral weight:** To be described later.

**3: Denominators:** $\mathcal{D}(k; P_K) = \Delta(k^2, M_u^2)\Delta((k - P_K)^2, M_s^2)\hat{\Delta}(k_{\omega-1}^2, \Lambda_K^2) \,,$

$$\text{where: } \Delta(s,t) = [s+t]^{-1}, \; \hat{\Delta}(s,t) = t\Delta(s,t) \,.$$

- **Algebraic** manipulation yields:

$$\chi_K^{(2)}(k_-^K; P_K) = \mathcal{M}(k; P_K) \int_0^1 d\alpha \, 2\boxed{\chi_K(\alpha; \sigma^3(\alpha))} \,, \; \sigma = (k - \alpha P_K)^2 + \Omega_K^2 \,,$$

- **$\rho_K(\omega)$** will play a **crucial role** in determining the meson's observables.
- Realistic **DSE predictions** will help us to shape it.

Scalar function:

$$\boxed{\chi_K(\alpha; \sigma^3) = \left[\int_{-1}^{1-2\alpha} d\omega \int_{1+\frac{2\alpha}{\omega-1}}^{1} dv + \int_{1-2\alpha}^{1} d\omega \int_{\frac{\omega-1+2\alpha}{\omega+1}}^{1} dv\right] \frac{\rho_K(\omega)}{n_K} \frac{\Lambda_K^2}{\sigma^3}}$$

# Beyond factorization: Nakanishi representation

## Modeling the LFWF:

S-S Xu et al., PRD 97 (2018) no.9, 094014.

- The **pseudoscalar LFWF** can be written:

$$f_K \psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \mathrm{tr}_{CD}\int_{dk_\parallel} \delta(n\cdot k - x n\cdot P_K)\gamma_5\gamma\cdot n\chi_K^{(2)}(k_-^K;P_K)\,.$$

- The **moments** of the distribution:

$$< x^m >_{\psi_K^{\uparrow\downarrow}} = \int_0^1 dx x^m \psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \frac{1}{f_K n\cdot P}\int_{dk_\parallel}\left[\frac{n\cdot k}{n\cdot P}\right]^m \gamma_5\gamma\cdot n\chi_K^{(2)}(k_-^K;P_K)$$

$$\int_0^1 d\alpha\alpha^m\left[\frac{12}{f_K}\mathcal{Y}_K(\alpha;\sigma^2)\right]\,, \qquad \mathcal{Y}_K(\alpha;\sigma^2) = [M_u(1-\alpha)+M_s\alpha]\mathcal{X}(\alpha;\sigma_\perp^2)\,.$$

**Uniqueness of Mellin moments** $\Rightarrow$

$$\boxed{\psi_K^{\uparrow\downarrow}(x,k_\perp^2) = \frac{12}{f_K}\mathcal{Y}_K(x;\sigma_\perp^2)}$$

- ✓ Compactness of this result is a merit of the algebraic model.

- The explicit form of $\rho_K(\omega)$ **controls** the shape of **PDAs, GPDs, PDFs**, etc.

$$\boxed{\psi_K^{\uparrow\downarrow}(x,k_\perp^2) \sim \int d\omega \cdots \rho_K(\omega)\cdots}$$

# Beyond factorization: Nakanishi representation

## Modeling the LFWF:

- *Asymptotic* model:

$$\rho_\pi(\omega) \sim (1-\omega^2) \longrightarrow \begin{cases} \phi(x) \sim x(1-x) & \textit{Asymptotic PDA} \\ q(x) \sim [x(1-x)]^2 & \textit{Free-scale PDF} \end{cases}$$

C. Mezrag et al., PLB 741 (2015) 190-196.
C. Mezrag et al., FBS 57 (2016) no.9, 729-772

- Experience and careful analysis lead us to the following **flexible** parametrization intended to a realistic description of meson DFs:

$$u_G \rho_G(\omega) = \frac{1}{2b_0^G}\left[\mathrm{sech}^2\left(\frac{\omega-\omega_0^G}{2b_0^G}\right) + \mathrm{sech}^2\left(\frac{\omega+\omega_0^G}{2b_0^G}\right)\right]\left[1+\omega\, v_G\right],$$

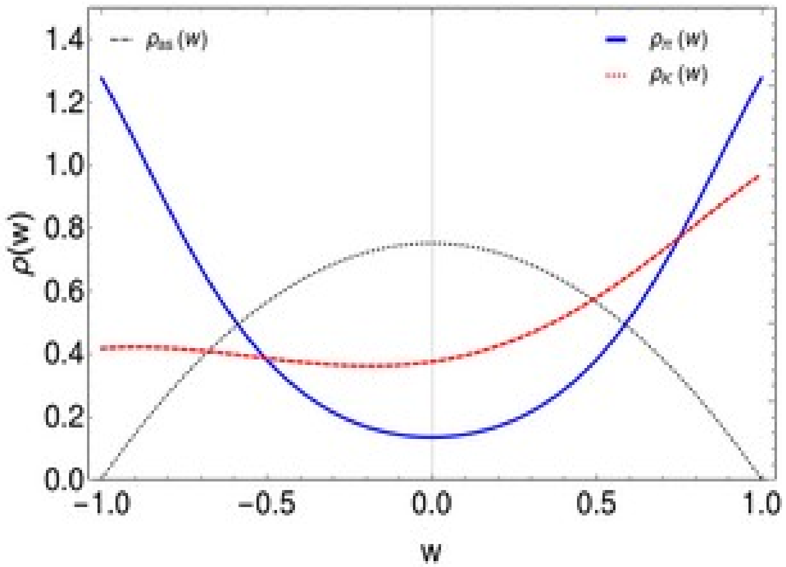
- Employing PDFs and PDAs as ***benchmarks:***

| $\Lambda_\pi$ | $b_0^\pi$ | $\omega_0^\pi$ | $\nu_\pi$ | $\Lambda_K$ | $b_0^K$ | $\omega_0^K$ | $\nu_K$ |
|---|---|---|---|---|---|---|---|
| $M_u$ | $0.275$ | $1.23$ | $0$ | $3M_s$ | $0.1$ | $0.625$ | $0.41$ |

$$m_\pi = 0.140\ \mathrm{GeV}, m_K = 0.49\ \mathrm{GeV}$$

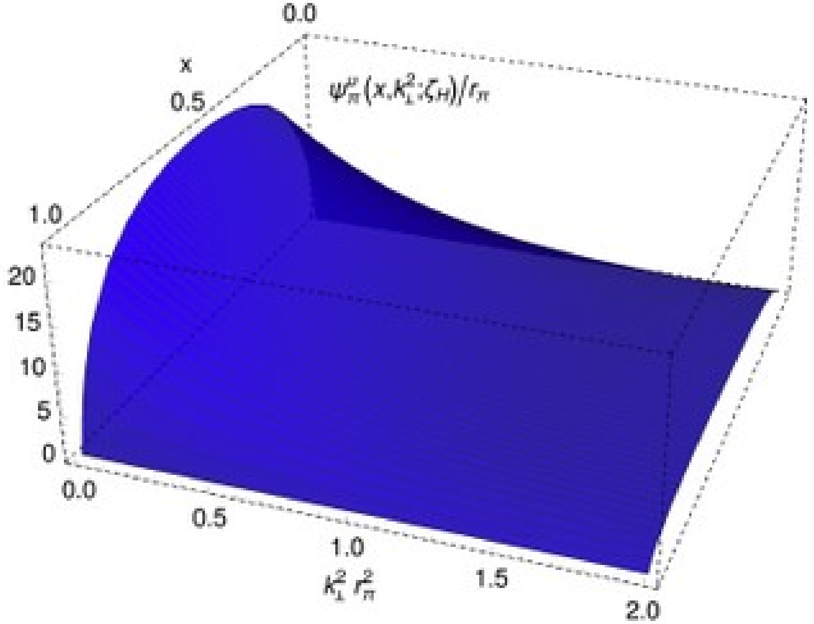
$$M_u = 0.31\ \mathrm{GeV}, M_s = 1.2\ M_u$$

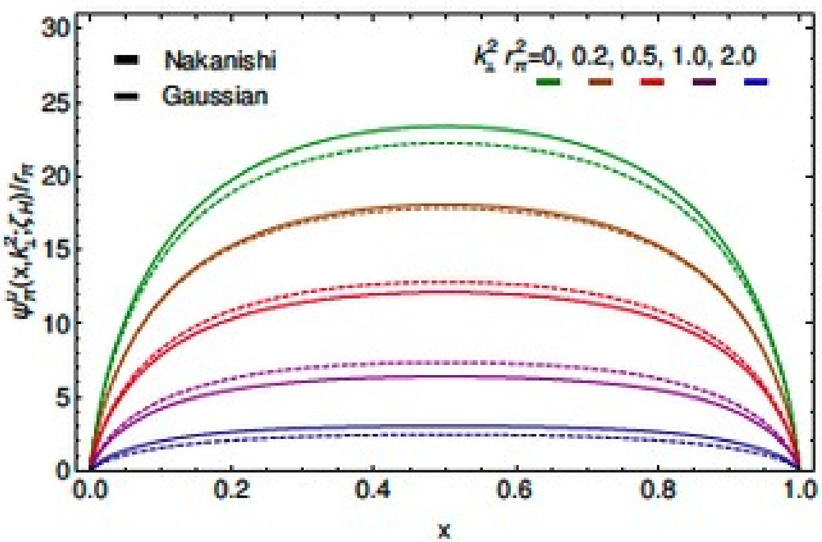
Typical values of **constituent** quark masses, from **realistic** DSEs **solutions**.

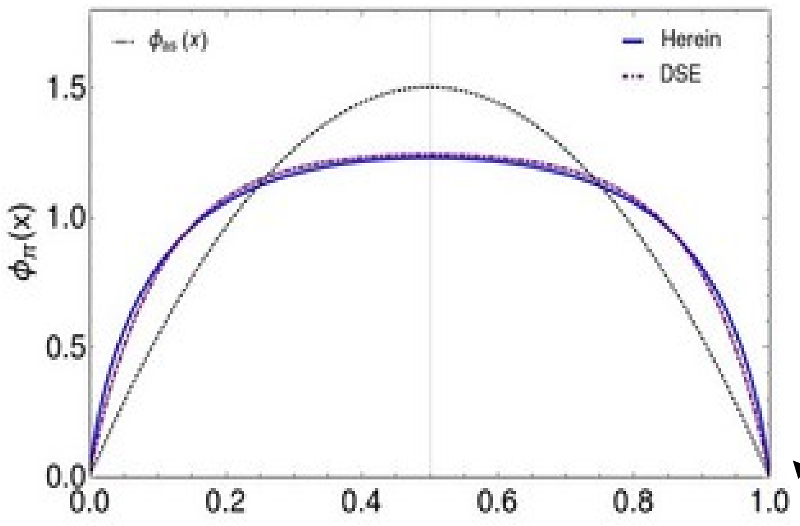


# GPDs from LFWFs

## Pion LFWFs: $\psi_{\pi}^{\uparrow\downarrow}(x,k_\perp^2) = \frac{12}{f_\pi}\mathcal{Y}_\pi(x;\sigma_\perp^2)$




$\sim (1-x)^{1+\gamma}$

| | Herein | DSE | Lat. A | Lat. B |
|---|---|---|---|---|
| $<(1-2x)^2>$ | 0.252 | 0.250 | 0.234(6)(6) | 0.244(30) |

**Lattice (A):** G.S. Bali et al., JHEP 1908 (2019) 065

**Lattice (B):** Rui Zhang et al., arXiv:2005.13955

**DSE:** L. Chang et al., PRL 110 (2013) no.13, 132001

Factorized gaussian ansatz:

$$\psi_{\pi u}^{\uparrow\downarrow}(x,k_\perp^2;\zeta_H) = \left(\frac{32\pi^2 r_\pi^2}{3\langle x^2\rangle_u^{\zeta_H}}u^\pi(x;\zeta_H)\right)^{1/2}\exp\left(-\frac{r_\pi^2 k_\perp^2}{3\langle x^2\rangle_u^{\zeta_H}}\right)$$

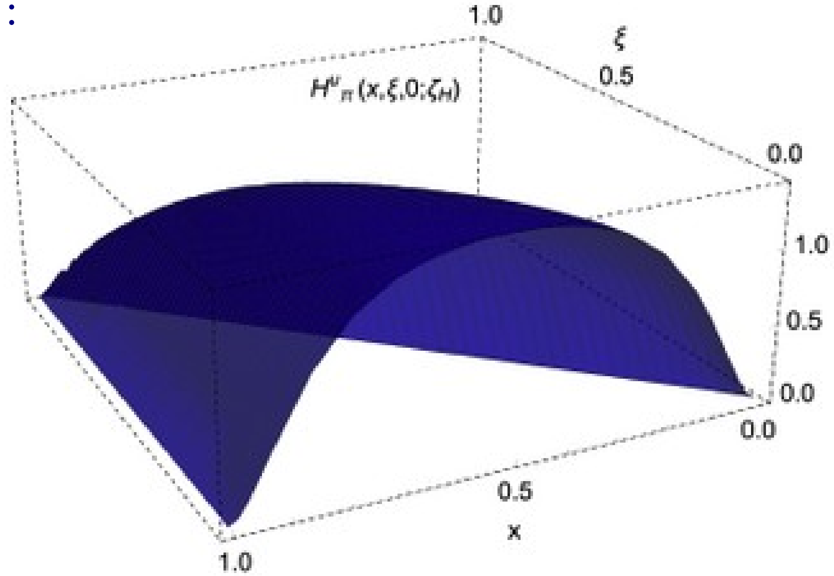
The only (additional) input needed to fix an approximated compact result is the pion charge radius

$PDG:\ r_\pi = 0.659(8)\,fm \qquad DSE:\ r_\pi = 0.69\,fm\,[Nakanishi]$

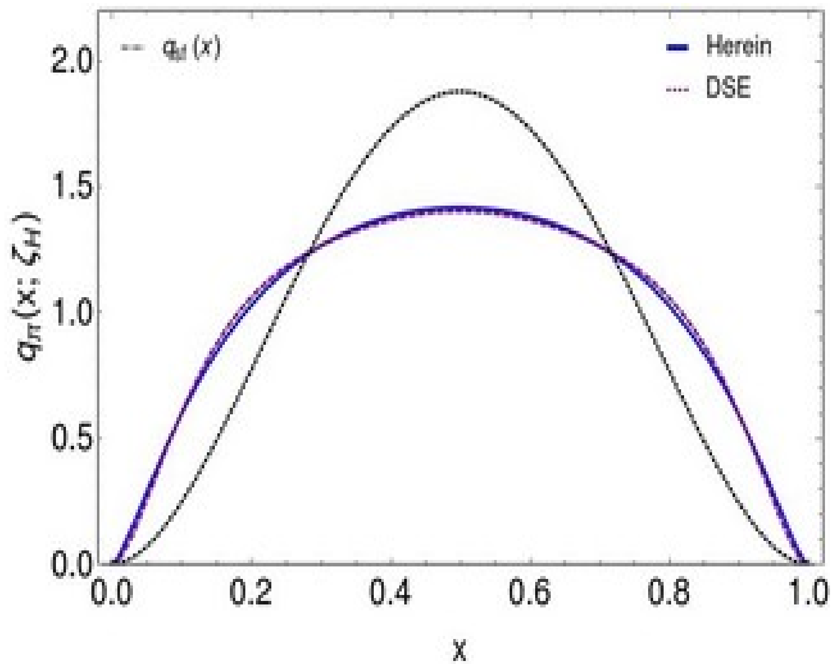
# GPDs from LFWFs

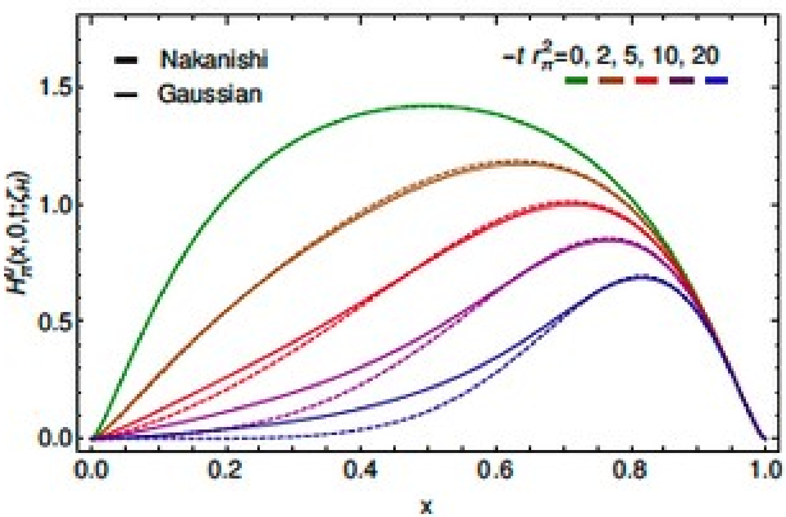
**Pion GPD:** $H_\pi^u\,(x,\xi,t;\zeta_H) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3}\,\psi^{\uparrow\downarrow *}_{\pi\,u}\left(\frac{x-\xi}{1-\xi},\left(\mathbf{k}_\perp + \frac{\mathbf{1-x}}{\mathbf{1-\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2;\zeta_H\right)\psi^{\uparrow\downarrow}_{\pi\,u}\left(\frac{x+\xi}{1+\xi},\left(\mathbf{k}_\perp - \frac{\mathbf{1-x}}{\mathbf{1+\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2;\zeta_H\right)$

