Positivity in Multi-field EFTs

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Based on 2005.03047 with S.-Y. Zhou (PRL 125, 201601),
2101.01191 with X. Li, H. Xu, C. Yang, and S.-Y. Zhou

Positivity bounds

- EFTs describe the IR behavior of some "UV completion", by integrating out its heavy dofs.
- But in a bottom-up view: not all EFTs have a UV completion!
- Using axiomatic principles of QFT, including causality, unitarity, Lorentz symmetry, etc., bounds can be placed on (the signs of) (combinations of) Wilson coefficients.
  - 2-to-2 amplitude $A_{2\to 2}(s,t) = c_0 + c_2 s^2 + c_{2,1} s^2 t + c_4 s^4 + \cdots + c_{n,m} s^n t^m$
  - $c_2 > 0$; or in SMEFT: $C^{(8)} > 0$ [A. Adams et al., JHEP 06]
  - More bounds on higher-s dependence. Talk by F. Riva.
  - Other recent developments [B. Bellazzini et al., 2011.00037] [A. Tolley et al., 2011.02400] [Caron-Huot and Van Duong, 2011.02957] [T. Trott, 2011.10058] [Arkani-Hamed et al., 2012.15849]

- We are interested in extracting the positivity bounds of:
  - Leading energy dependence only, s^2. (Dim-8)

$$A(s,0) = c_0 + c_2 s^2 + c_4 s^4 + \cdots$$

  - EFTs with more than one fields/particles. (e.g. **SMEFT operators;** or those involving multiplet particles, chiral PT, spin-2 EFTs, ...)
- Somewhat orthogonal to talk by F. Riva yesterday (one field but with all dimensions > 8).
- Motivation: phenomenologically interesting scenarios, **SMEFT**, massive gravity, chiral PT, inflation... etc.

Main result:

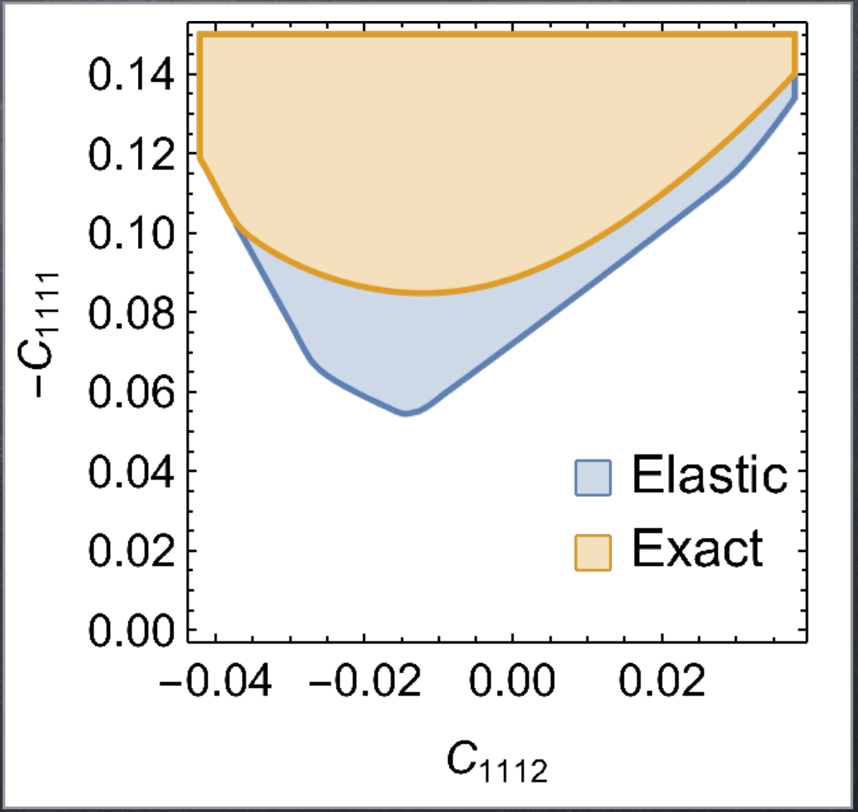
- Finding bounds is a geometric problem:
  finding the extremal rays of a spectrahedron.

  - Small number of fields: analytical solution.

  - Large number of fields: solve by semi-definite programming (SDPs).

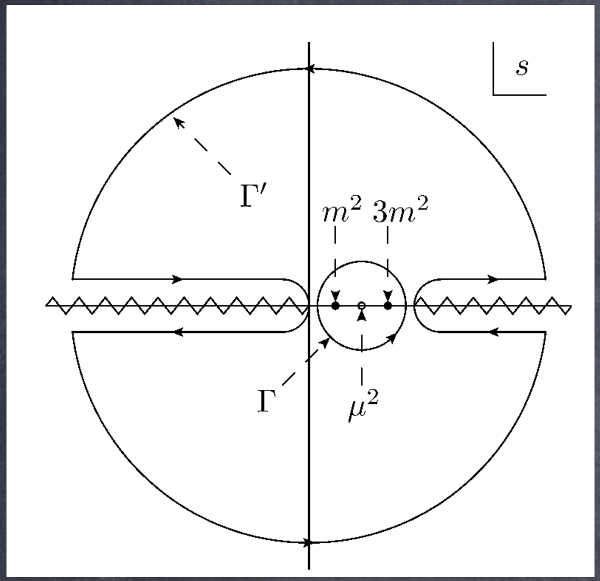
2-scalar EFT

$$
\begin{aligned}
&C_{1111} \geq 0 \quad \text{and} \quad 4C_{1111}C_{1212} - C_{1112}^2 \geq 0 \\
&\text{and} \quad \Big\{ C_{1112}C_{1122}C_{1222} - C_{1111}C_{1222}^2 - C_{1112}^2 C_{2222} + C_{1212}\left(-C_{1122}^2 + 4C_{1111}C_{2222}\right) \geq 0 \\
&\text{or} \quad \Big[\Delta \equiv 3\left(4C_{1111}C_{2222} - C_{1112}C_{1222}\right) + \left(C_{1122} + C_{1212}\right)^2 \geq 0 \\
&\text{and} \quad \frac{3C_{1112}^2}{4C_{1111}} - 2\left(C_{1122} + C_{1212}\right) \leq \sqrt{\Delta} \leq C_{1212} - 2C_{1122} \\
&\text{and} \quad 2\Delta^{3/2} \geq 27\left(C_{1111}C_{1222}^2 + C_{1112}^2 C_{2222}\right) - 9\left(C_{1122} + C_{1212}\right)\left(8C_{1111}C_{2222} + C_{1112}C_{1222}\right) \\
&\qquad\qquad + 2\left(C_{1122} + C_{1212}\right)^3 \Big] \Big\}
\end{aligned}
$$



Dispersion relation

$$f = \frac{1}{2\pi i}\oint_{\Gamma} \mathrm{d}s \frac{A(s,0)}{(s-\mu^2)^3}$$ (not necessarily elastic)

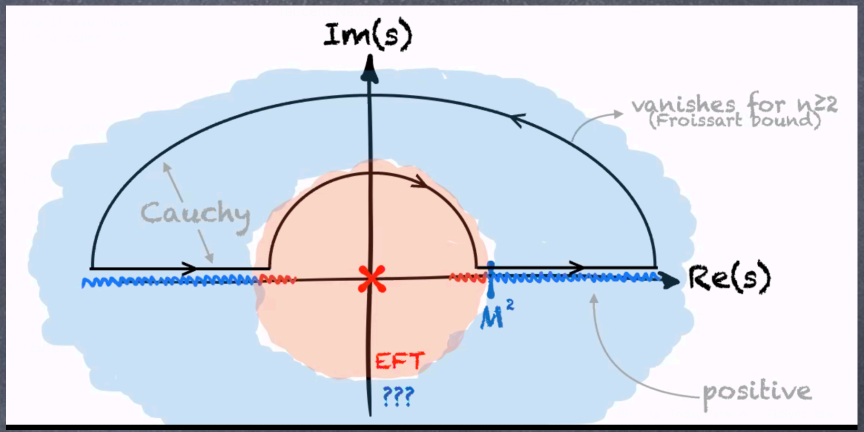


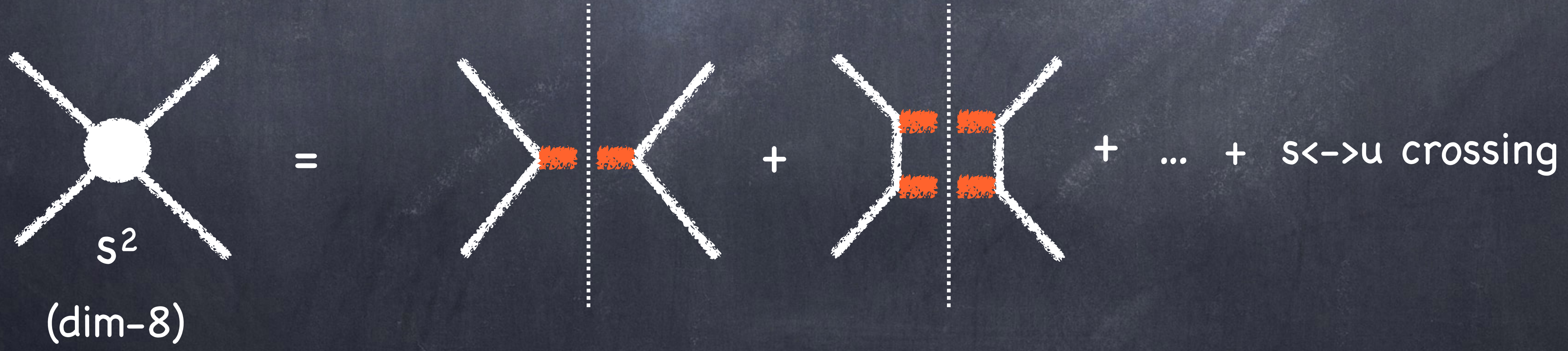
Causality -> Cauchy's integral formula

Unitarity -> Optical theorem
Froissart bound

$$A(s,0) \leq \mathcal{O}(s \ln^2 s)$$

See also talk by Riva





ijkl: particle index

$1 \le i, j, k, l \le n$

$M^2 = m_i^2 + m_j^2 + m_k^2 + m_l^2$

Forward scattering amp, at **low energy** (calculable in EFT)