$t=0$:




Valence-quark overlap GPD and forward PDF limit




Factorized gaussian ansatz:

$$\boxed{H_\pi^u(x,\xi,t;\zeta_H) \;=\; \theta(x-\xi)\sqrt{u^\pi\left(\frac{x-\xi}{1-\xi}\right)u^\pi\left(\frac{x+\xi}{1+\xi}\right)}\;\exp\left(-\frac{-t\,r_\pi^2(1-x)^2}{6\langle x^2\rangle_u^{\zeta_H}(1-\xi^2)}\right)}$$

The only (additional) input needed to fix an approximated compact result is the pion charge radius

$PDG:\; r_\pi = 0.659(8)\,fm \qquad DSE:\; r_\pi = 0.69\,fm\,[Nakanishi]$

# GPDs from LFWFs

**Pion GPD:** $H_\pi^u(x,\xi,t;\zeta_H) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3} \psi^{\uparrow\downarrow *}_{\pi u}\left(\frac{x-\xi}{1-\xi}, \left(\mathbf{k}_\perp + \frac{\mathbf{1-x}}{\mathbf{1-\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2; \zeta_H\right) \psi^{\uparrow\downarrow}_{\pi u}\left(\frac{x+\xi}{1+\xi}, \left(\mathbf{k}_\perp - \frac{\mathbf{1-x}}{\mathbf{1+\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2;\zeta_H\right)$

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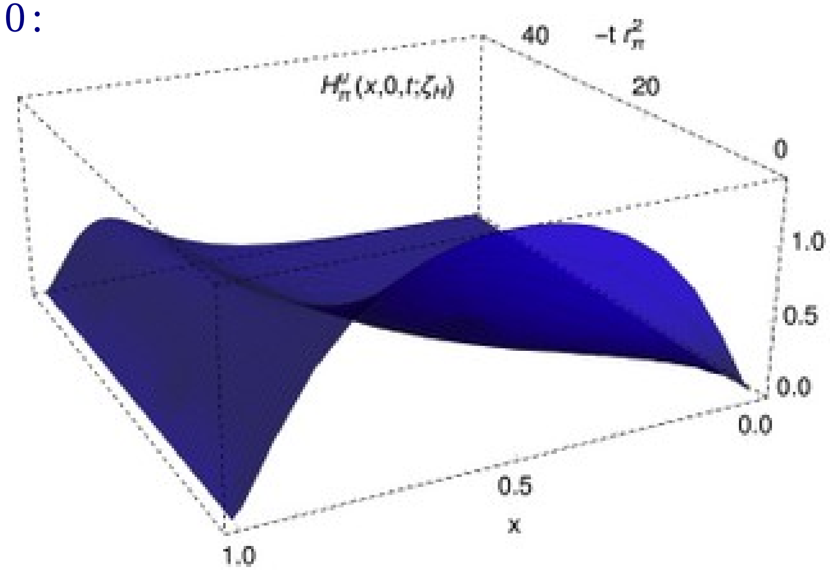
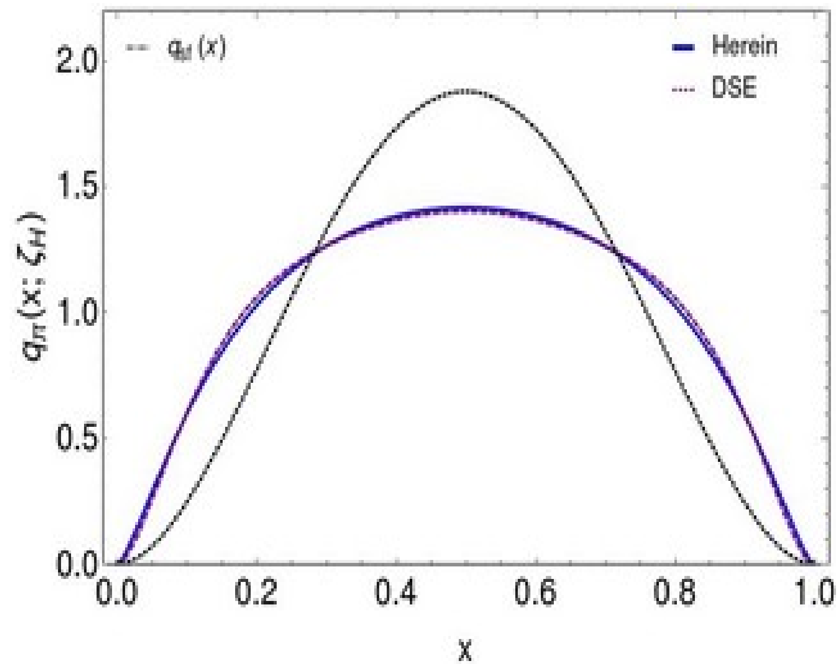
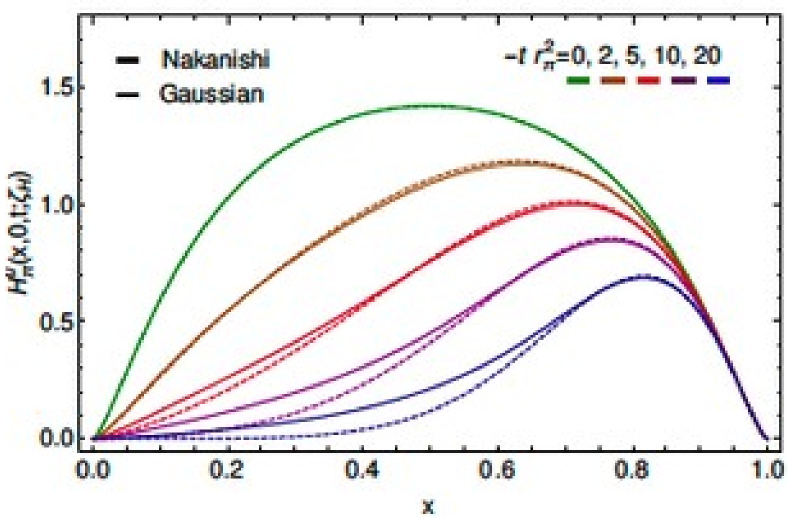



Valence-quark overlap GPD and forward PDF limit




Factorized gaussian ansatz:

$$\boxed{H_\pi^u(x,\xi,t;\zeta_H) = \theta(x-\xi)\sqrt{u^\pi\left(\frac{x-\xi}{1-\xi}\right)u^\pi\left(\frac{x+\xi}{1+\xi}\right)}\;\exp\left(-\frac{-t\, r_\pi^2(1-x)^2}{6\langle x^2\rangle_u^{\zeta_H}(1-\xi^2)}\right)}$$

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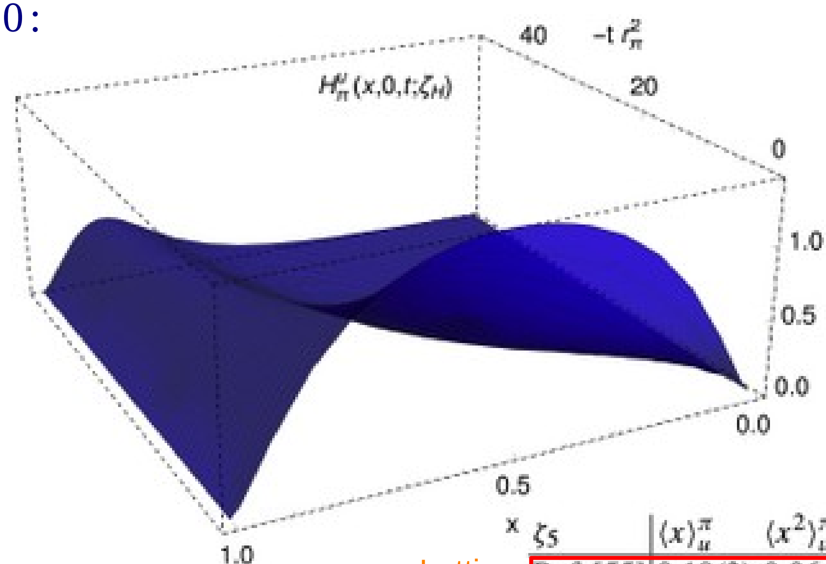
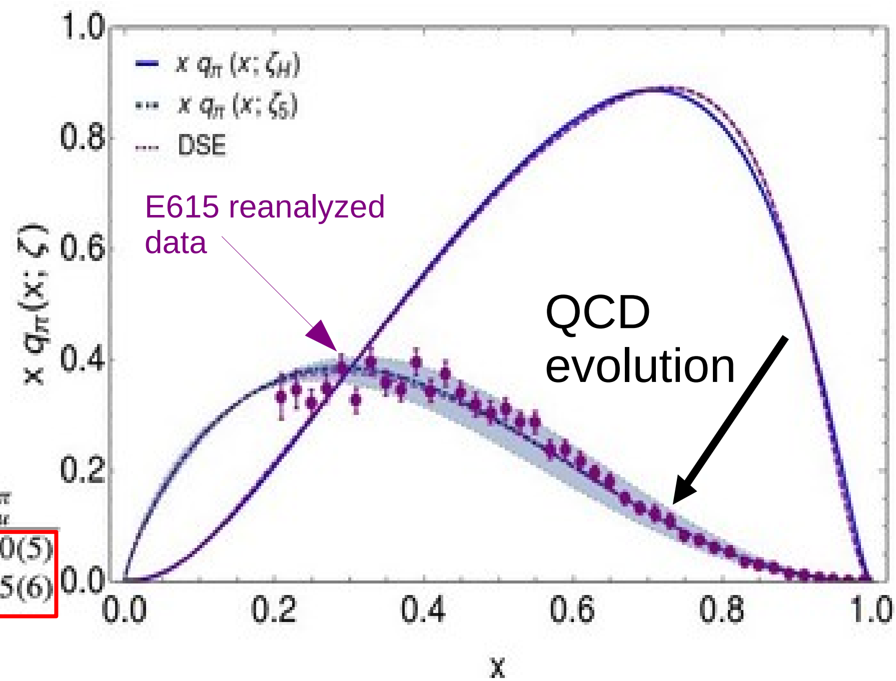
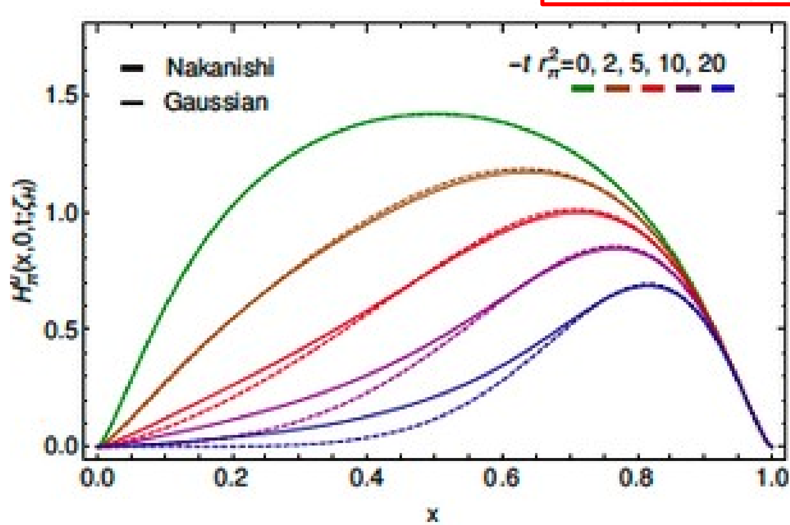



Valence-quark overlap GPD and forward PDF limit




Lattice:

| $\zeta_5$ | $\langle x\rangle_u^\pi$ | $\langle x^2\rangle_u^\pi$ | $\langle x^3\rangle_u^\pi$ |
|---|---|---|---|
| Ref. [55] | 0.18(3) | 0.064(10) | 0.030(5) |
| Herein | 0.20(2) | 0.074(10) | 0.035(6) |

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$$H_\pi^u(x,\xi,t;\zeta_H) = \theta(x-\xi)\sqrt{u^\pi\left(\frac{x-\xi}{1-\xi}\right)u^\pi\left(\frac{x+\xi}{1+\xi}\right)}\,\exp\left(-\frac{-t\, r_\pi^2(1-x)^2}{6\langle x^2\rangle_u^{\zeta_H}(1-\xi^2)}\right)$$

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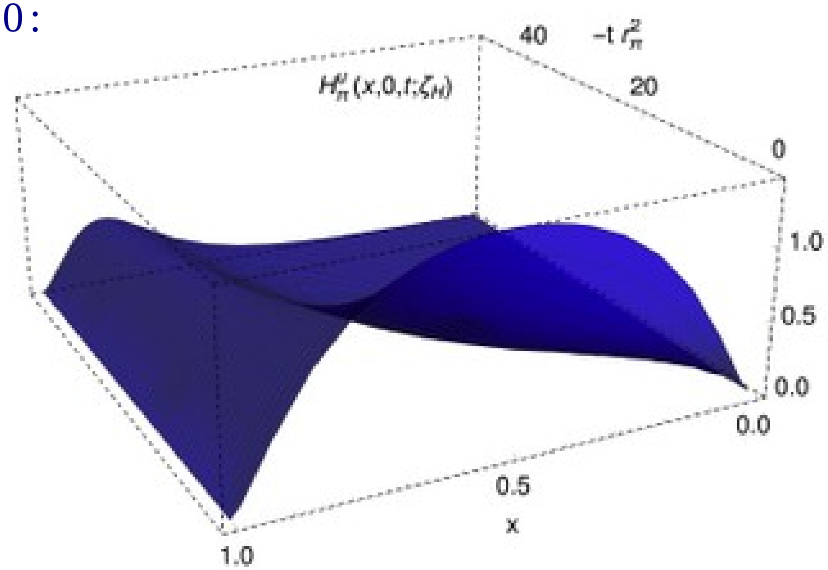
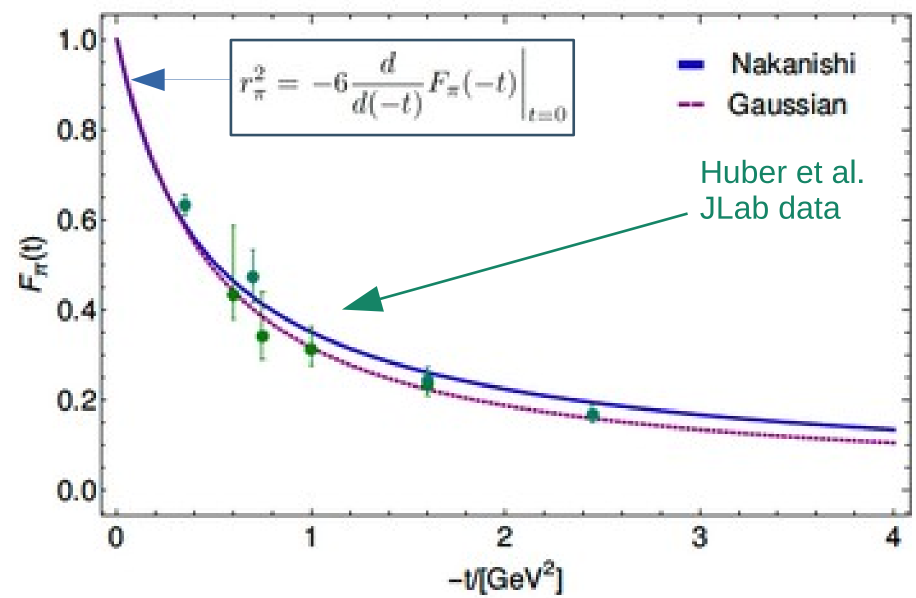
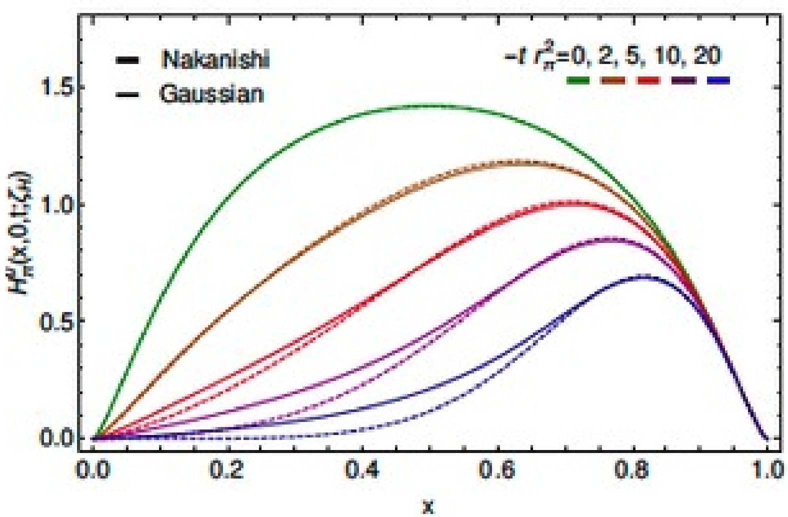



Valence-quark overlap GPD and the EM form factor

$$F_M(-t) = e_u F^u_M(-t) + e_{\bar h} F^{\bar h}_M(-t)$$




Factorized gaussian ansatz:

$$\boxed{H^u_\pi(x,\xi,t;\zeta_H) = \theta(x-\xi)\sqrt{u^\pi\left(\frac{x-\xi}{1-\xi}\right)u^\pi\left(\frac{x+\xi}{1+\xi}\right)}\ \exp\left(-\frac{-t\, r_\pi^2(1-x)^2}{6\langle x^2\rangle_u^{\zeta_H}(1-\xi^2)}\right)}$$

The only (additional) input needed to fix an approximated compact result is the pion charge radius

$PDG:\ r_\pi = 0.659(8)\, fm \qquad DSE:\ r_\pi = 0.69\, fm\,[Nakanishi]$

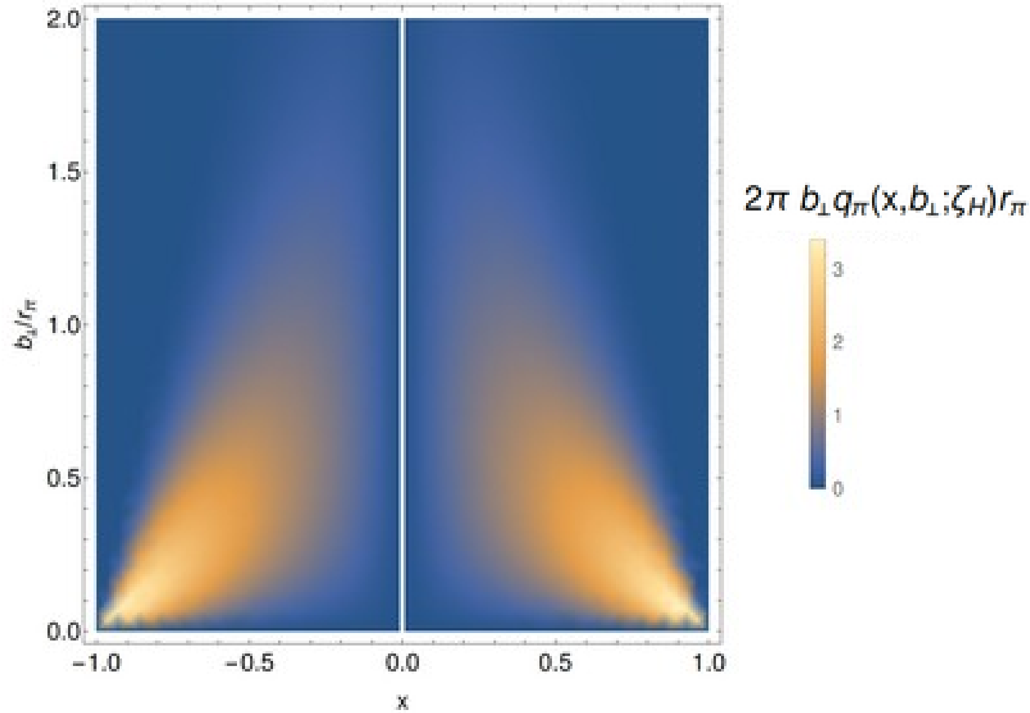
# GPDs from LFWFs

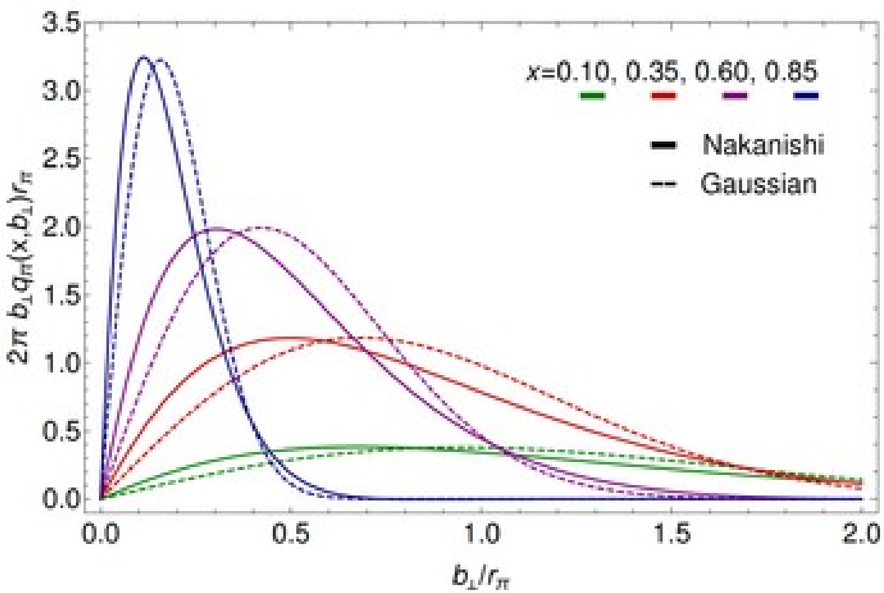
## Pion IPD GPD:

$$u^{\pi'}(x, b_\perp^2; \zeta_H) = \int_0^\infty \frac{d\Delta_\perp}{2\pi} \Delta_\perp J_0(b_\perp \Delta_\perp)\; H_\pi^u(x,\xi,t;\zeta_H)|_{\xi=0}$$

The probability of finding the pion's u-quark (x>0) or d-antiquark (x<0) at a distance $b_\perp$ away from the CoTM peaks up at a small but non-zero value and at |x| near 1.

This probability density at x=cte. peaks around a maximum at non-zero $b_\perp$; the larger is x, the smaller $b_\perp$ and the narrower the distribution.
The larger is the momentum fraction carried by the parton, the more it bears on the CoTM definition.




Factorized gaussian ansatz:

$$q^{\pi}(x, b_\perp^2; \zeta_H) = \frac{\gamma_{\pi'}(\zeta_H)}{\pi r_\pi^2} \frac{q^{\pi'}(|x|;\zeta_H)}{(1-|x|)^2} \exp\left(-\frac{\gamma_{\pi}(\zeta_H)}{(1-|x|)^2}\frac{b_\perp^2}{r_\pi^2}\right)$$

$$\gamma_{\pi'}(\zeta_H) = \frac{3\langle x^2\rangle_u^{\zeta_H}}{2} \qquad (q = u[x \geq 0], d[x \leq 0])$$

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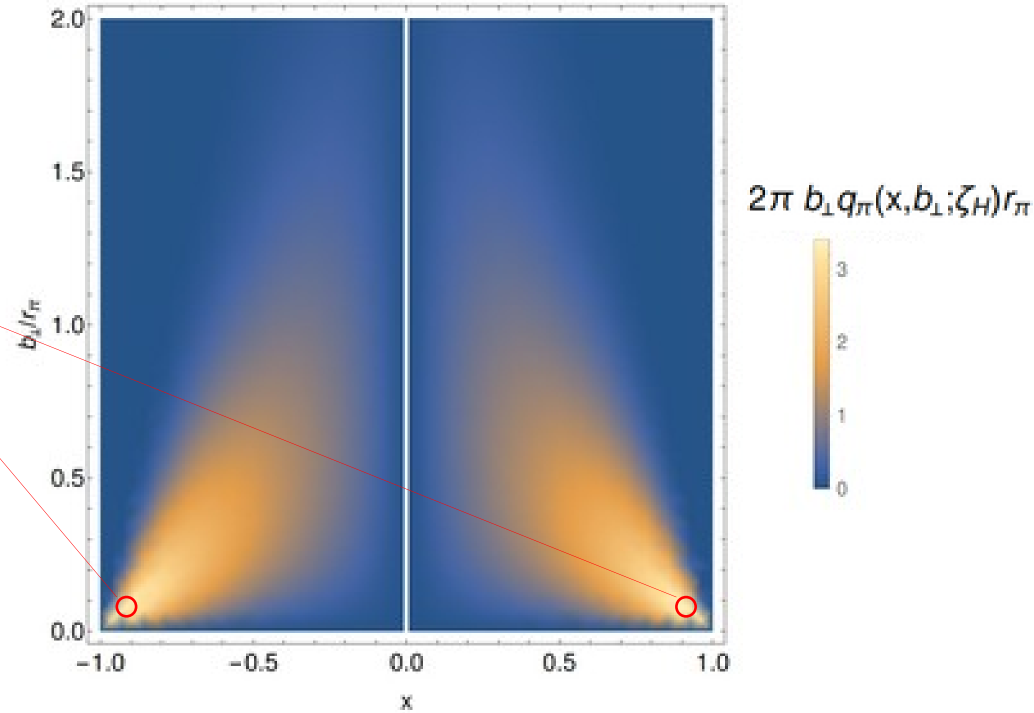
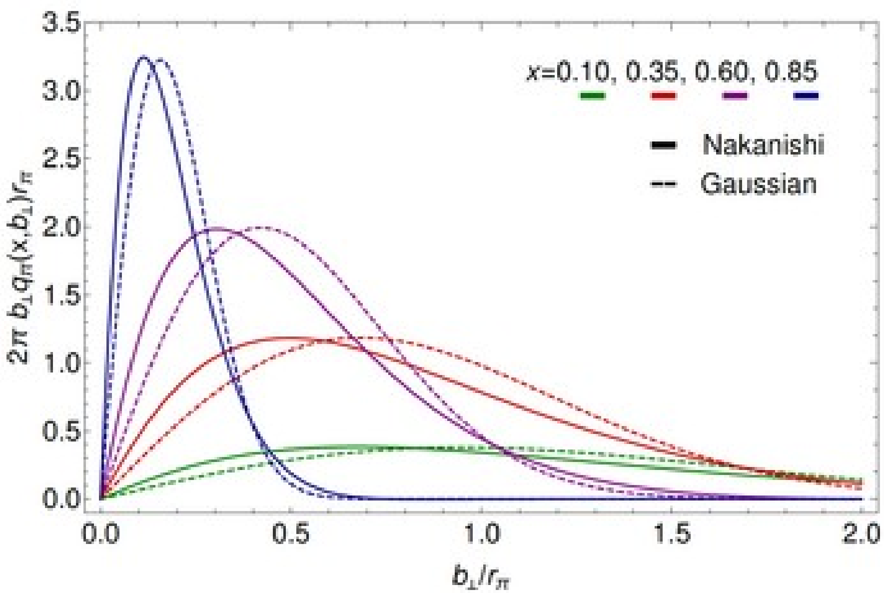
$(|x|,b_\perp/r_\pi)=(0.91,0.065)$

This probability density at x=cte. peaks around a maximum at non-zero $b_\perp$; the larger is x, the smaller $b_\perp$ and the narrower the distribution.
The larger is the momentum fraction carried by the parton, the more it bears on the CoTM definition.




Factorized gaussian ansatz:

$$q^{\pi}(x,b_\perp^2;\zeta_H)=\frac{\gamma_{\pi}(\zeta_H)}{\pi r_\pi^2}\frac{q^{\pi}(|x|;\zeta_H)}{(1-|x|)^2}\exp\left(-\frac{\gamma_\pi(\zeta_H)}{(1-|x|)^2}\frac{b_\perp^2}{r_\pi^2}\right)$$

$$\gamma_{\pi}(\zeta_H)=\frac{3\langle x^2\rangle_u^{\zeta_H}}{2}\qquad (q=u[x\geq 0],d[x\leq 0])$$

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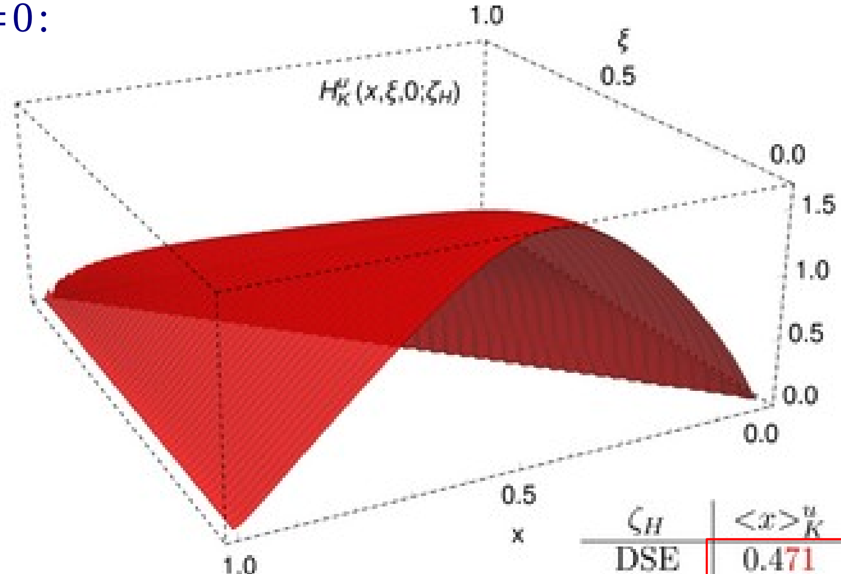
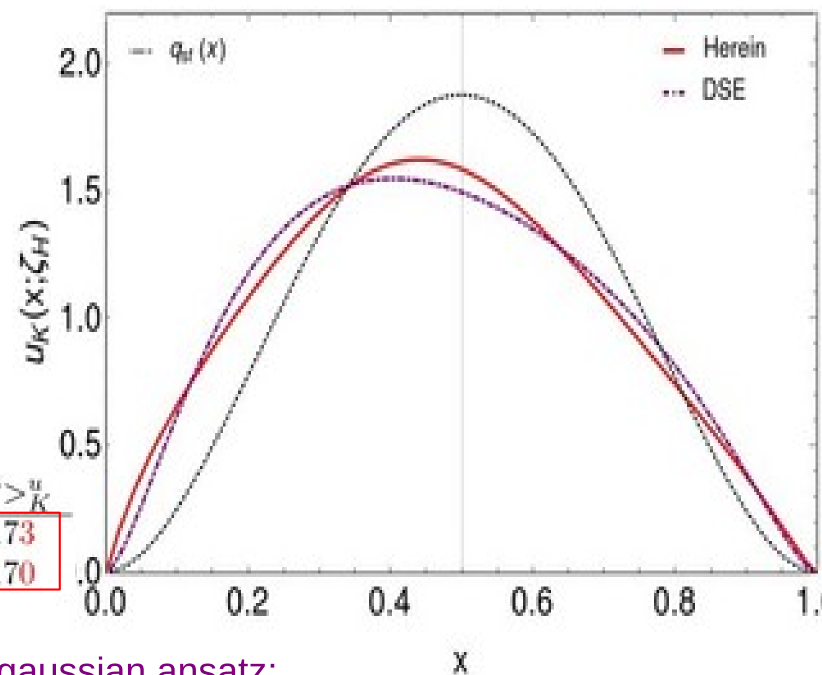
# GPDs from LFWFs

**Kaon GPD:** $H_K^u(x,\xi,t;\zeta_H) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3} \psi^{\uparrow\downarrow *}_{K^u}\left(\frac{x-\xi}{1-\xi}, \left(\mathbf{k}_\perp + \frac{\mathbf{1-x}}{\mathbf{1-\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2;\zeta_H\right)\psi^{\uparrow\downarrow}_{K^u}\left(\frac{x+\xi}{1+\xi},\left(\mathbf{k}_\perp - \frac{\mathbf{1-x}}{\mathbf{1+\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2;\zeta_H\right)$

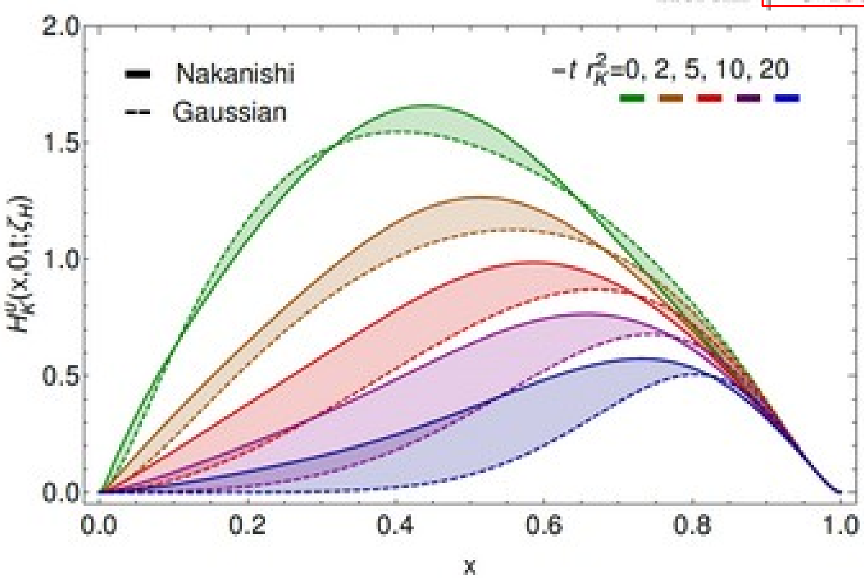
$t=0$:




Valence-quark overlap GPD and forward PDF limit




| $\zeta_H$ | $\langle x\rangle_K^u$ | $\langle x^2\rangle_K^u$ | $\langle x^3\rangle_K^u$ |
|---|---|---|---|
| DSE | 0.471 | 0.270 | 0.173 |
| Herein | 0.468 | 0.266 | 0.170 |




Factorized gaussian ansatz:

$$H_K^u(x,\xi,t;\zeta_H) = \theta(x-\xi)\sqrt{u^K\left(\frac{x-\xi}{1-\xi}\right)u^K\left(\frac{x+\xi}{1+\xi}\right)} \times \exp\left(-\frac{-t\,r_K^2(1-x)^2}{\left(4\langle x^2\rangle_{\bar{s}}^{\zeta_H}+2(1+\delta)\langle x^2\rangle_u^{\zeta_H}\right)(1-\xi^2)}\right)$$

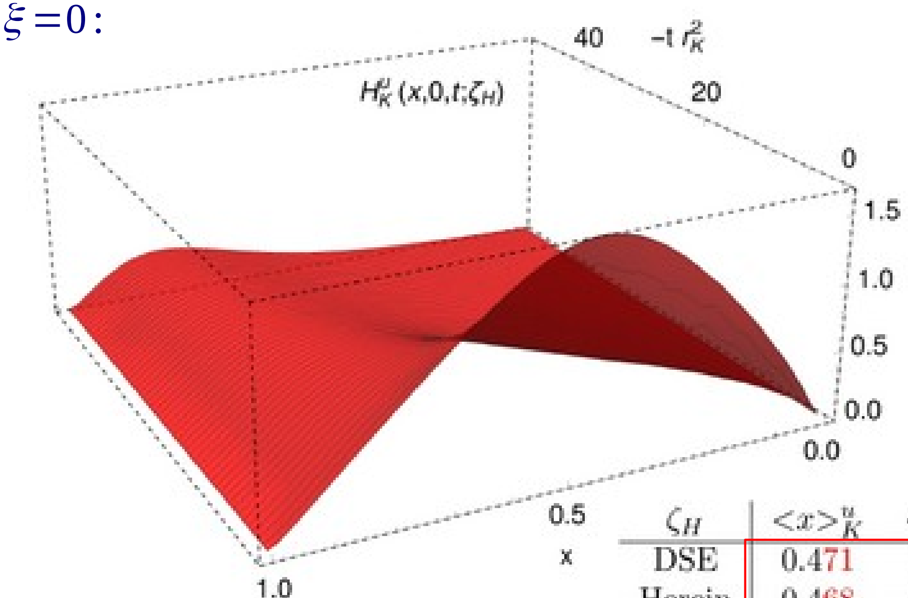
The only (additional) input needed to fix an approximated compact result is the pion charge radius

$PDG:\ r_K = 0.560(31)\,fm \qquad DSE:\ r_K = 0.56\,fm\,[Nakanishi]$

# GPDs from LFWFs

**Kaon GPD:** $H_K^u(x,\xi,t;\zeta_H) = \int \frac{d^2\mathbf{k}_\perp}{16\pi^3}\psi_{K^u}^{\uparrow\downarrow *}\left(\frac{x-\xi}{1-\xi},\left(\mathbf{k}_\perp + \frac{\mathbf{1-x}}{\mathbf{1-\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2;\zeta_H\right)\psi_{K^u}^{\uparrow\downarrow}\left(\frac{x+\xi}{1+\xi},\left(\mathbf{k}_\perp - \frac{\mathbf{1-x}}{\mathbf{1+\xi}}\frac{\mathbf{\Delta}_\perp}{\mathbf{2}}\right)^2;\zeta_H\right)$

$\xi = 0:$




| $\zeta_H$ | $\langle x \rangle_K^u$ | $\langle x^2 \rangle_K^u$ | $\langle x^3 \rangle_K^u$ |
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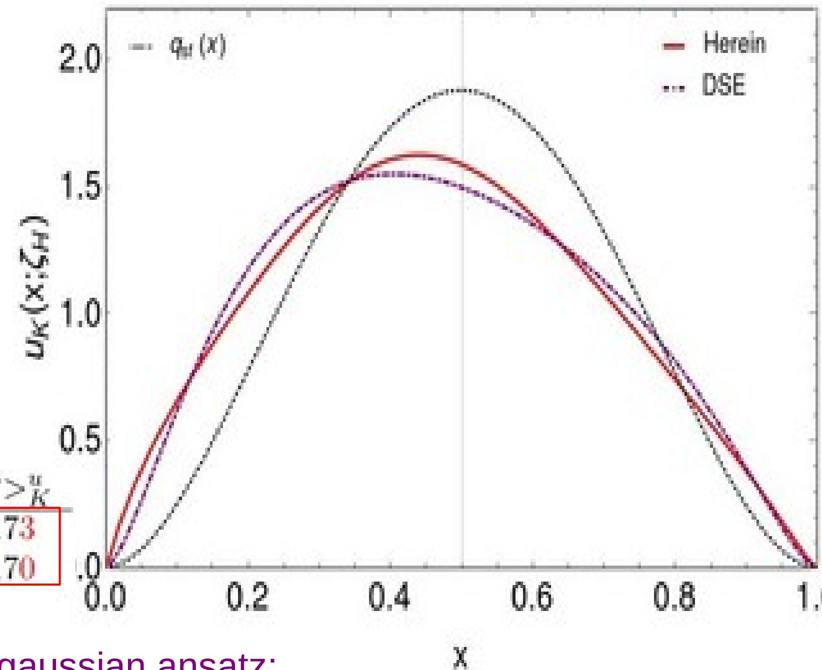
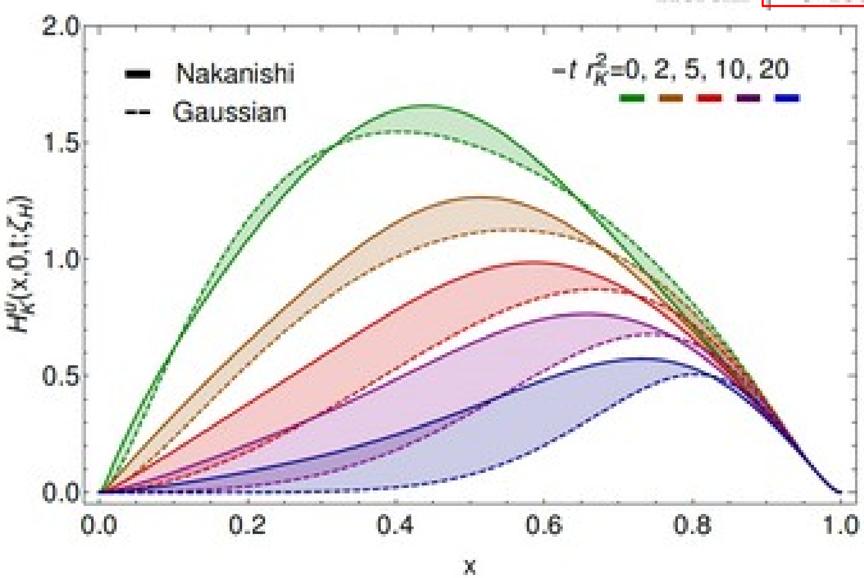
Valence-quark overlap GPD and forward PDF limit

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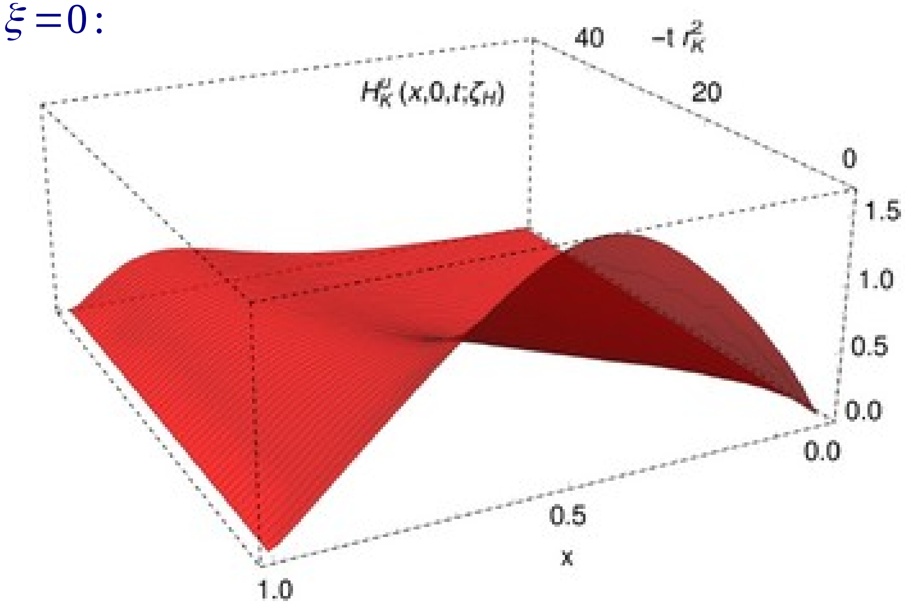
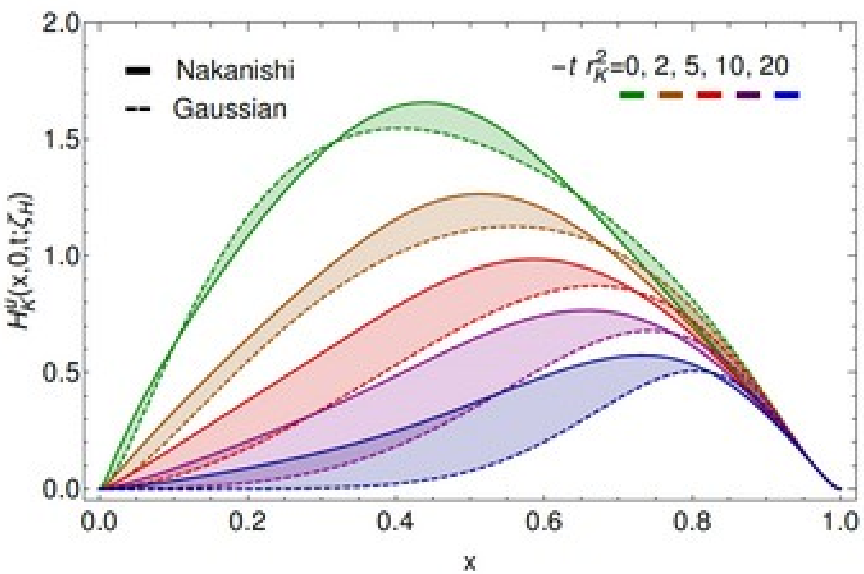
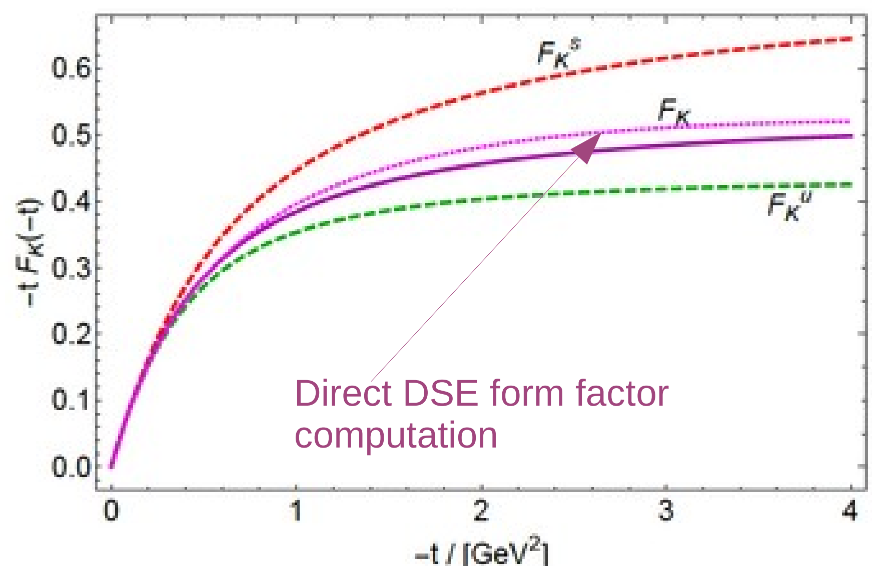



Valence-quark overlap GPD and the EM form factor

$$F_M(-t) = e_u F_M^u(-t) + e_{\bar{h}} F_M^{\bar{h}}(-t)$$

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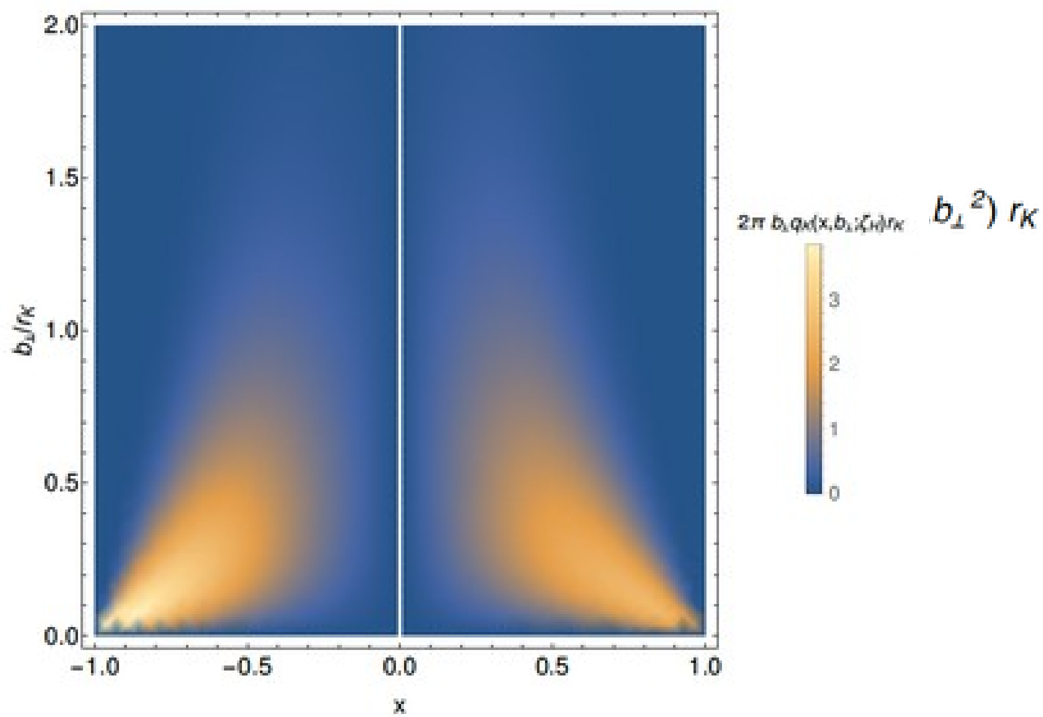
$$u^{K}(x, b_\perp^2; \zeta_H) = \int_0^\infty \frac{d\Delta_\perp}{2\pi} \Delta_\perp J_0(b_\perp \Delta_\perp)\; H_K^u(x, \xi, t; \zeta_H)|_{\xi=0}$$

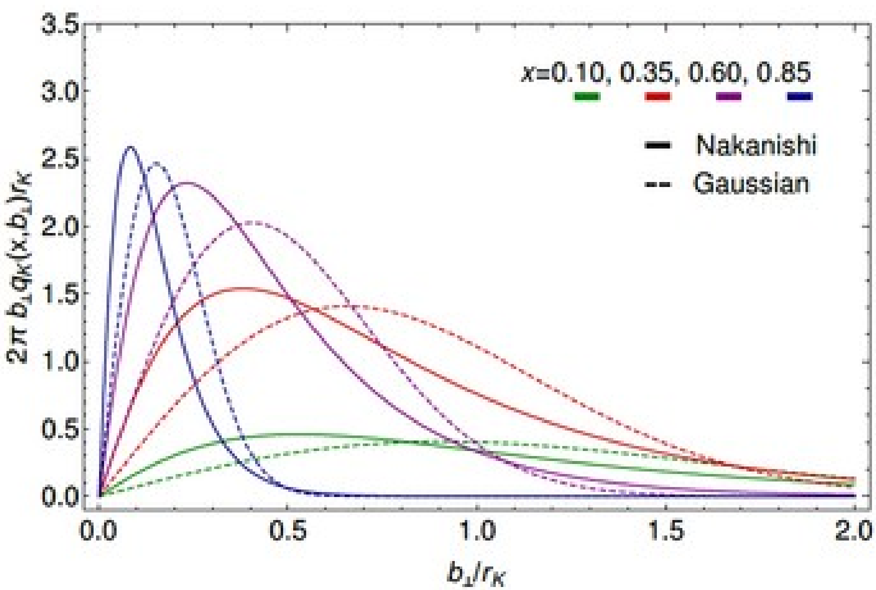
Nakanishi LFWF

The flavor asymmetry is made manifest by the comparison of u-quark (x>0) and s-antiquark (x<0) probability densities: the heavier parton, carrying a larger momentum fraction, is more probably found close to the CoTM, to the definition of which it contributes more than the lighter.