$$\frac{d^2}{ds^2} M_{ij\to kl}\left(s = \frac{1}{2}M^2, t = 0\right)$$

$$= \sum_X{}' \int_{(\epsilon\Lambda)^2}^{\infty} \frac{ds\, M_{ij\to X}(s, \Pi_X) M^*_{kl\to X}(s, \Pi_X)}{\pi\left(s - \frac{1}{2}M^2\right)^3} + (j \leftrightarrow l)$$

s<->u crossing

$\epsilon < 1$

X = BSM states summation & PS integration

Amplitude of SM -> X

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$$M^{ijkl} = \sum_X{}' \int_{(\epsilon\Lambda)^2}^{\infty} \frac{d\mu\, {m_X}^{ij} {m_X}^{kl}}{\pi(\mu - \frac{1}{2}M^2)^3} + (j \leftrightarrow l)$$

where $M^{ijkl} \equiv \frac{d^2}{ds^2} M_{ij\to kl}\left(\frac{1}{2}M^2\right), \quad m_X^{ij} \equiv M_{ij\to X}(\mu, \Pi_X)$

M^{ijkl} contains all scattering info at D8, and is calculable in SMEFT.

$$M^{ijkl} = \sum_\alpha C_\alpha^{(8)}/\Lambda^4 M_\alpha^{ijkl}$$

The standard approach: elastic positivity

$$M^{ijkl} = {\sum_X}' \int_{(\epsilon\Lambda)^2}^{\infty} \frac{d\mu \, {m_X}^{ij} {m_X}^{kl}}{\pi(\mu - \frac{1}{2}M^2)^3} + (j \leftrightarrow l)$$

Simplest generalization from 1-scalar EFT to multi-fields:

- When i=k, j=l, (i j -> i j),
  RHS -> $\mathrm{Tr}\left(mm^T\right) \geq 0$ i.e. $M^{ijij} \geq 0$

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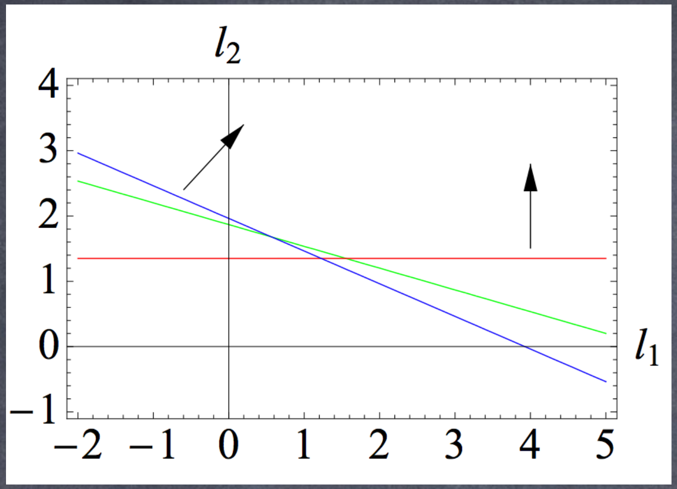
Dim-8 coef. space

- Superposition: $M(|u\rangle + |v\rangle \ \rightarrow \ |u\rangle + |v\rangle) = u^i v^j u^{*k} v^{*l} \cdot M^{ijkl}$
  with superposed states: $|u\rangle = u^i \, |i\rangle \, , \ |v\rangle = v^i \, |i\rangle$
  RHS -> $|u \cdot m_X \cdot v|^2 + |u \cdot m_X \cdot v^*|^2 \geq 0$
  i.e. $u^i v^j u^{*k} v^{*l} M^{ijkl} \geq 0$

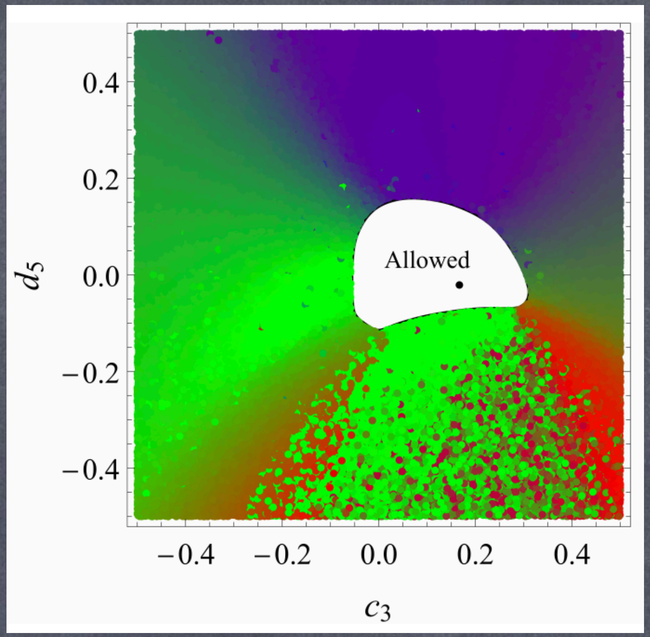
Dim-8 coef. space

Chiral PT



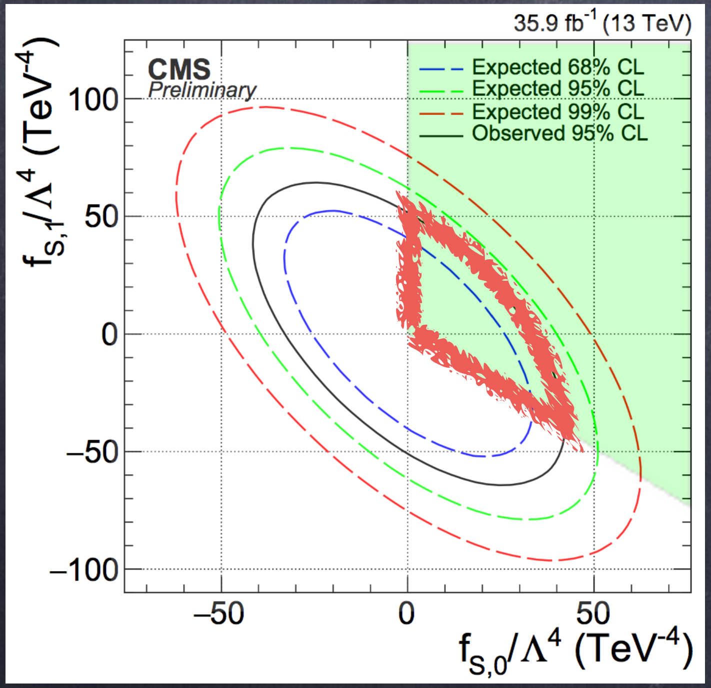
[0801.3222, Manohar & Mateu]

dRGT massive gravity



[1601.04068, Cheung & Remmen]

Anomalous quartic-gauge boson couplings



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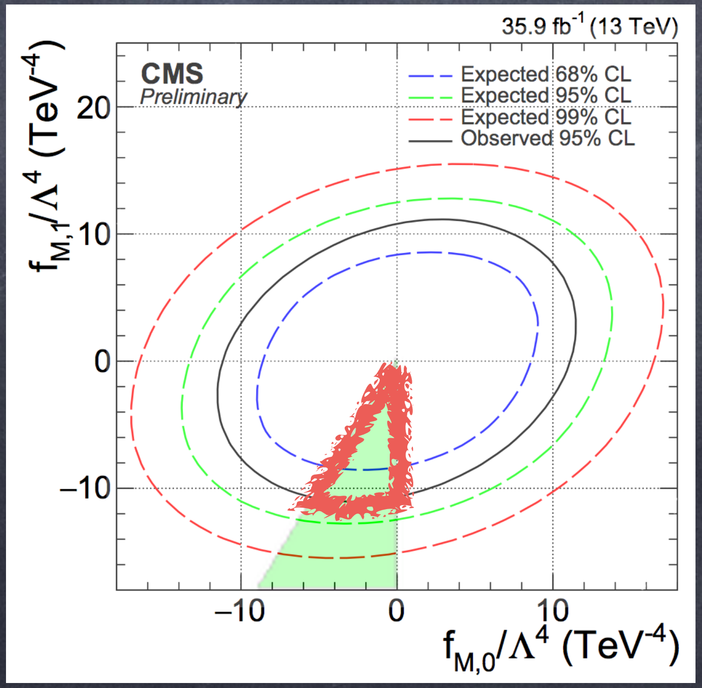
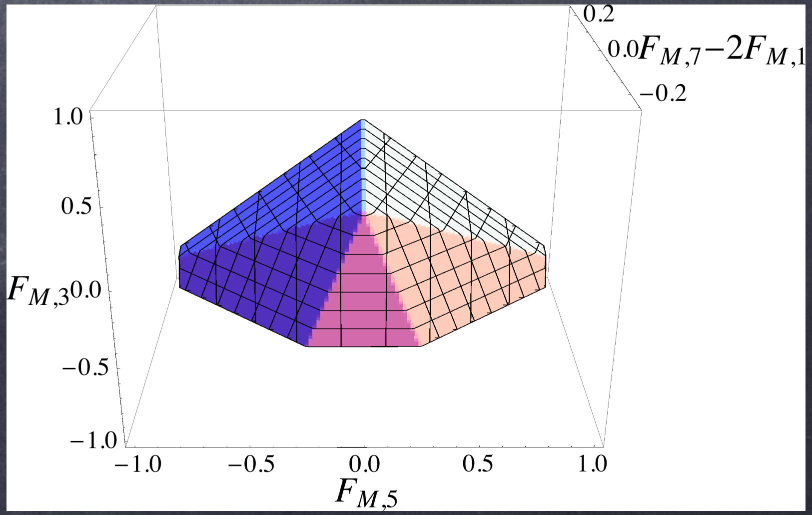


[1902.08977, Q. Bi, CZ, S.-Y. Zhou]



Beyond elastic positivity

- Motivations for an improved approach.
  - $u^i v^j u^{*k} v^{*l} M^{ijkl} \geq 0$ is however to the determination of degree-4 positive semidefinite (PSD) polynomial, which is NP hard.
    - e.g. u,v are 12-dimensional in the aQGC problem.
  - Connection to UV physics is not clear.
  - Elastic bounds are not the optimal.

$$M^{ijkl} = \sum_X{}' \int_{(\epsilon\Lambda)^2}^{\infty} \frac{d\mu\, m_X{}^{ij} m_X{}^{kl}}{\pi(\mu - \frac{1}{2}M^2)^3} + (j \leftrightarrow l)$$

- Being completely ignorant about m, the only essential math structure we need is

i(j|k|l): j,l symmetrized

$$M^{ijkl} \in \mathbf{C}^{n^4} = \mathrm{cone}\left(\left\{m^{i(j}m^{|k|l)}, m \in \mathbb{R}^{n^2}\right\}\right)$$

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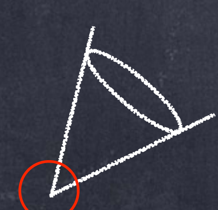
  - Conical hull: The set of all positive linear combinations of elements of X = {x}, is a convex cone, denoted by $C = \text{cone}(X)$
  - Salient cone: if the cone C does not contain any straight line.

    $x \in \mathbf{C}$ and $-x \in \mathbf{C} \Rightarrow x = 0$



$$\mathcal{M}^{ijkl} = \sum_\alpha m_\alpha^{i(j} m_\alpha^{|k|l)} \Rightarrow \mathcal{M}^{ijij} = 2\sum_\alpha m_\alpha^{ij} m_\alpha^{ij} \geq 0$$

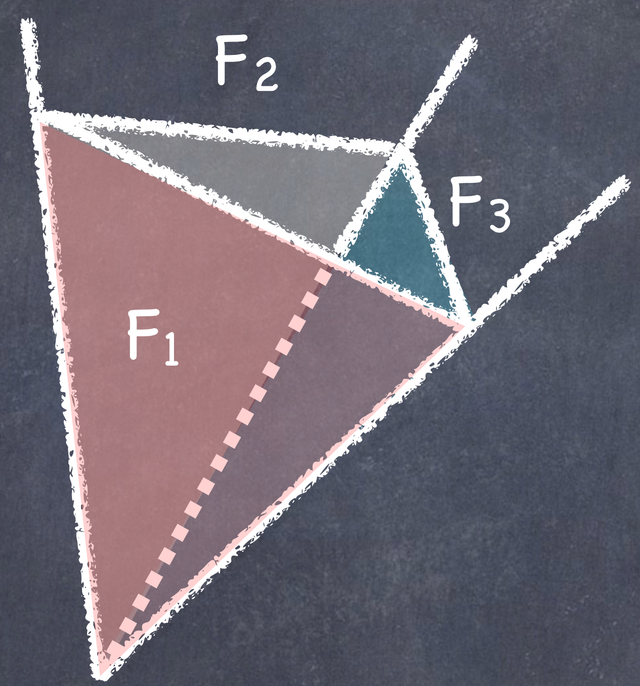
$$-\mathcal{M}^{ijij} = 2\sum_\beta m_\beta^{ij} m_\beta^{ij} \geq 0$$

$$\Rightarrow \mathcal{M}^{ijij} = 2\sum_\alpha m_\alpha^{ij} m_\alpha^{ij} = 0$$

$$\Rightarrow m_\alpha = 0 \Rightarrow \mathcal{M}^{ijkl} = 0$$

Representations of polyhedral cones

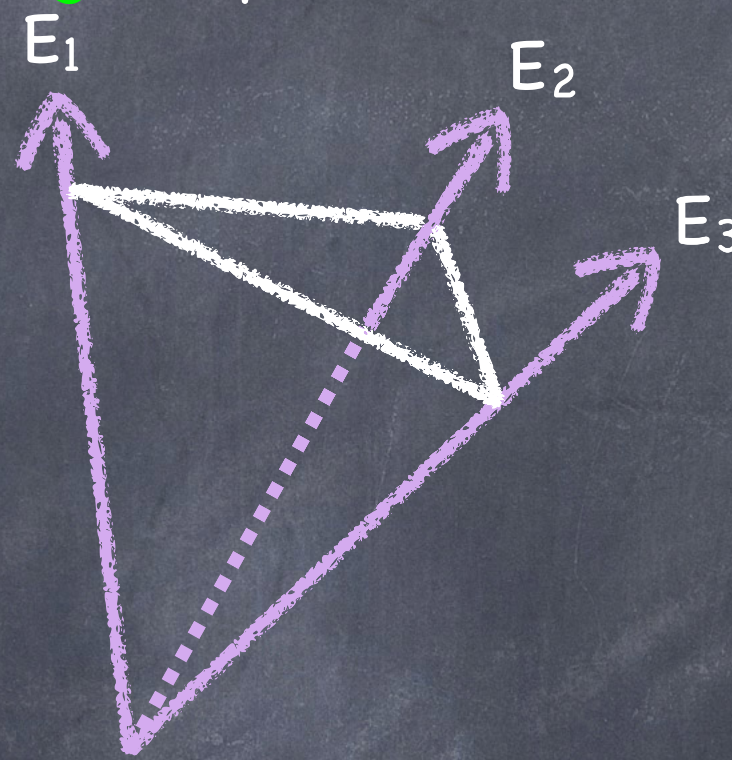
face-representation



The polyhedral cone {x} is bounded by hyper-planes:

$$\vec{x}: \quad \begin{aligned} \vec{n}_1 \cdot \vec{x} &\geq 0 \\ \vec{n}_2 \cdot \vec{x} &\geq 0 \\ &\vdots \\ \vec{n}_N \cdot \vec{x} &\geq 0 \end{aligned}$$

edge-representation



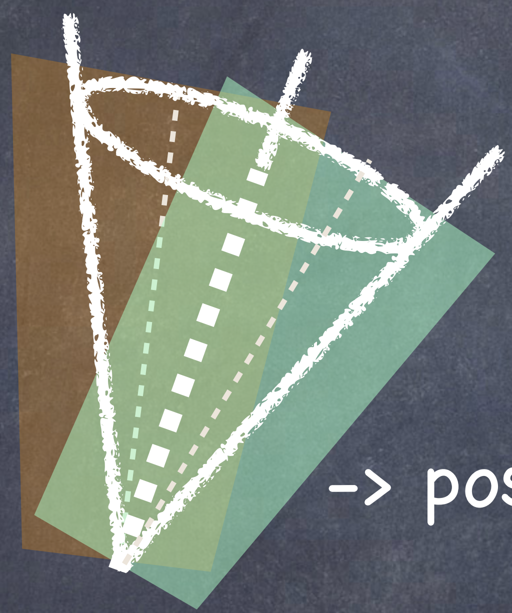
Edges are the generators of {x}

$$\vec{x}: \quad \vec{x} = \sum_i w_i \vec{E}_i \qquad (w_i \geq 0)$$

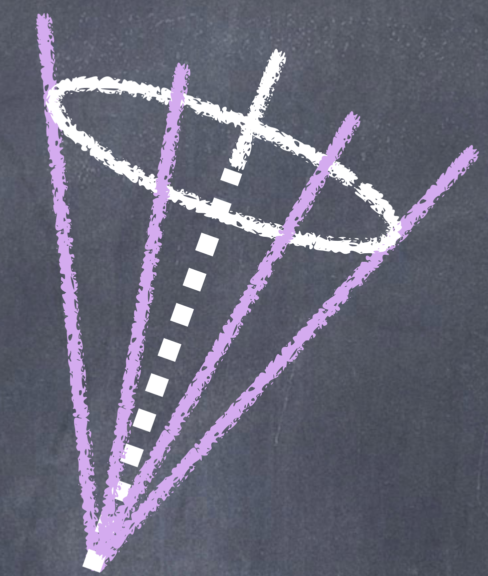
$$\{\vec{x}\} = \mathrm{cone}\left(\left\{\vec{E}_i\right\}\right)$$ (conical hull)