$(|x|, b_\perp/r_K)$=(0.83,0.094)

$(|x|, b_\perp/r_K)$=(0.94,0.041)




Factorized gaussian ansatz:

$$q^K(x, b_\perp^2; \zeta_H) = \frac{\gamma_K(\zeta_H)}{\pi r_K^2} \frac{q^K(|x|; \zeta_H)}{(1-|x|)^2} \exp\left(-\frac{\gamma_K(\zeta_H)}{(1-|x|)^2} \frac{b_\perp^2}{r_K^2}\right)$$

$$\gamma_K(\zeta_H) = \langle x^2 \rangle_{\bar{s}}^{\zeta_H} + \frac{1+\delta}{2} \langle x^2 \rangle_u^{\zeta_H} \qquad (q = u[x \geq 0], s[x \leq 0])$$

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# GPDs from LFWFs

## Kaon IPD GPD:

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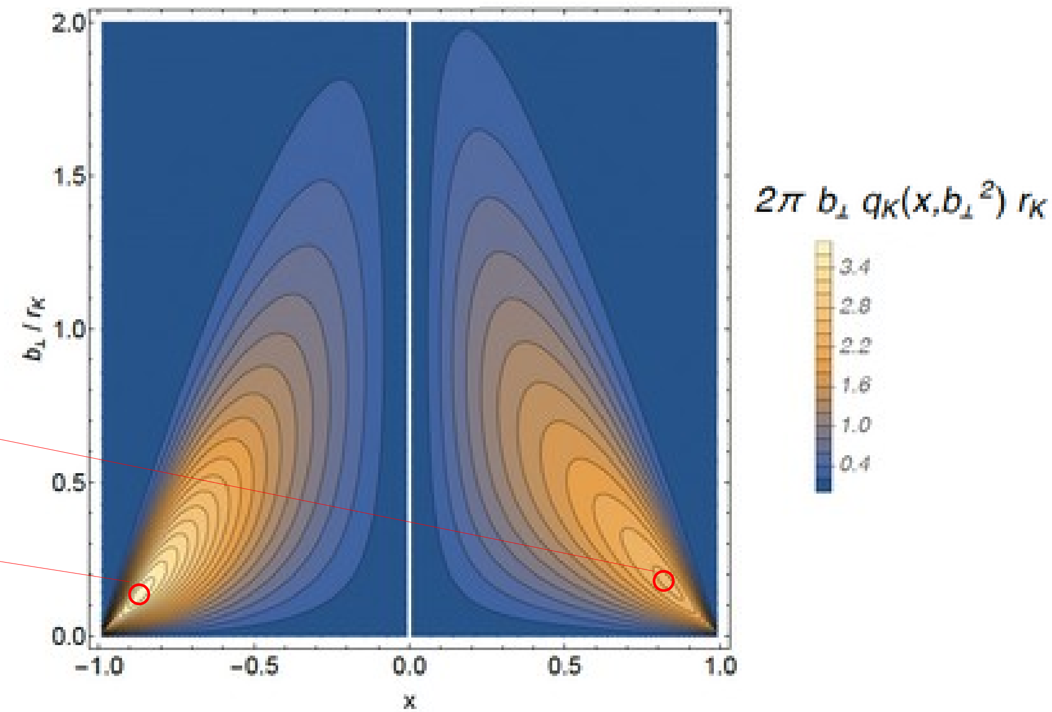
Gaussian LFWF

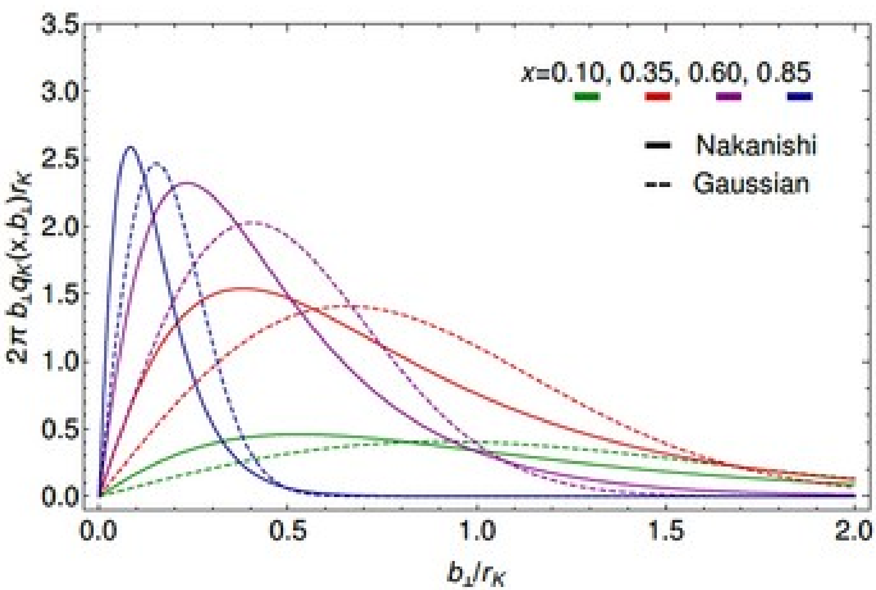



The flavor asymmetry is made manifest by the comparison of u-quark (x>0) and s-antiquark (x<0) probability densities: the heavier parton, carrying a larger momentum fraction, is more probably found close to the CoTM, to the definition of which it contributes more than the lighter.

$$(|x|, b_\perp/r_K) = (0.84, 0.17)$$

$$(|x|, b_\perp/r_K) = (0.87, 0.13)$$

Same picture: qualitative and semi-quantitative agreement!




Factorized gaussian ansatz:

$$q^K(x,b_\perp^2;\zeta_H) = \frac{\gamma_K(\zeta_H)}{\pi r_K^2}\frac{q^K(|x|;\zeta_H)}{(1-|x|)^2}\exp\left(-\frac{\gamma_K(\zeta_H)}{(1-|x|)^2}\frac{b_\perp^2}{r_K^2}\right)$$

$$\gamma_K(\zeta_H) = \langle x^2\rangle_{\bar s}^{\zeta_H} + \frac{1+\delta}{2}\langle x^2\rangle_u^{\zeta_H} \qquad (q=u[x\geq 0], s[x\leq 0])$$

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# Pion’s gravitational Form Factors

- **Gravitational** form factors connect with Energy-momentum tensor and are obtained from the **t-dependence** of the **GPD’s 1-st Mellin moment**:

$$\int_{-1}^{1} dx 2x H_\pi^u(x,\xi,t) = \boxed{\theta_2^\pi(-t)} - \xi^2 \boxed{\theta_1^\pi(-t)}$$

(isospin symmetry implemented)

mass distribution

pressure distribution

Owing to GPD’s polynomiality:

$$\theta_2^\pi(-t) = \int_{-1}^{1} dx 2x H_\pi^u(x,0,t)$$

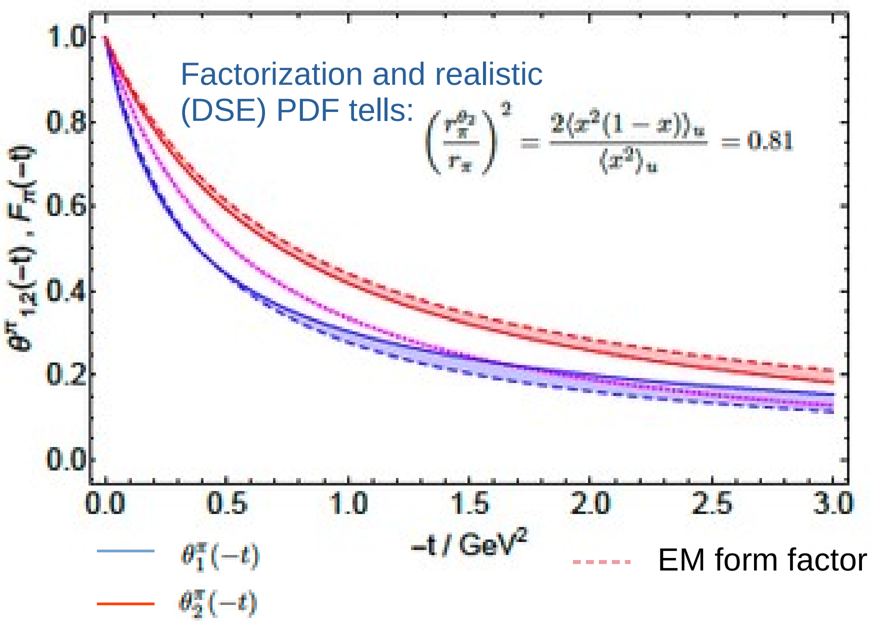
One needs both DGLAP ( $|x| \geq \xi$ ) and ERBL ( $|x| \leq \xi$ ) GPD to derive the pressure distribution. Radon transform inversion [N. Chouika et al. EPJC 77(2017)906] can be used. This can be illustrated with the Nakanishi asymptotic case:

$$\rho_\pi(\omega) \sim (1-\omega^2)$$

Also a factorized LFWF!!!

$$\boxed{\psi_{\pi u}^{\uparrow\downarrow}(x, k_\perp^2; \zeta_H) = 8\sqrt{15}\pi \frac{M^3}{(k_\perp^2 + M^2)^2} x(1-x)}$$

[N. Chouika et al. PLB780(2018)787]




Needs further ERBL completion (related to the D-term ambiguity) and can use the soft pion theorem for this:

$$H_\pi^u(x,\xi,t) = \boxed{\hat{H}_\pi^u(x,\xi,t)} + \boxed{P_\sigma(-t)} \,\mathrm{sign}(\xi) \left( \boxed{D^-\left(\frac{x}{\xi}\right) + \frac{1}{\xi} D^+\left(\frac{x}{\xi}\right)} \right)$$

Satisfies polynomiality but not soft-pion theorem

$$\boxed{\begin{aligned} D^-(z) &= \frac{1}{2}\left(\hat{H}_\pi^u(-z,1,0) - \hat{H}_\pi^u(z,1,0)\right) \\ D^+(z) &= \frac{1}{2}\left(\varphi_\pi^u\left(\frac{1+z}{2}\right) - \hat{H}_\pi^u(z,1,0) - \hat{H}_\pi^u(-z,1,0)\right) \end{aligned}}$$

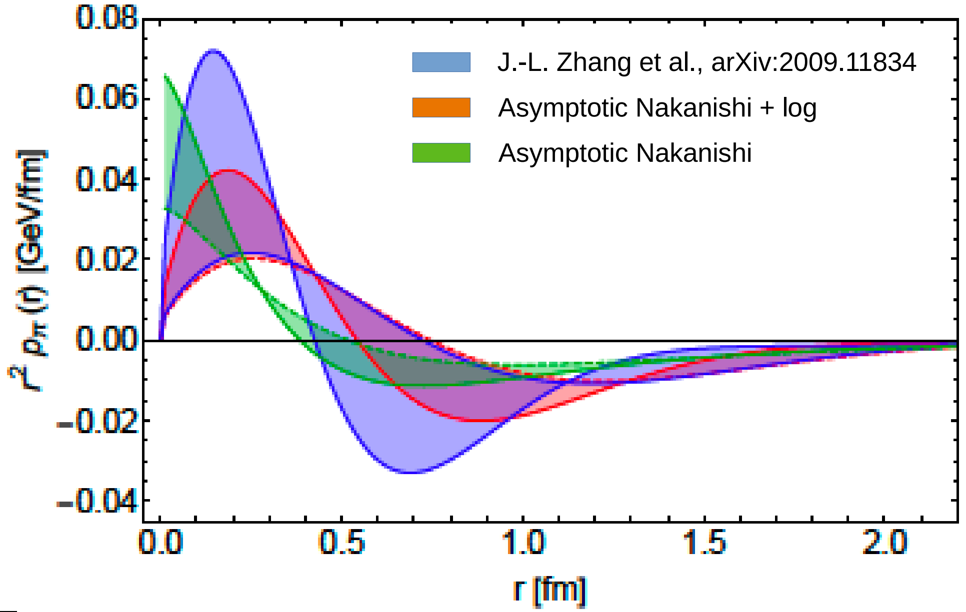
$$\boxed{P_\sigma(-t) = \left[1 + \frac{-t}{M^2}\right]^{-1}}$$

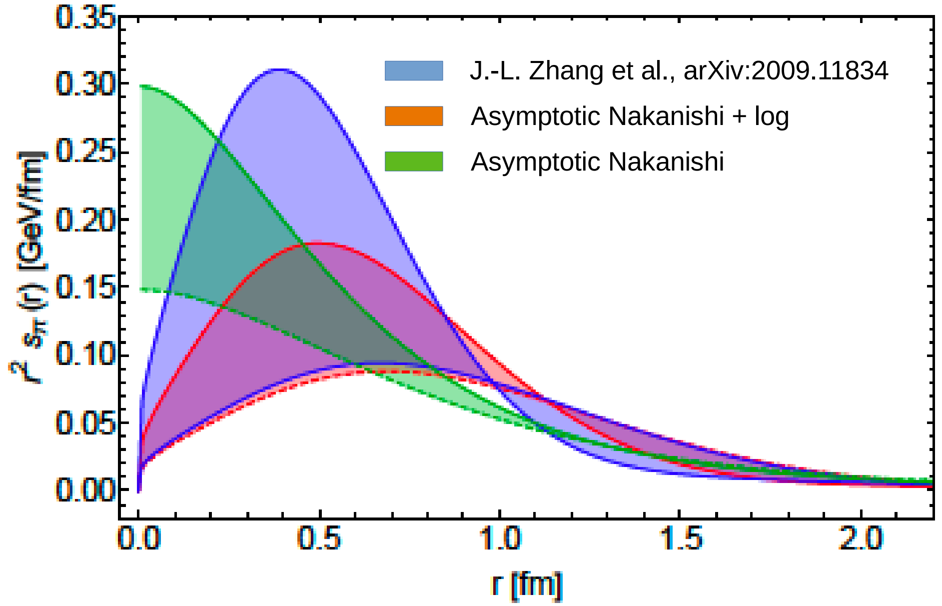
[J-L. Zhang et al., ArXiv:2009.11834]

# Pion’s gravitational Form Factors

Thus, the normal pressure can be sketched via the Fourier transform of the GFF:

$$p_\pi(r) = \frac{1}{6\pi^2 r}\int_0^\infty d\Delta \frac{\Delta}{2E(\Delta)}\sin(\Delta r)\left[\Delta^2\theta_1\left(\Delta^2\right)\right]$$




And a shear pressure as:

$$s_\pi(r) = \frac{3}{16\pi^2}\int_0^\infty d\Delta \frac{\Delta}{2E(\Delta)}\Delta j_2(\Delta r)\left[\Delta^2\theta_1\left(\Delta^2\right)\right]$$