Representations of general cones

inequality-representation



extremal-representation



- **Extremal Ray (ER)**: An element x is an extremal ray of cone C, if it **cannot** be split into two other elements linearly independent:

$$\text{if } x = u + v \text{ and } u, v \in C, \text{ then } x = \lambda u \text{ or } x = \lambda v, \lambda > 0$$

  - Denote the set of ERs by ext({x})
  - (Krein-Milman theorem): a salient cone C is a conical hull of its **ERs,** C = cone(ext(C)), i.e. generators = ERs

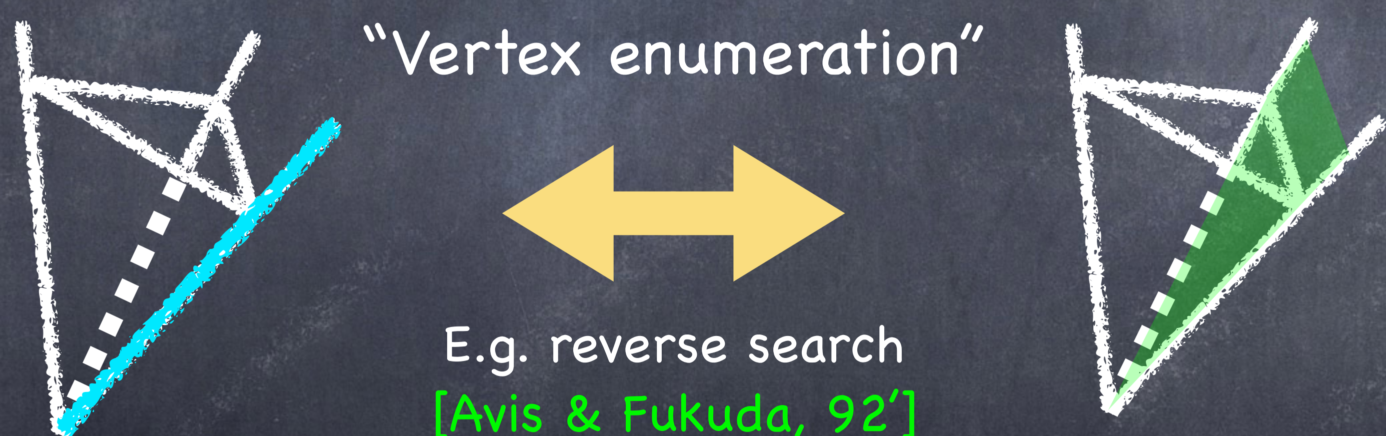
- **Solution #1**: if the EFT has symmetries, m^{ij} is determined by symm.
  => Use the "extremal representation approach": [**CZ** and S.-Y. Zhou PRL 125, 201601]

  - The ERs/generators of the cone are "projectors"
    [see also 1405.2960 Bellazzini et al.]

$$\text{cone}\left(\left\{m^{i(j}m^{|k|l)}, m \in \mathbb{R}^{n^2}\right\}\right)$$

$$\rightarrow \quad \text{cone}\left(\left\{P_r^{i(j|k|l)}\right\}\right) \qquad P_r^{ijkl} = \sum_\alpha C_{i,j}^{r,\alpha}\left(C_{k,l}^{r,\alpha}\right)^*$$

  - "vertex enumeration" to get facets.

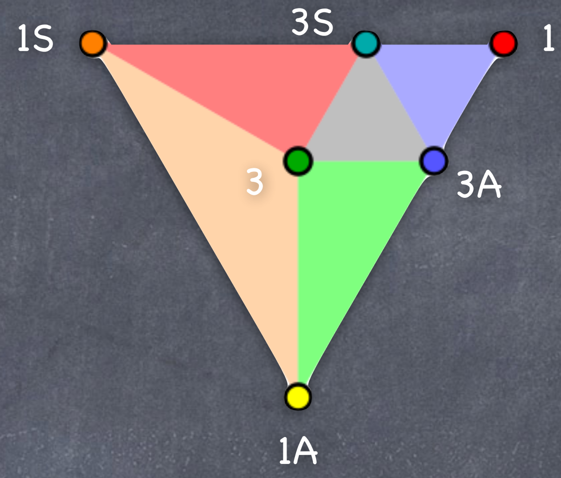


  - ERs <=> UV states.
    (For pheno applications see [B. Fuks, Y. Liu, **CZ**, S.-Y. Zhou 2009.02212])

4-Higgs operators:

$$O_{S,0} = [(D_\mu\Phi)^\dagger D_\nu\Phi] \times [(D^\mu\Phi)^\dagger D^\nu\Phi] \qquad f_{S,0} \geq 0$$
$$O_{S,1} = [(D_\mu\Phi)^\dagger D^\mu\Phi] \times [(D_\nu\Phi)^\dagger D^\nu\Phi] \qquad f_{S,0} + f_{S,2} \geq 0$$
$$O_{S,2} = [(D_\mu\Phi)^\dagger D_\nu\Phi] \times [(D^\nu\Phi)^\dagger D^\mu\Phi] \qquad f_{S,0} + f_{S,1} + f_{S,2} \geq 0$$

Triangular cone

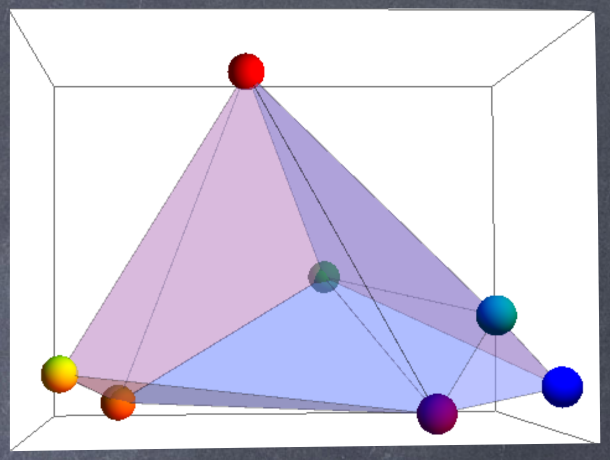


4-W operators:

$$O_{T,0} = \mathrm{Tr}[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}]\mathrm{Tr}[\hat{W}_{\alpha\beta}\hat{W}^{\alpha\beta}]$$
$$O_{T,2} = \mathrm{Tr}[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}]\mathrm{Tr}[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}]$$
$$O_{T,1} = \mathrm{Tr}[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}]\mathrm{Tr}[\hat{W}_{\mu\beta}\hat{W}^{\alpha\nu}]$$
$$O_{T,10} = \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\mathrm{Tr}[\hat{W}_{\alpha\beta}\tilde{W}^{\alpha\beta}]$$

$$F_{T,2} \geq 0,$$
$$4F_{T,1} + F_{T,2} \geq 0,$$
$$F_{T,2} + 8F_{T,10} \geq 0,$$
$$8F_{T,0} + 4F_{T,1} + 3F_{T,2} \geq 0,$$
$$12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10} \geq 0,$$
$$4F_{T,0} + 4F_{T,1} + 3F_{T,2} + 12F_{T,10} \geq 0.$$

6-facet 4D cone

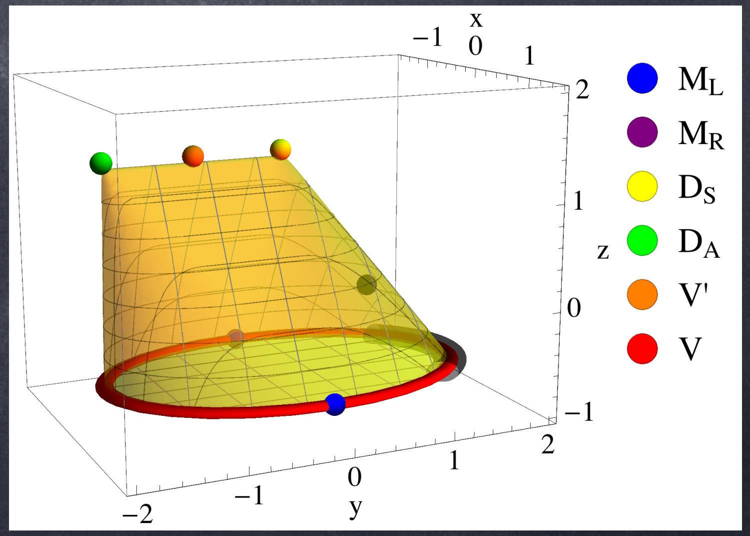


4-electron operators:

$$O_1 = \partial^\alpha(\bar{e}\gamma^\mu e)\partial_\alpha(\bar{e}\gamma_\mu e)\ ,$$
$$O_2 = \partial^\alpha(\bar{e}\gamma^\mu e)\partial_\alpha(\bar{l}\gamma_\mu l)\ ,$$
$$O_3 = D^\alpha(\bar{e}l)\ D_\alpha(\bar{l}e),$$
$$O_4 = \partial^\alpha(\bar{l}\gamma^\mu l)\ \partial_\alpha(\bar{l}\gamma_\mu l)\ ,$$

$$C_1 \leq 0,\ C_3 \geq 0,\ C_4 \leq 0$$
$$2\sqrt{C_1C_4} \geq C_2,$$
$$2\sqrt{C_1C_4} \geq -(C_2 + C_3).$$

4D "circular cone"



[CZ and S.-Y. Zhou 2005.03047]

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$$f_{S,0} \geq 0$$
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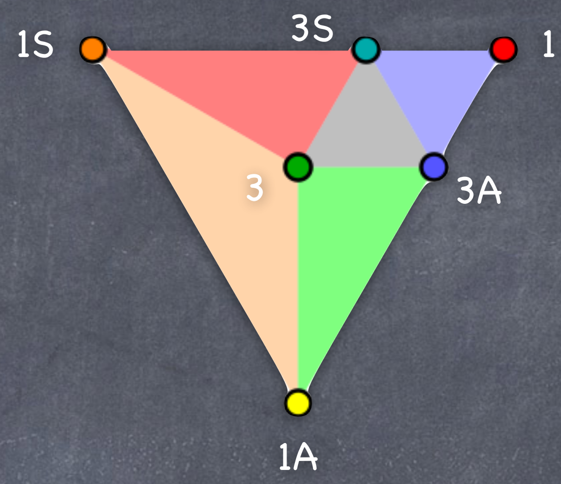
Triangular cone



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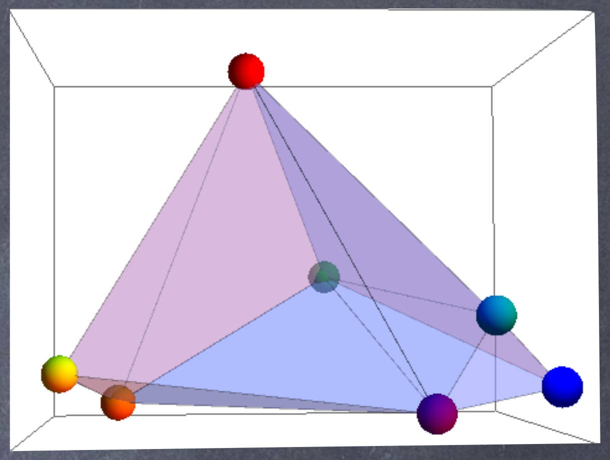
Beyond elastic positivity!

6-facet 4D cone



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$$C_1 \leq 0,\ C_3 \geq 0,\ C_4 \leq 0$$
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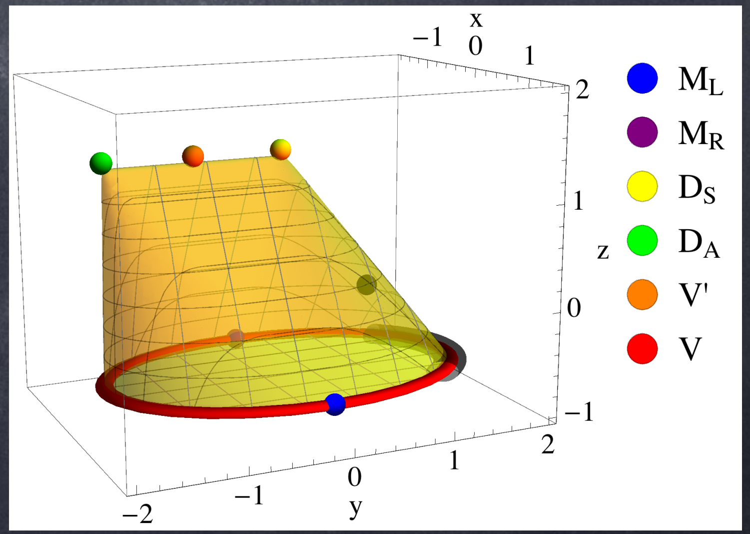
$$2\sqrt{C_1 C_4} \geq -(C_2 + C_3).$$

4D "circular cone"



[CZ and S.-Y. Zhou 2005.03047]

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Solution #2: use duality.

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  - Are they enough to carve out exactly the original cone C?

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  - i.e. set of **all** valid linear bounds.
  - Are they enough to carve out exactly the original cone C?
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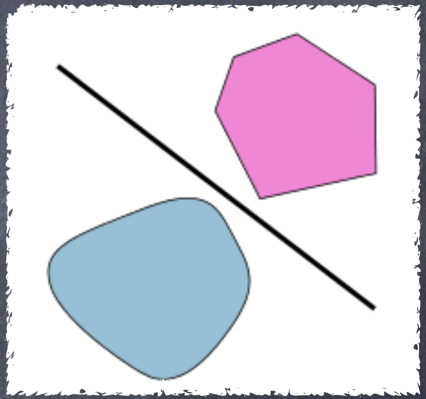

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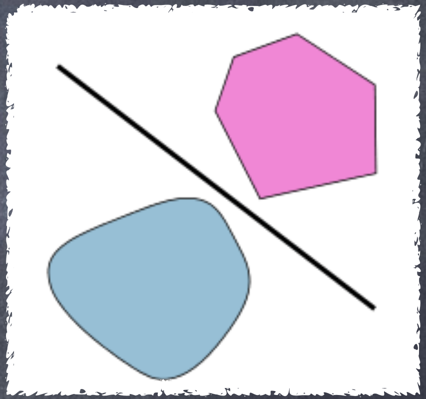

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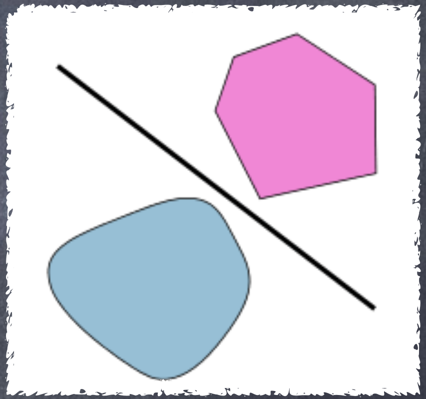

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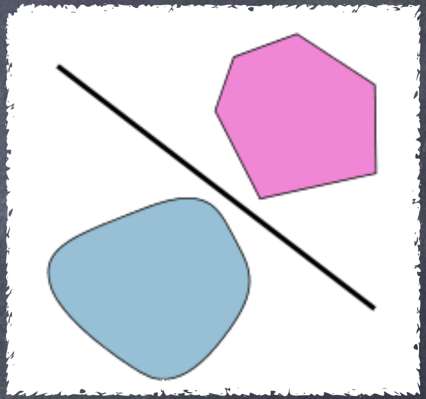

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  - Assuming the C^{n4*} is salient.

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$$\mathbf{C}^{n^4} \subset \overrightarrow{\mathbf{S}}^{n^4} \equiv \{\mathcal{T} \mid \mathcal{T}^{ijkl} = \mathcal{T}^{ilkj} = \mathcal{T}^{kjil} = \mathcal{T}^{jilk}\}$$

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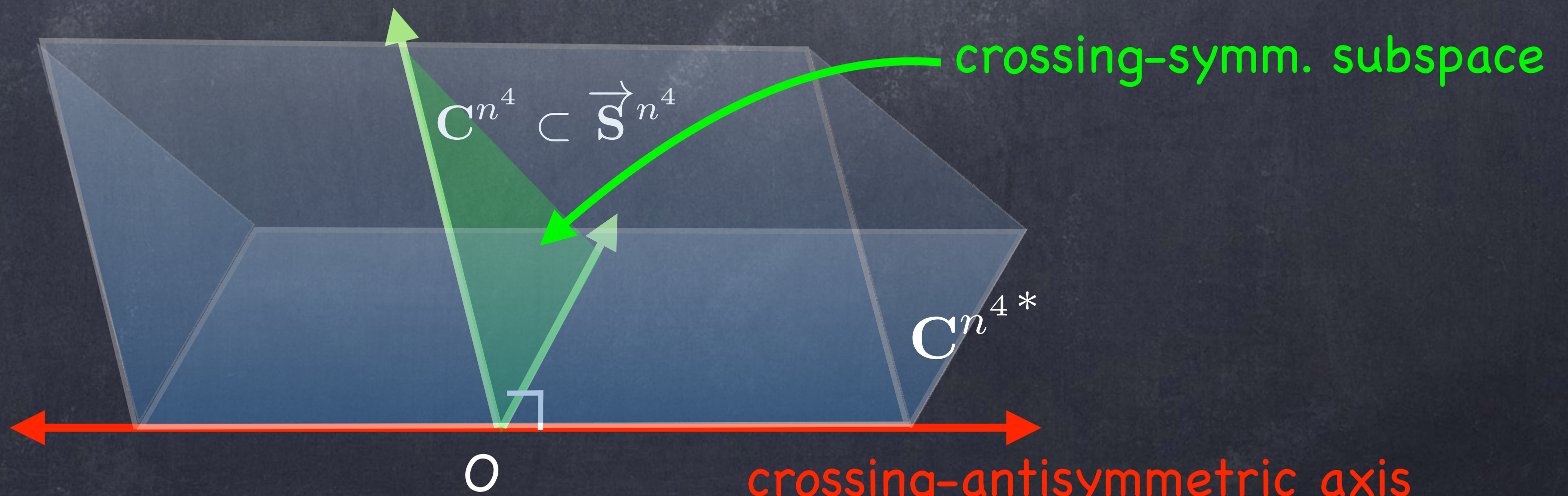
  - $\mathbf{C}^{n^4\,*}$ contains straight lines perpendicular to this subspace, and thus not salient.
  - Not needed!