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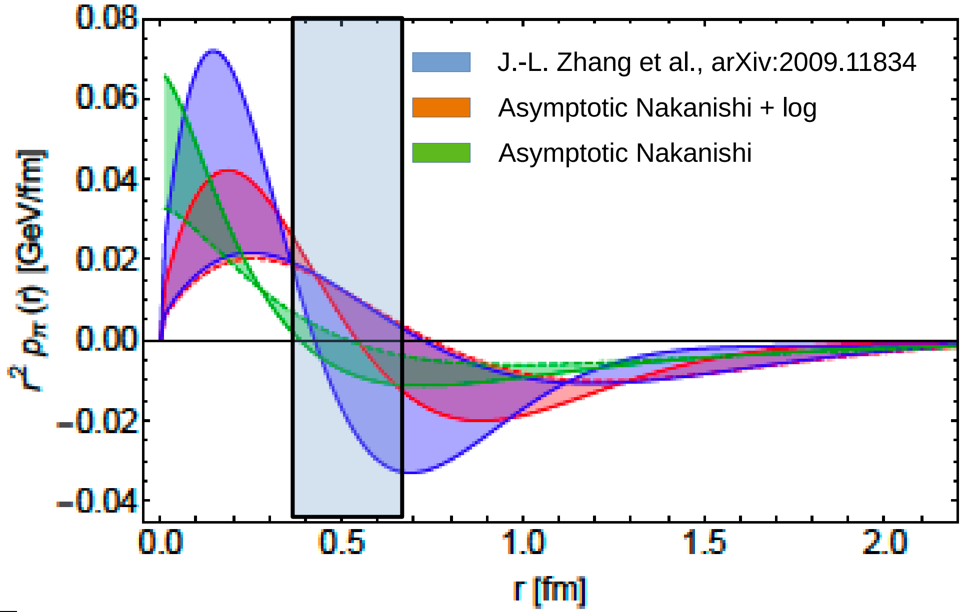
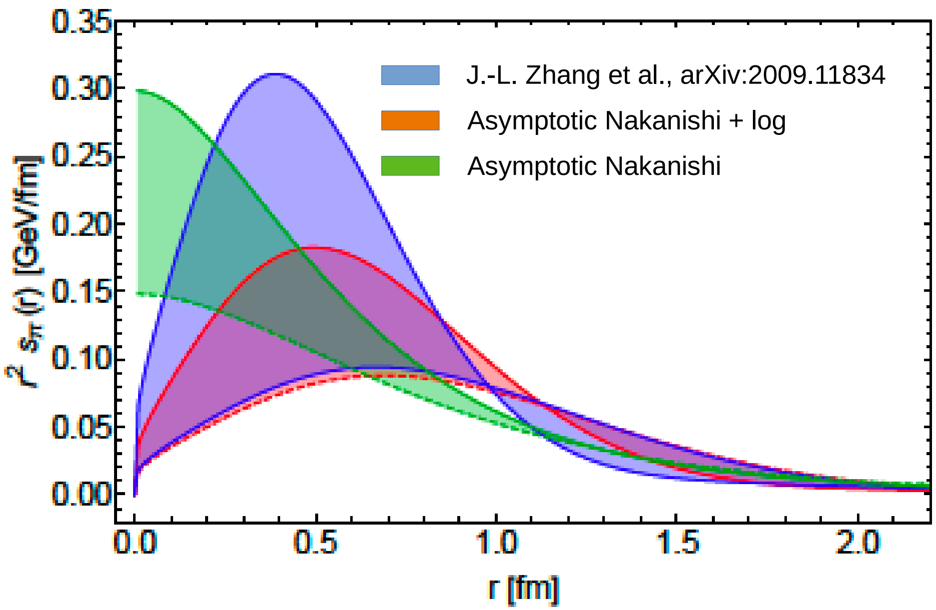
Unmodified and logarithmically suppressed pressures display a zero crossing which are commensurate and lie around 0.5 fm, indicating where "repulsive" forces (positive pressure) become "confining" (negative).

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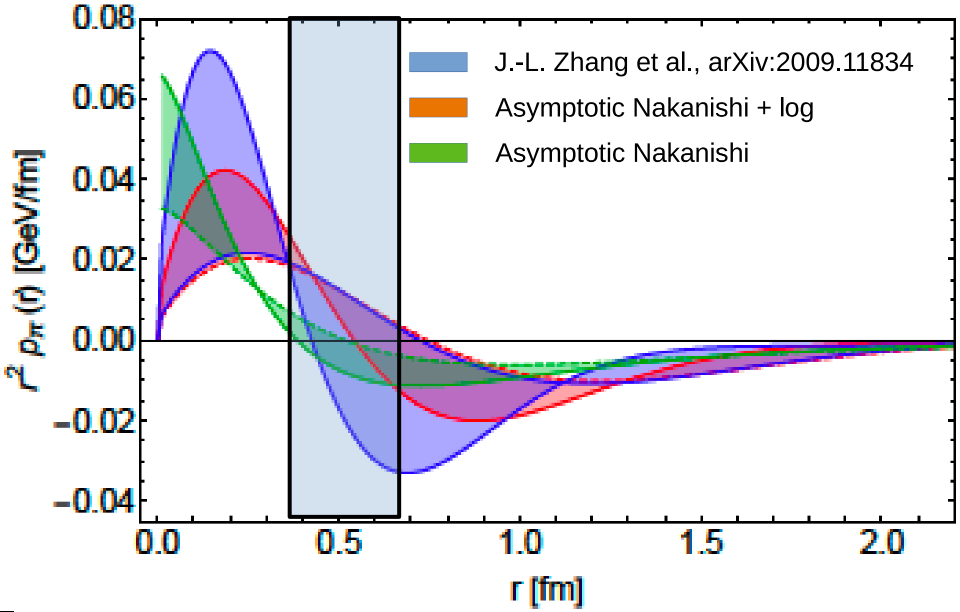


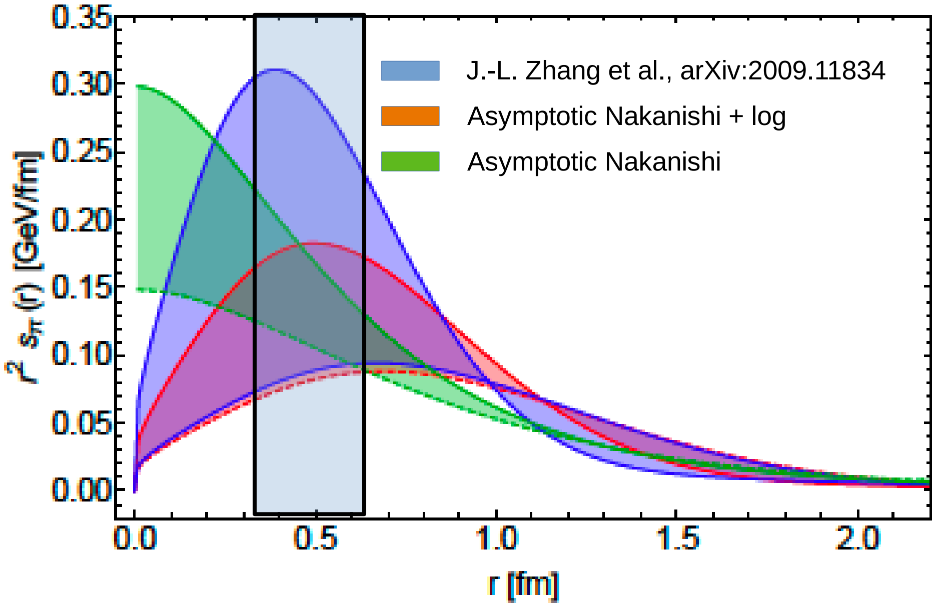


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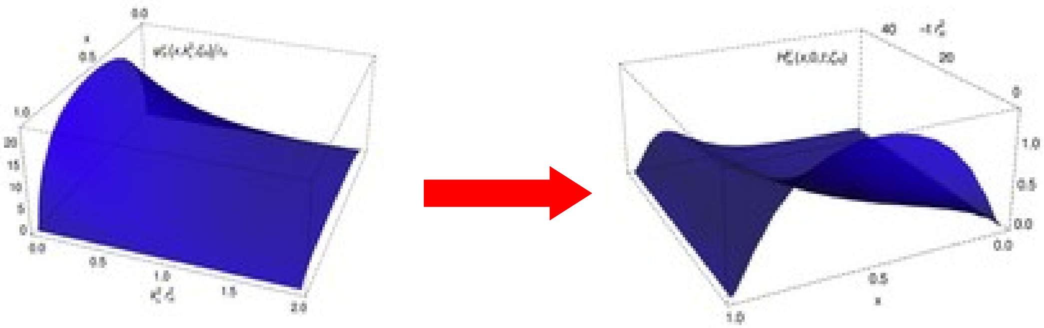
The logarithmically suppressed shear pressure peaks up roughly where the normal pressure takes its zero, indicating that "repulsive" and "confining" forces maximally interfere with each other.

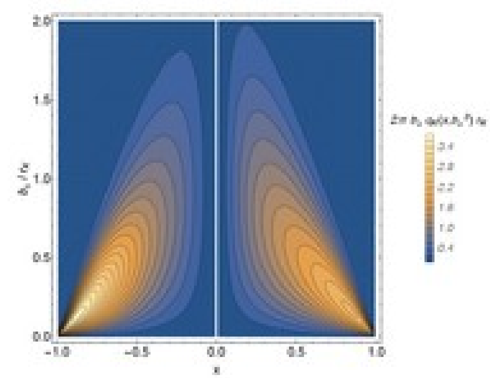
# Epilogue

Pion and kaon Bethe-Salpeter wave functions have been modeled, with the help of either factorization approximation or Nakanishi representation, on the ground of a realistic DSE estimate of PDAs and used to obtain LFWFs.

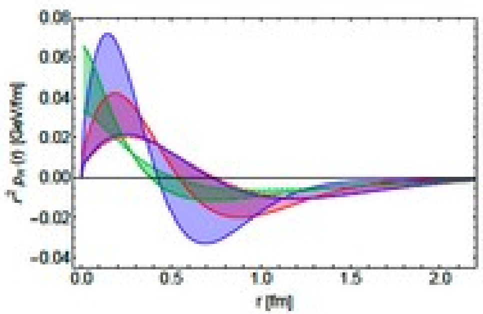
Pion and kaon GPDs are then estimated, within the DGLAP kinematic domain, from the overlap representation of their LFWFs



Particularly, within the framework of the factorization approximation, DSE computations of PDAs and FFs deliver an estimate for the DGLAP GPD.



The impact-parameter dependent GPDs have been also obtained and displayed, thus featuring the spatial distribution of the partons inside the mesons



Gravitational FFs and pressure distributions have been illustrated in the case of the asymptotic Nakanishi (also factorized) LFWF, for which ERBL completion is available.

# Backslides

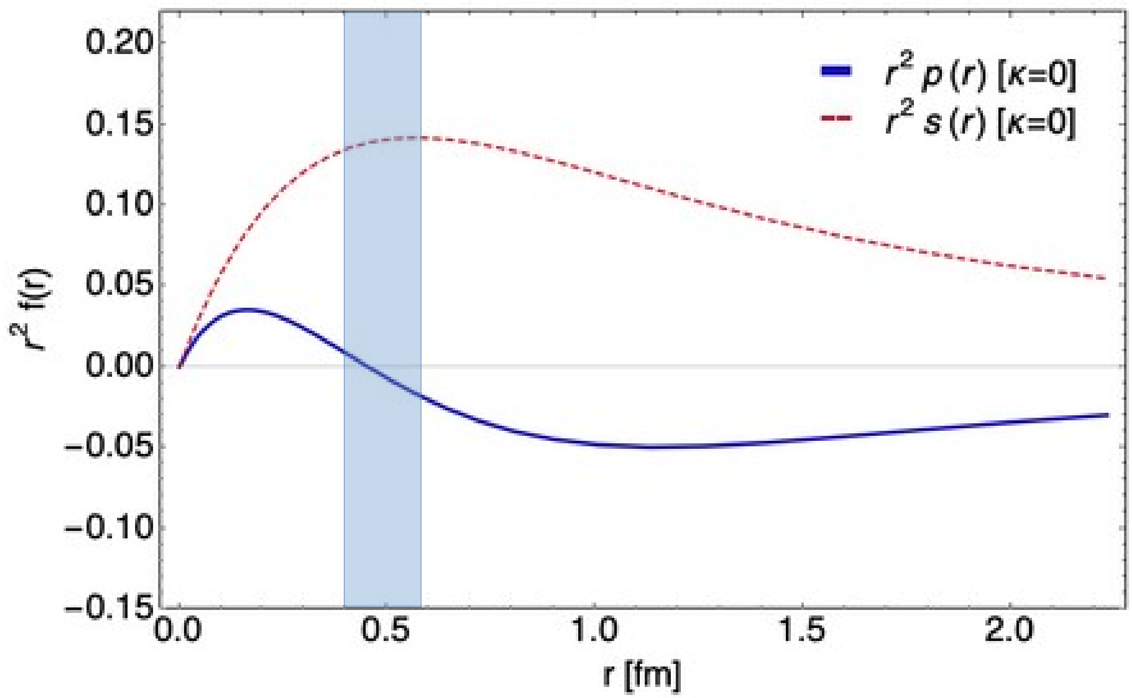
Very preliminary results: 2-d FT pressure distributions

$$p(r) = \frac{1}{3}\int \frac{d^2\Delta}{(2\pi)^2}\frac{e^{i\Delta\cdot r}}{2E(\Delta)}[\Delta^2\theta_1(\Delta)] = \frac{1}{3(2\pi)^2}\int_0^\infty d\Delta \frac{\Delta}{2E(\Delta)}[\Delta^2\theta_1(\Delta)]\int_0^{2\pi} d\phi\, e^{i\Delta r\cos(\phi)}$$

$$\Rightarrow\ p(r) = \frac{1}{6\pi}\int_0^\infty d\Delta \frac{\Delta}{2E(\Delta)}[\Delta^2\theta_1(\Delta)]J_0(\Delta r) \quad (1)$$

$$s(r) = -\frac{3}{4}\int \frac{d^2\Delta}{(2\pi)^2}\frac{e^{i\Delta\cdot r}}{2E(\Delta)}P_2(\hat{\Delta}\cdot\hat{r})[\Delta^2\theta_1(\Delta)] = -\frac{3}{4(2\pi)^2}\int_0^\infty d\Delta \frac{\Delta}{2E(\Delta)}[\Delta^2\theta_1(\Delta)]\left(2\pi J_0(\Delta r) - \frac{3\pi J_1(\Delta r)}{\Delta r}\right)$$

$$\Rightarrow\ s(r) = -\frac{3}{4}p(r) + \frac{9}{16\pi r}\int_0^\infty d\Delta \frac{\Delta}{2E(\Delta)}[\Delta^2\theta_1(\Delta)]\frac{J_1(\Delta r)}{\Delta} \quad (2)$$



# QCD evolution

## DGLAP "at al orders" and effective coupling

➢ **Aprroach:** Define an effective coupling such that the equations below are exact.

$$\frac{d}{dt}q(x;t) = -\frac{\alpha(t)}{4\pi}\int_x^1 \frac{dy}{y} q(y;t) P\left(\frac{x}{y}\right)$$

\- or -

$$\frac{d}{dt}M_n(t) = -\frac{\alpha(t)}{4\pi}\gamma_0^n M_n(t)$$

*i.e.* no LO, NLO, etc:
**all orders are there**

… and identify, not tune, the (*initial*) **hadron scale $\zeta_H$.**
Fully dressed quasiparticles are the correct degrees of freedom.

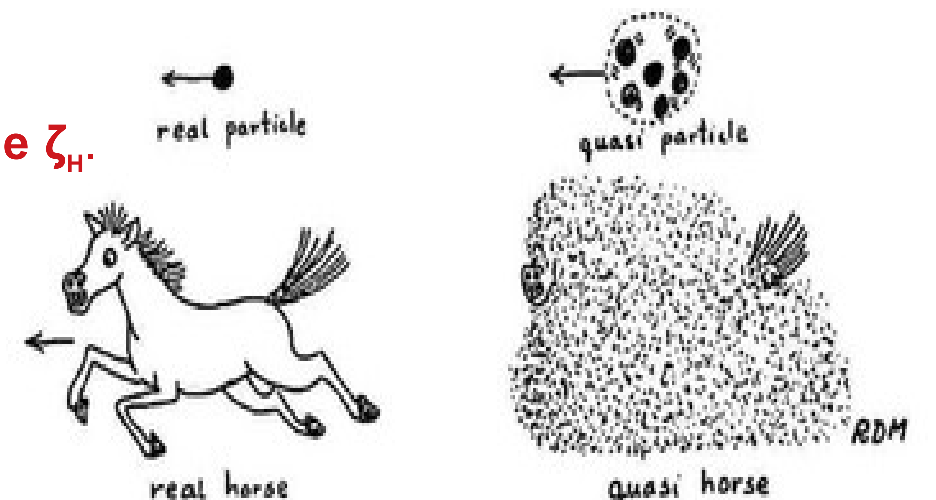



Fig. 0.4 *Quasi Particle Concept*

# QCD evolution

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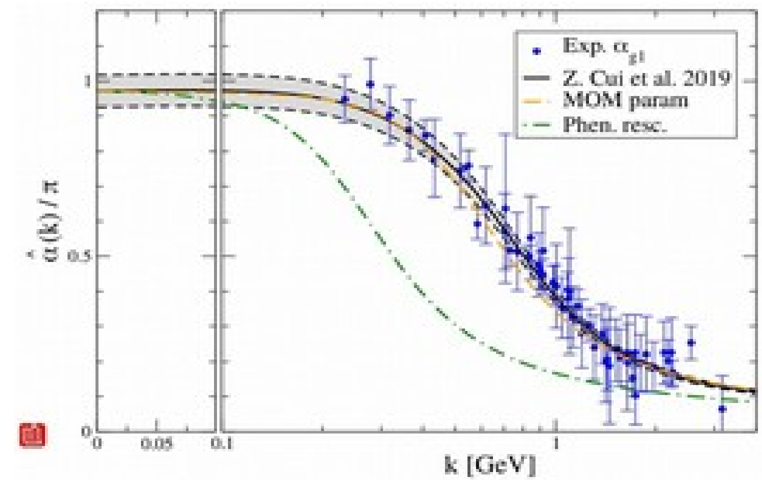



… and identify, not tune, the (*initial*) **hadron scale $\zeta_H$**.
Fully dressed quasiparticles are the correct degrees of freedom.