- Knowing that physical amplitudes are crossing symmetric,

$$\mathbf{Q}^{n^4} \equiv \mathbf{C}^{n^4 *} \cap \overrightarrow{\mathbf{S}}^{n^4}$$
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  - Finding positivity bounds = finding ERs of some spectrahedron.

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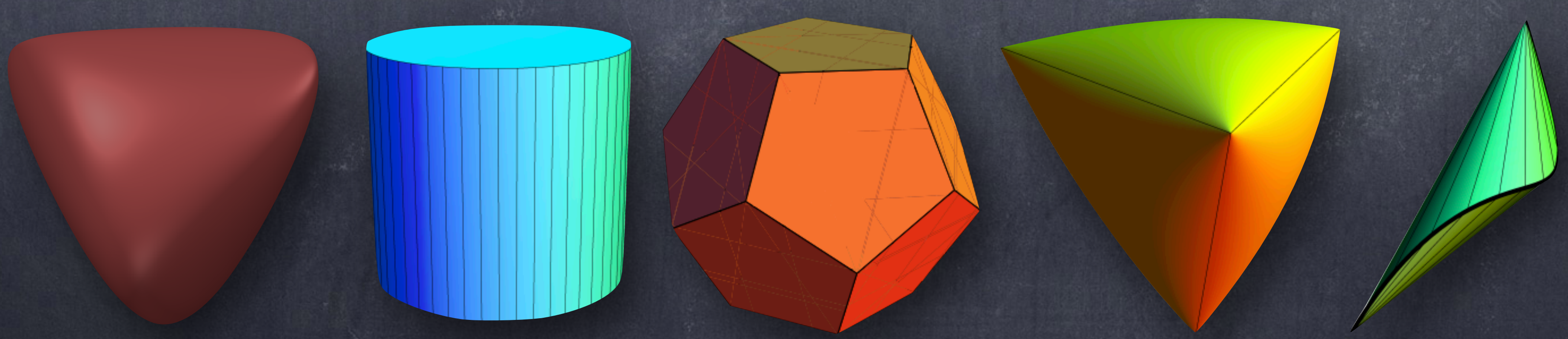
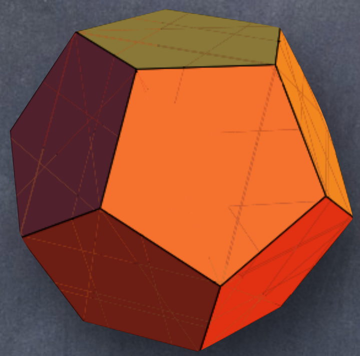
Spectrahedron?

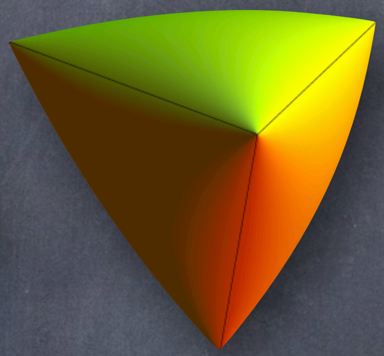
- Wiki: the set of n $\times$ n positive semidefinite matrices forms a convex cone, and a spectrahedron is a shape that can be formed by intersecting this cone with a linear affine subspace.
  - Let $Q_i,\ i = 0, 1, ..., N$ be the basis matrices of the affine space $Q(x) = Q_0 + x_i Q_i$
  - The spectrahedron $G = \{x \mid Q(x) \succeq 0\}$
  - How do they look like? From google:



Spectrahedron?

- Each point x in a spectrahedron is contained in (the relative interior of) a unique facet, F(x)





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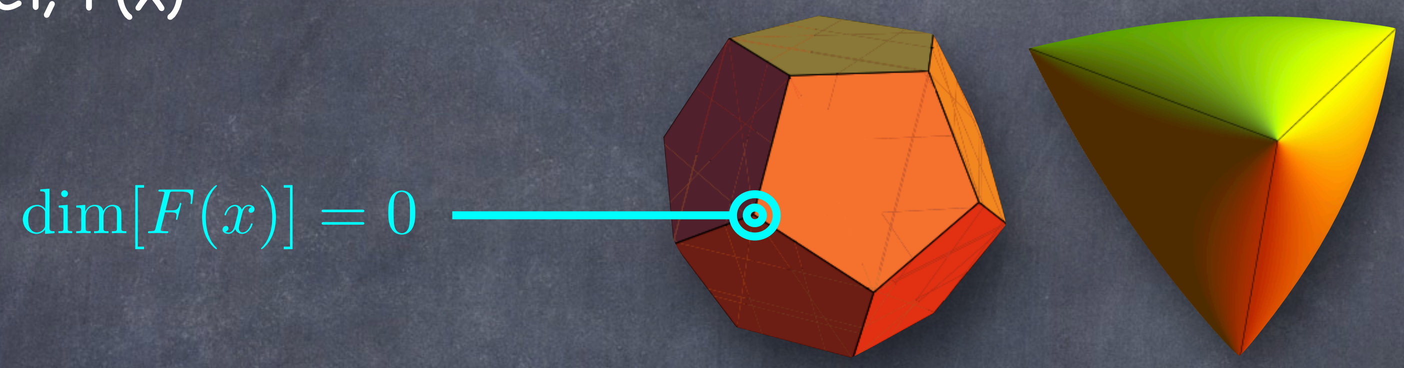

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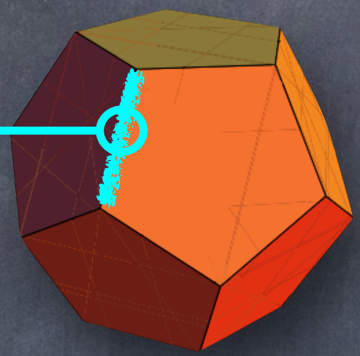
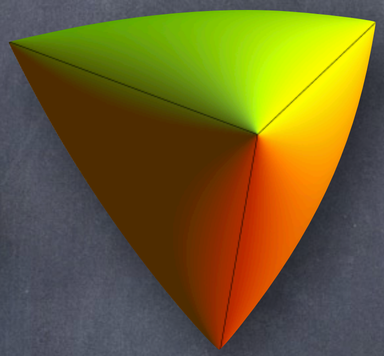
$\dim[F(x)] = 1$





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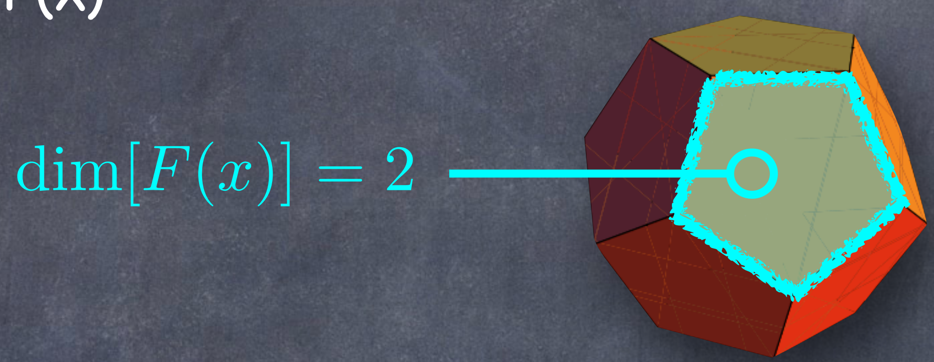

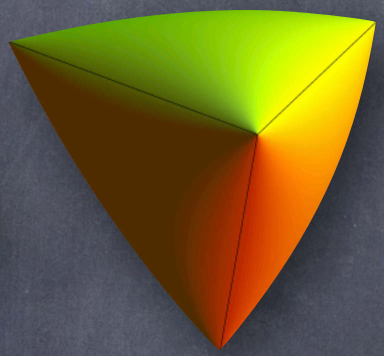

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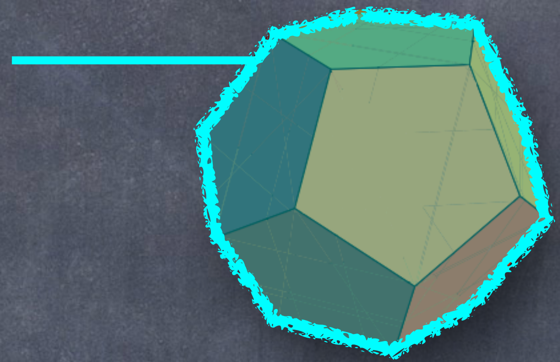
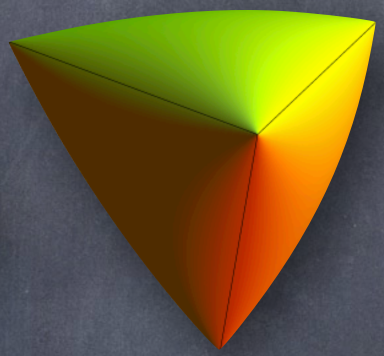
$$\dim[F(x)] = 3$$





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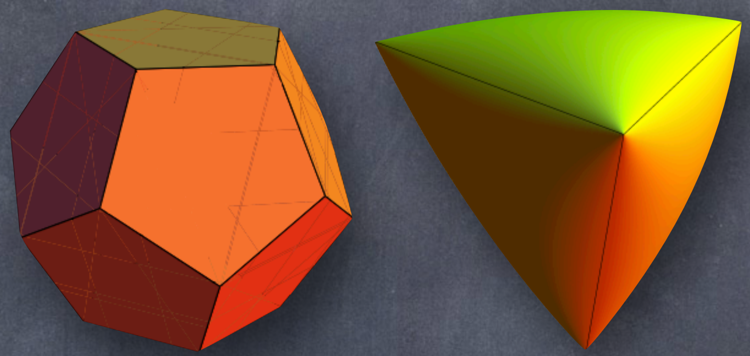

- The null space of Q(x) is constant on F(x) -> numerically identify F(x) for any x [Ramana & Goldman '95]
  - Let $\{u_i\}$ be basis of Null(Q(x)), then Null(B) is the linear span of F(x)

$$B = \begin{bmatrix} Q_1 u_1 & \cdots & Q_m u_1 \\ \vdots & \ddots & \vdots \\ Q_1 u_k & \cdots & Q_m u_k \end{bmatrix}$$

General 2-scalar case: $\mathcal{L} \supset \frac{1}{\Lambda^4} C_{ijkl} O_{ijkl}, \quad O_{ijkl} = \partial_\mu \phi_i \partial^\mu \phi_j \partial_\nu \phi_k \partial^\nu \phi_l$

6 Ops: C1111, C2222, C1212, C1122, C1112, C1222.

$$\mathcal{M}_{\text{scalar}} = \begin{bmatrix} 4C_{1111} & C'_{1122} & C_{1112} & C_{1112} \\ C'_{1122} & 4C_{2222} & C_{1222} & C_{1222} \\ C_{1112} & C_{1222} & C_{1212} & C'_{1122} \\ C_{1112} & C_{1222} & C'_{1122} & C_{1212} \end{bmatrix},$$

$$\mathbf{Q}^{2^4} \ni \mathcal{Q} = \begin{array}{c|cccc} & \text{kl=11} & 22 & 12 & 21 \\ \hline \text{ij=11} & a & b & e & e \\ 22 & b & c & f & f \\ 12 & e & f & d & b \\ 21 & e & f & b & d \end{array}$$

$$C'_{1122} \equiv C_{1122} + \frac{1}{2} C_{1212}$$

$$\{Q_i, 1 \le i \le 6\} = \left\{ \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&0&1&1\\0&0&0&0\\1&0&0&0\\1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&1\\0&1&0&0\\0&1&0&0 \end{pmatrix} \right\}$$

$$\mathcal{Q}(x) = x_i Q_i$$

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$\boxed{\text{For illustration, impose a } Z_2}$

$a \geq 0, \ c \geq 0,$

$ac \geq b^2, \ d \geq |b|$

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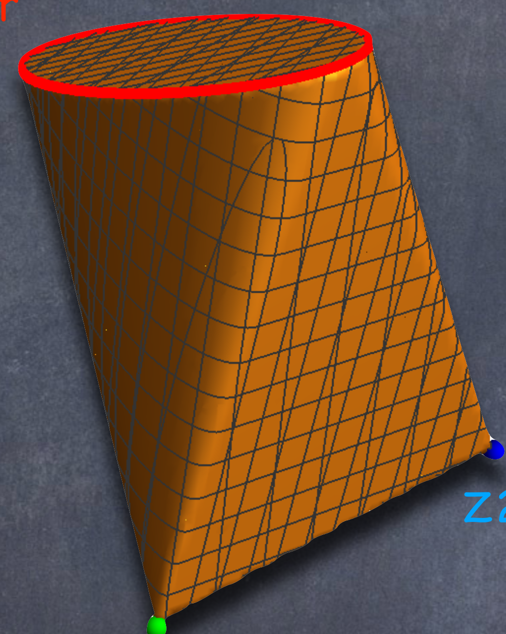
3D "cross section" of 4D cones

Amplitude space

$\mathbf{C}^{2^4}$



ERs = UV particles

Dual space (spectrahedron)

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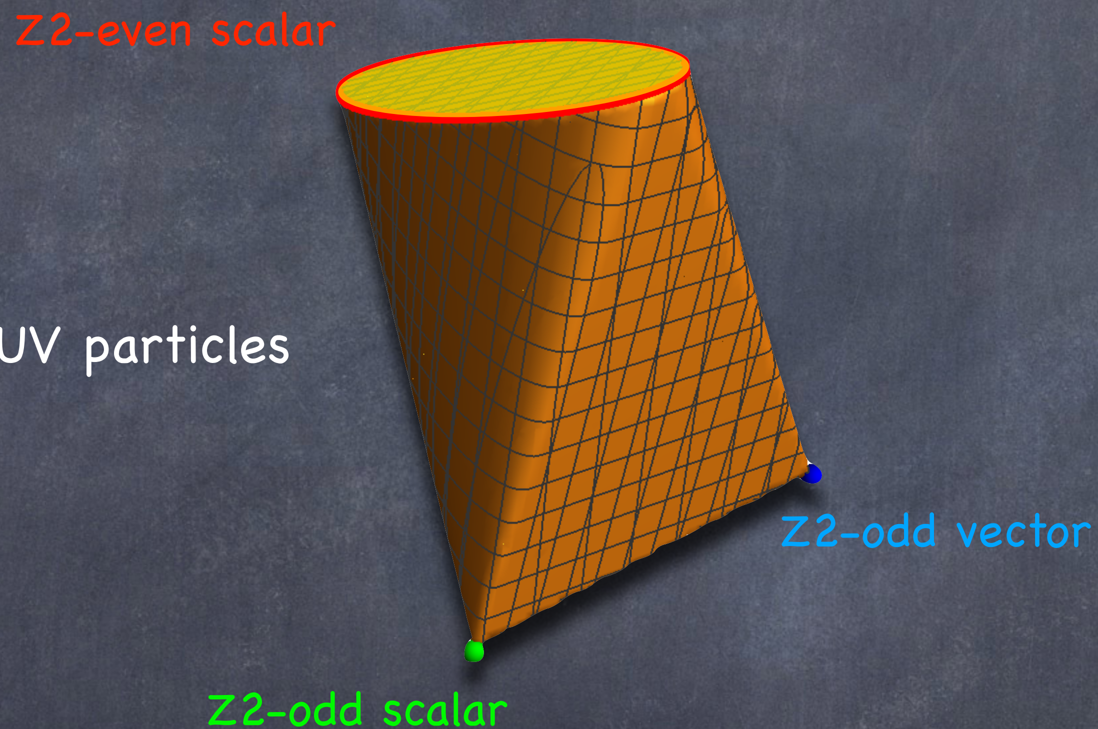

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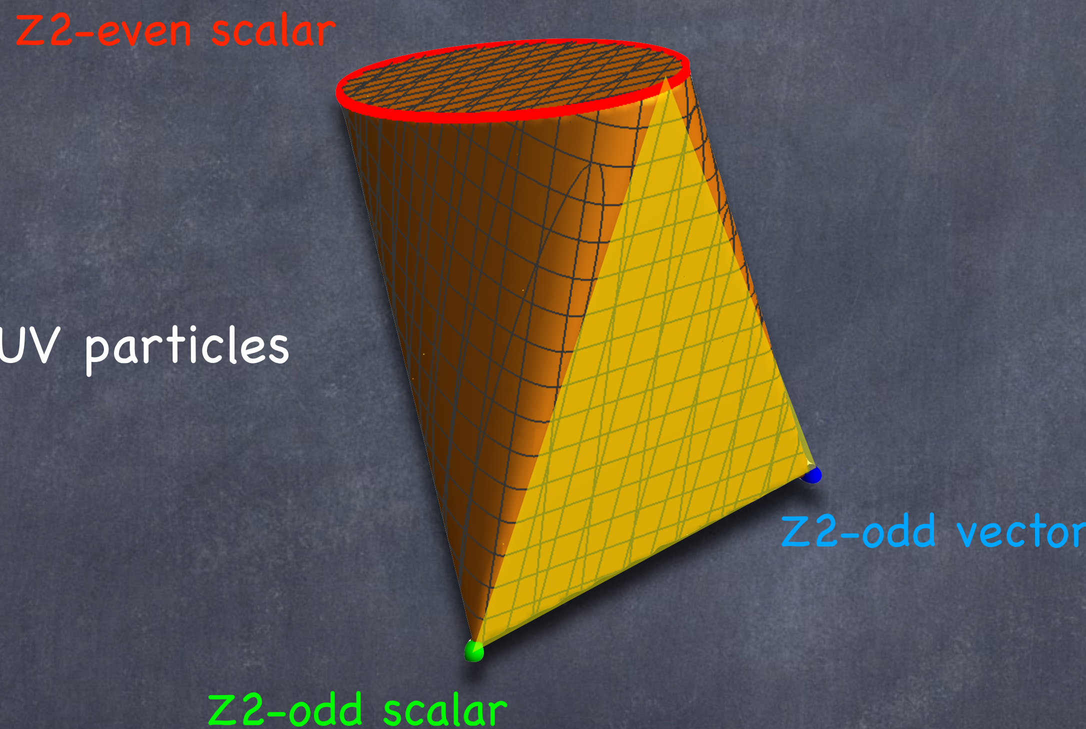