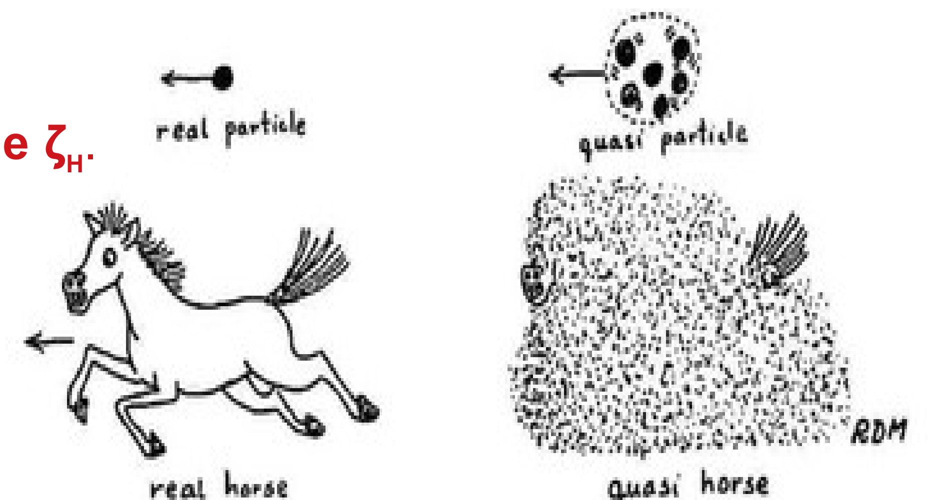



Fig. 0.4 Quasi Particle Concept

- Features of the **PI effective** charge lead to the **answer**.

J. R-Q et al., arXiv: 1909.13802; D. B. et al., PRD 96 (2017) no.5, 054026. J. R-Q. et al., FBS 59 (2018) no.6, 121; Z-F Cui et al., CPC 44 (2020) 8, 083102

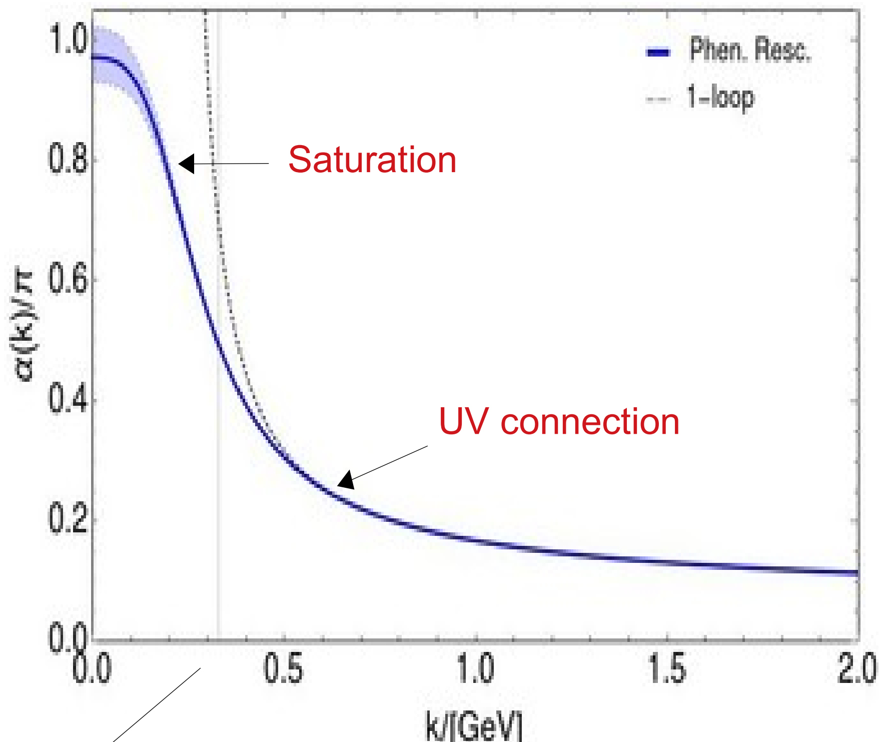
# QCD evolution

## The IR fixed point from PI effective charge:

- **Approach:** Define an effective coupling such that:

$$\frac{d}{dt}q(x;t) = -\frac{\alpha(t)}{4\pi}\int_x^1 \frac{dy}{y} q(y;t) P\left(\frac{x}{y}\right)$$

**"All orders hypothesis"**




- The ***coupling***:

$$\alpha(k^2) = \frac{\gamma_m \pi}{\ln\left[\frac{\mathcal{M}^2(k^2)}{\Lambda^2_{\rm QCD}}\right]} \; ; \; \alpha(0) = 0.97(4)$$

Where $\mathcal{M}(k^2 = \Lambda^2_{\rm QCD}) := m_G = 0.331(2)\ \text{GeV}$ defines a *screening* mass.

→We identify: $\zeta_H := m_G(1 \pm 0.1)$ 10% uncertainty

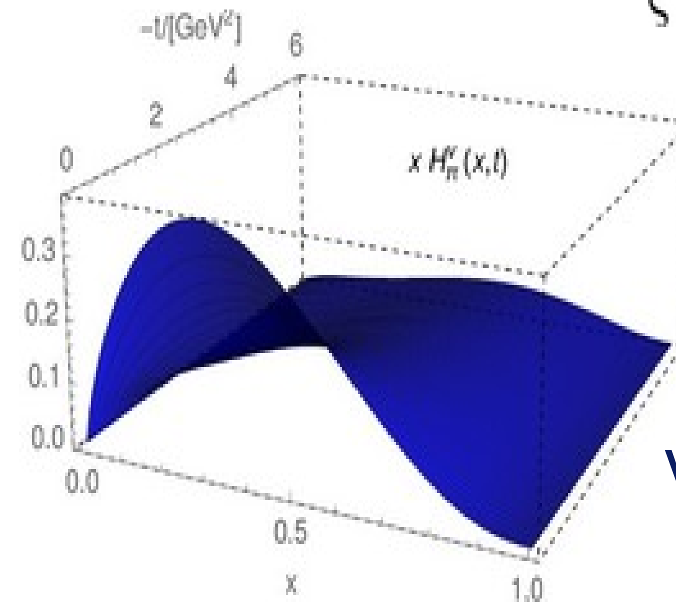
# QCD evolution

## Evolved GPDs:

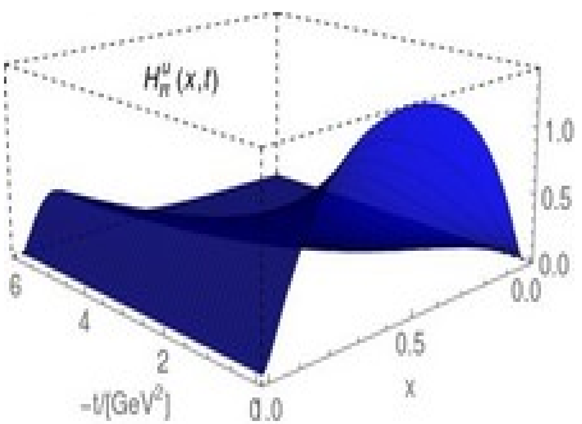
- Starting with **valence** distributions, at *hadron scale*, and generate **gluon** and **sea** distributions via evolution equations.

- Thus **gluon** and **sea** **GPDs** are obtained.

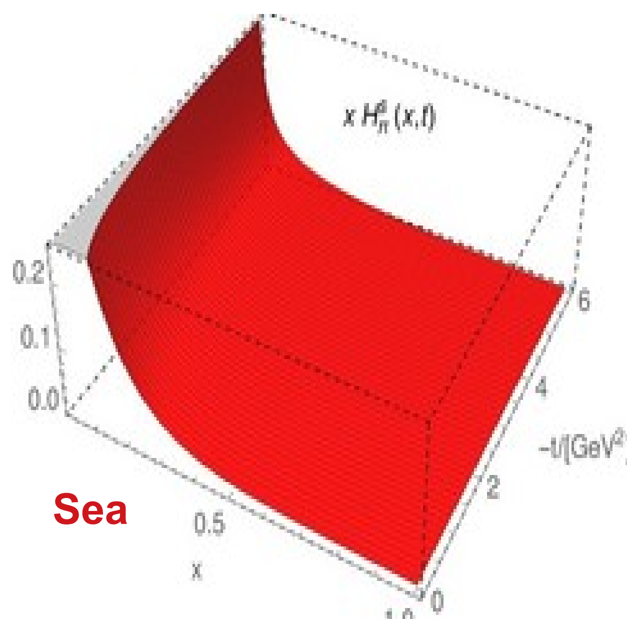
$\zeta = 5.2\ \mathrm{GeV}$



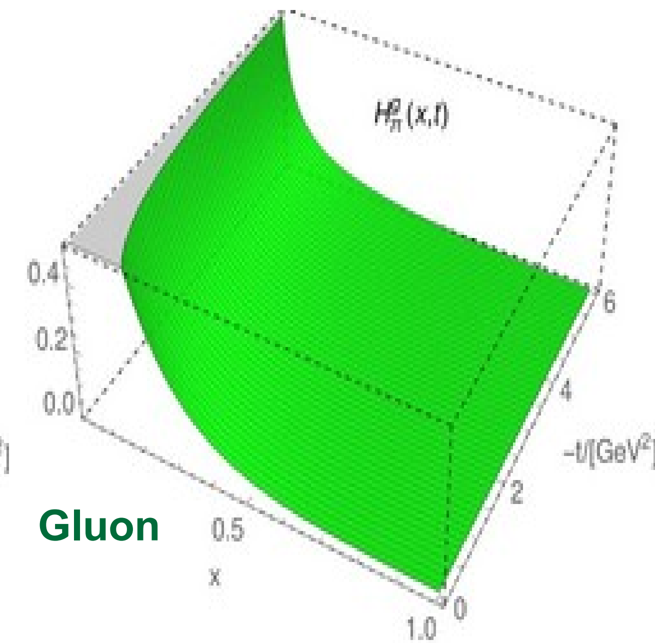

**Valence**




$\zeta_H = 0.331\ \mathrm{GeV}$




**Sea**



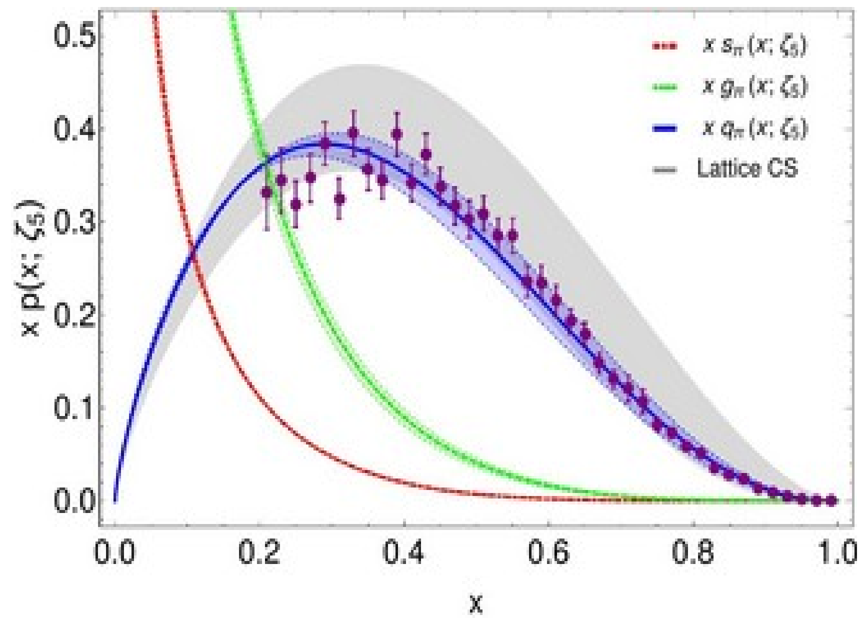

**Gluon**

# QCD evolution

## Evolved PDFs:

- Starting with **valence** distributions, at *hadron scale*, and generate **gluon** and **sea** distributions via evolution equations.
- Thus **gluon** and **sea** **GPDs** are obtained.
- The *forward* limit corresponds to the **PDFs**.

$\zeta = 5.2\ \mathrm{GeV}$




| $\zeta$ | $<x>_{\rm V}$ | $<x>_{\rm G}$ | $<x>_{\rm S}$ |
|---|---|---|---|
| 2 GeV | 0.483(42) | 0.411(24) | 0.106(18) |
| 5.2 GeV | 0.412(36) | 0.449(19) | 0.138(17) |

**Data**: M. Aicher et al., PRL 105 (2010) 252003

**Lattice**: R.S. Sufian et al., arXiv:2001.04960

# QCD evolution

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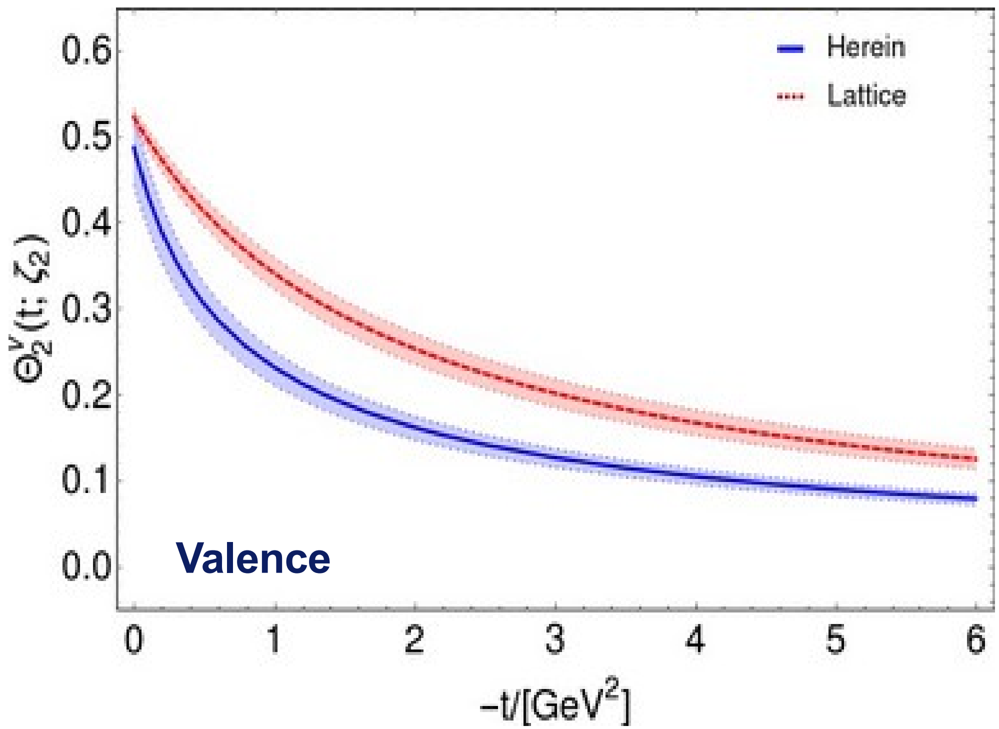
- Thus **gluon** and **sea** **GPDs** are obtained.
- The *forward* limit corresponds to the **PDFs**.
- **GFF** $\theta_2(t)$ comes from the *off-forward* <x>.

$\zeta = 2\ \mathrm{GeV}$




**Herein:** softer GFF,

**Lattice**: D. Brommel, PhD. Thesis 2007

$$\Theta_2(0)/2 = 0.26(4) \quad (m_\pi^2 > 0.3\ \mathrm{GeV}^2)$$

# QCD evolution

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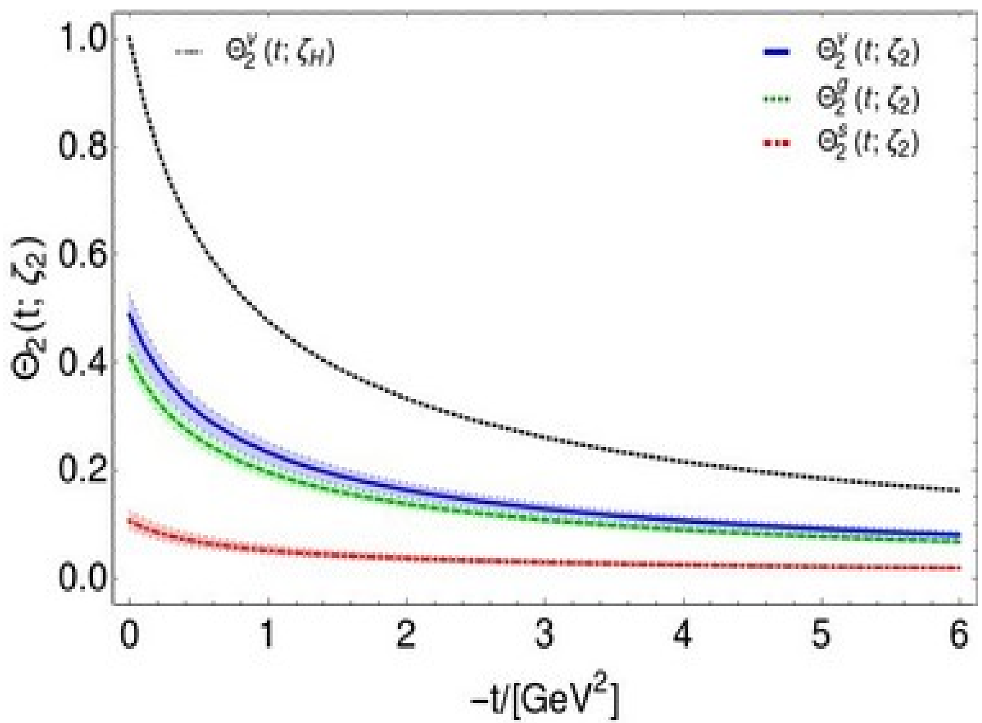

# QCD evolution

## Evolved GFFs:

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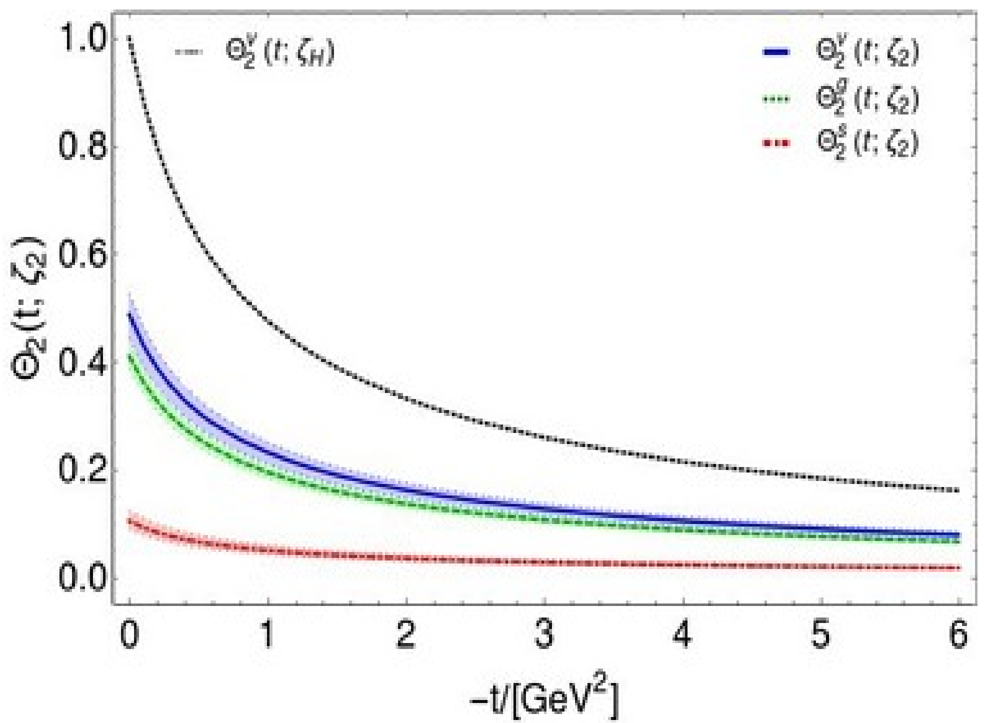

# QCD evolution

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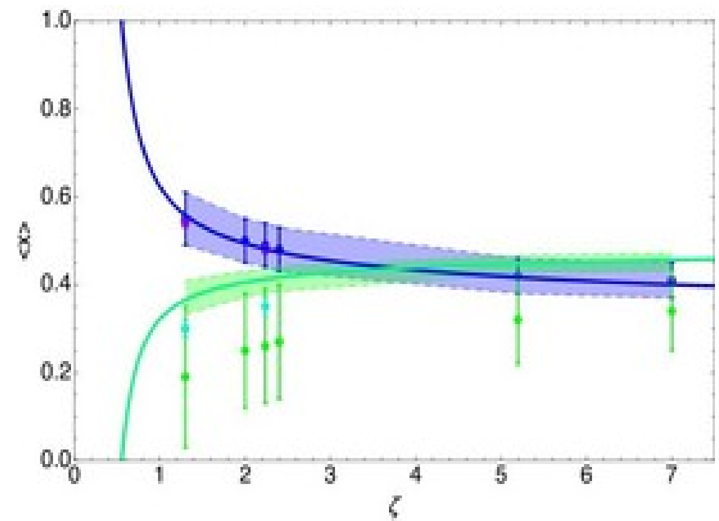
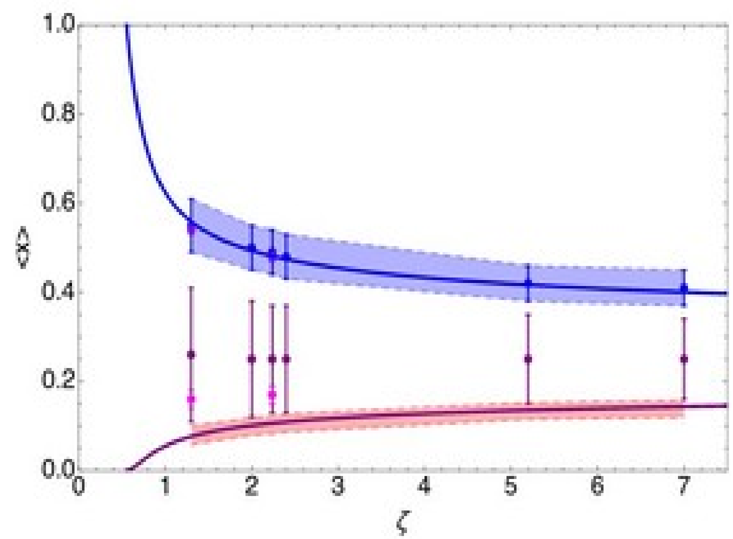
- The *forward* limit corresponds to the **PDFs**.

- **GFF** $\theta_2(t)$ comes from the *off-forward* <x>

- One can also test the evolution with the scale, for instance for the momentum fraction



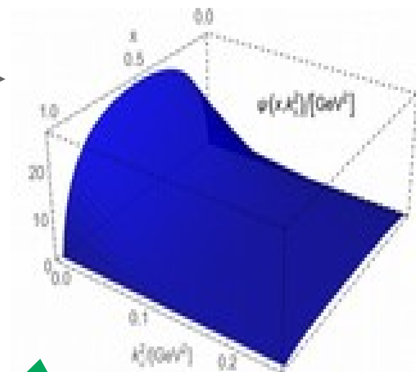

I. Novikov et al, arXiv: 2002.02902 "**xFitter**"

# Summary: Pion

- Using our **DSE prediction** of pion PDF as **benchmark**, we modeled the pion **BSWF**.

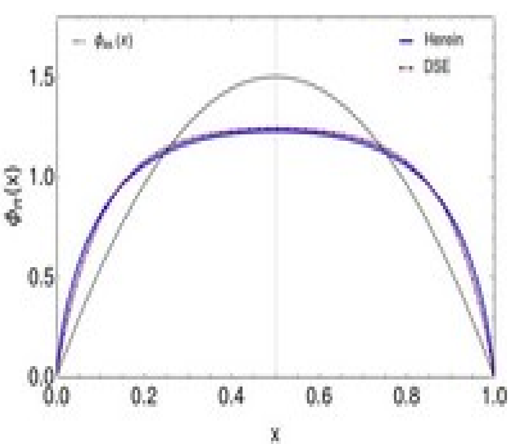
DSE: CPC 44 (2020) no.3, 031002, PRD 101 (2020) no.5, 054014

**BSWF**

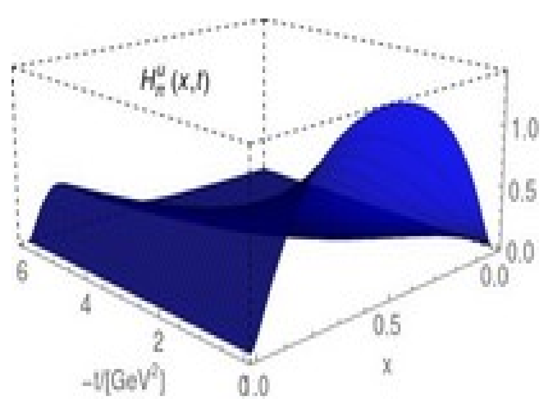


$- \int dk_\perp$

**LFWF**




**PDA**



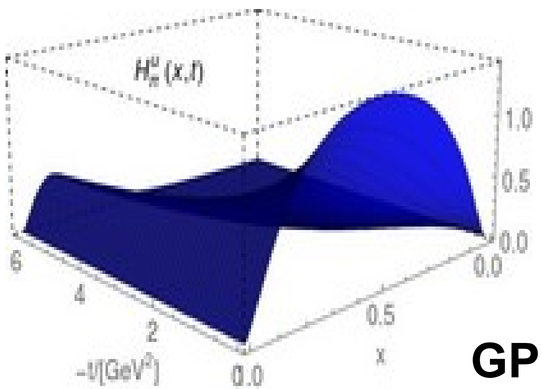

**GPD**

- **Consistent** features of the **PDA**:
  - ✔ Broad and concave at real world scales.
  - ✔ Correct endpoint behavior.
  - ✔ Agreement with **Lattice** and **DSE** results.

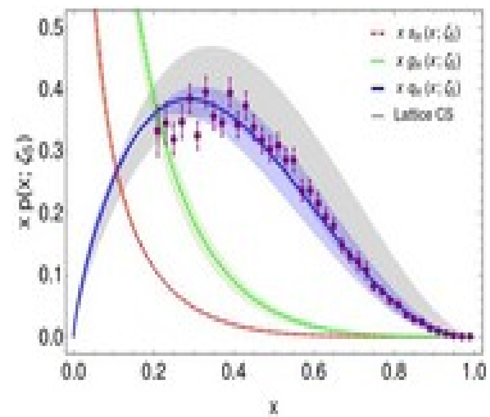
- The valence **GPD** is obtained from the **overlap** representation.
  - ✔ Limited to the **DGLAP** region.
  - ✔ **Gluon** and **sea** obtained from *evolution* equations.
  - ✔ **Extension** to ERBL region is possible.

(but insufficient)

# Summary: Pion



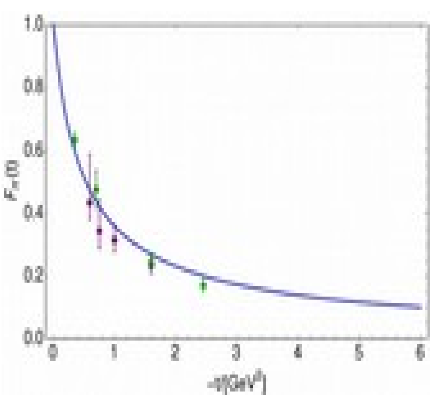

GPD



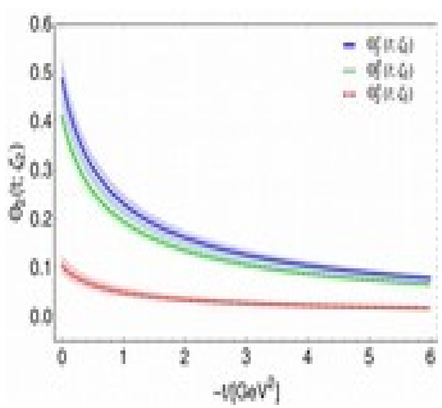

PDF

$\int dx$

$t = 0, \xi = 0$




EFF

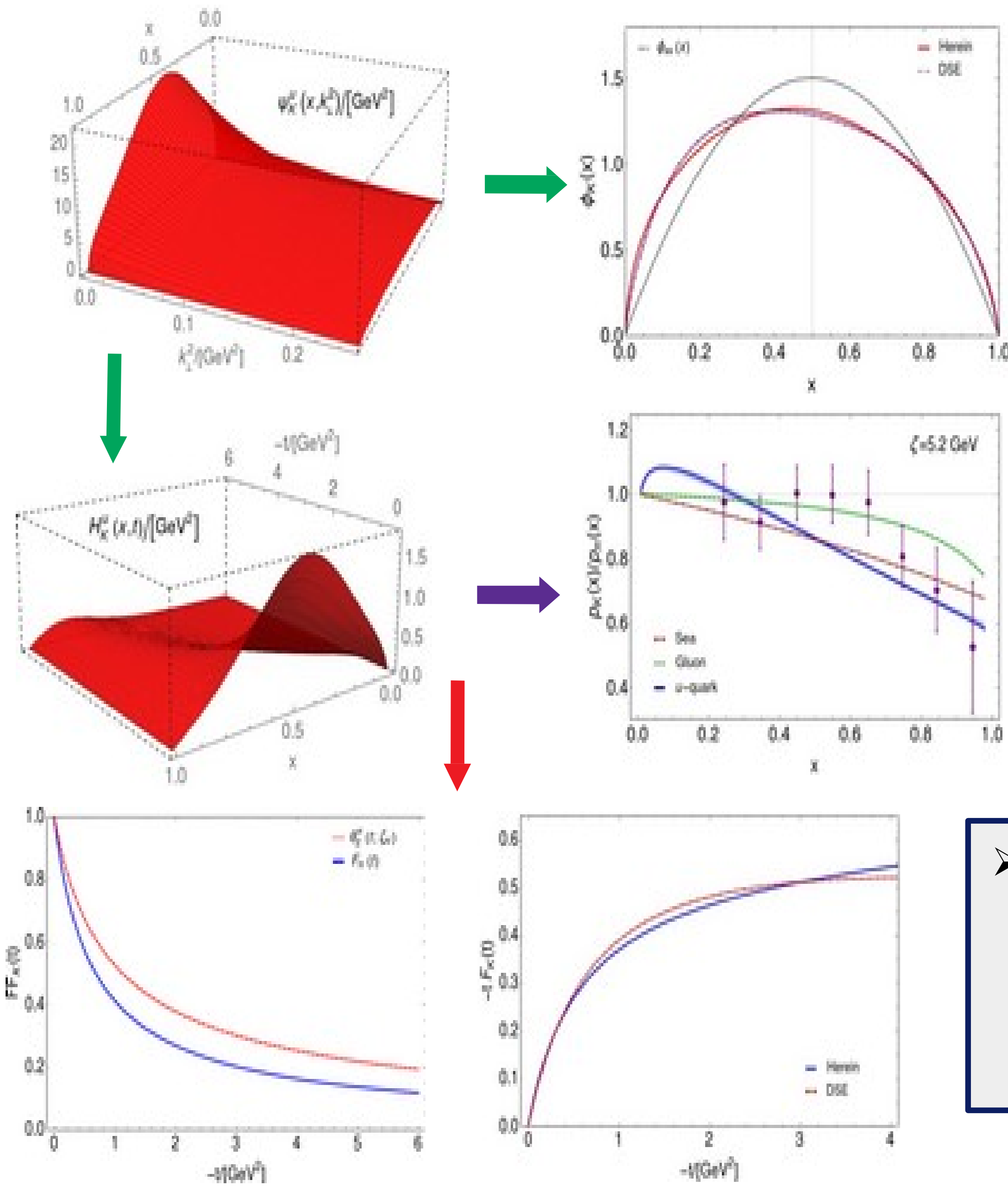



GFF

- Connection **PDF** with **DSE predictions** implies:
  - ✔ Keen agreement with reanalyzed data.
  - ✔ Large-x behavior as predicted by **pQCD**.
  - ✔ Compatible with novel **Lattice** results.
- **EFF *consistent*** with empirical data.
  - ✔ One can trust the off-forward quantities.
  - ✗ ERBL region + D-term **needed.**
- Intimate ***connection*** with the **running coupling**:
  - ✔ **PI** effective charge → effective coupling for **evolution**.
  - ✔ Specific ***definition*** of the hadron scale.

➔Both **LFWF** and **GPD** are **promising candidates** to be the real objects.

# Summary: Kaon




➢ Connection with **DSE predictions** implies:

- ✔ **Qualitiative** features of the distributions are properly captured.
- ✔ *Large-x* behavior of the **PDA** and **PDF** as predicted by **pQCD**.
- ✔ **K/pi PDF** ratio in agreement with data.

We still need new experiments !!!

✔ Computed **Gluon** and **Sea** Kaon **PDFs**

the **GPDs** are available too !!!

➢ Next steps:

- ✔ **Impact** parameter distributions
- ✔ **Transverse** momentum distributions **(TMDs)**