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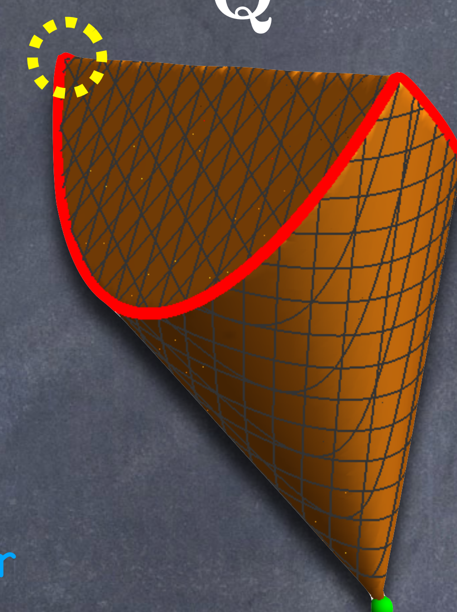

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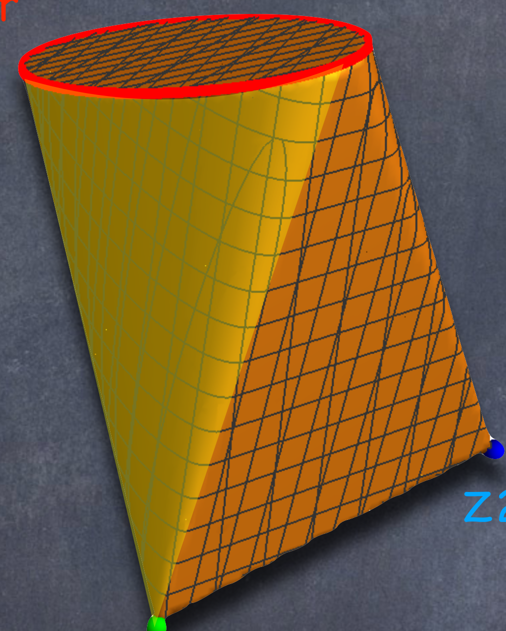

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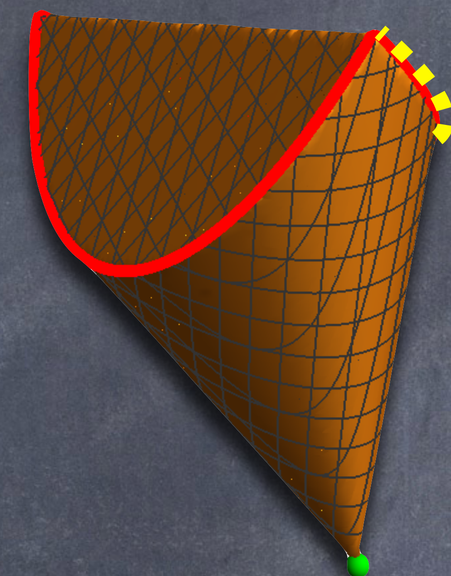

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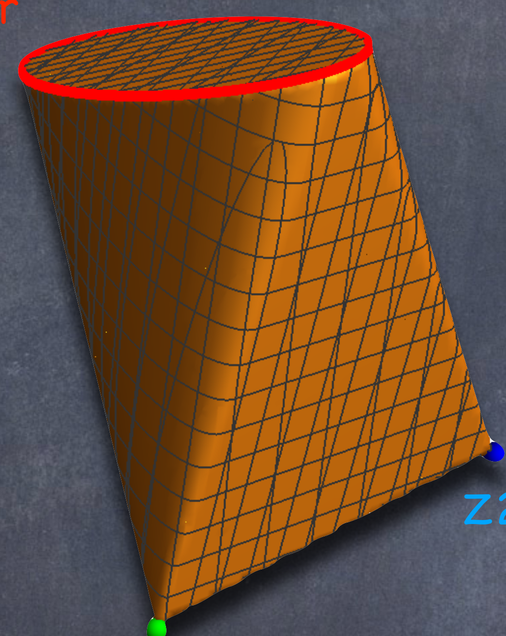

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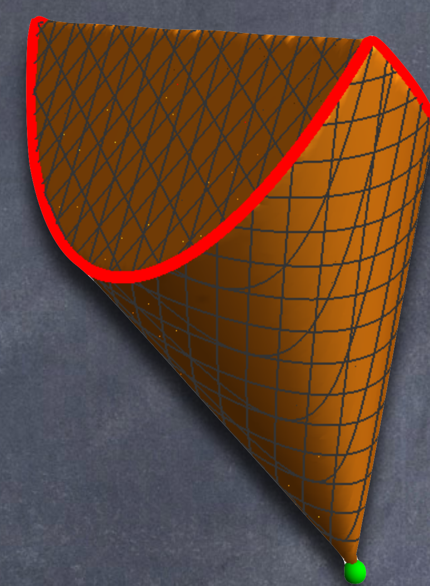


ERs = posi. bounds

Bounds

$$C_{1111} \geq 0, \;\; C_{2222} \geq 0, \;\; C_{1212} \geq 0$$
$$4\sqrt{C_{1111}C_{2222}} \geq \pm(2C_{1122} + C_{1212}) - C_{1212}$$

General 2-scalar case: $\mathcal{L} \supset \frac{1}{\Lambda^4} C_{ijkl} O_{ijkl}, \quad O_{ijkl} = \partial_\mu \phi_i \partial^\mu \phi_j \partial_\nu \phi_k \partial^\nu \phi_l$

6 Ops: C1111, C2222, C1212, C1122, C1112, C1222.

$$\mathcal{M}_{\rm scalar} = \begin{bmatrix} 4C_{1111} & C'_{1122} & C_{1112} & C_{1112} \\ C'_{1122} & 4C_{2222} & C_{1222} & C_{1222} \\ C_{1112} & C_{1222} & C_{1212} & C'_{1122} \\ C_{1112} & C_{1222} & C'_{1122} & C_{1212} \end{bmatrix}, \qquad \mathbf{Q}^{2^4} \ni \mathcal{Q} = \begin{array}{c|cccc} & kl=11 & 22 & 12 & 21 \\ \hline ij=11 & a & b & e & e \\ 22 & b & c & f & f \\ 12 & e & f & d & b \\ 21 & e & f & b & d \end{array}$$

$$C'_{1122} \equiv C_{1122} + \frac{1}{2} C_{1212}$$

$\boxed{\text{Without Z2 constraint:}}$

ERs of $\mathbf{Q}^{2^4}$ ->

$$\begin{array}{c|cccc} & kl=11 & 22 & 12 & 21 \\ \hline ij=11 & a^2 & ab & ac & ac \\ 22 & ab & b^2 & bc & bc \\ 12 & ac & bc & 2c^2 - ab & ab \\ 21 & ac & bc & ab & 2c^2 - ab \end{array} \qquad c^2 \geq ab$$

- To check these are ERs, use the B matrix.
- Can prove these are complete set of ERs.

Linear bounds for each (a, b, c) real, c^2 >= ab:

$$\begin{bmatrix} a & c & b \end{bmatrix} \cdot D \cdot \begin{bmatrix} a & c & b \end{bmatrix}^T \geq 0 \quad \forall c^2 \geq ab$$
$$\text{where } D = \begin{bmatrix} 2C_{1111} & C_{1112} & C_{1122} \\ C_{1112} & 2C_{1212} & C_{1222} \\ C_{1122} & C_{1222} & 2C_{2222} \end{bmatrix}.$$

To remove a,b,c dependence:

$$f(r,s,w) \equiv \begin{bmatrix} w^2 & \frac{rw+sw}{2} & rs \end{bmatrix} \cdot D \cdot \begin{bmatrix} w^2 & \frac{rw+sw}{2} & rs \end{bmatrix}^T$$
$$\geq 0 \quad \forall r,s,w \in \mathbb{R},$$

- i.e. a quartic form in (r,s,w) is PSD
    - Can be determined by completing squares (for at most 3 variables, Hilbert 1888)

Positivity bounds for general 2-scalar EFTs:

$$
\begin{aligned}
& C_{1111} \geq 0 \quad \text{and} \quad 4C_{1111}C_{1212} - C_{1112}^2 \geq 0 \\
& \text{and} \quad \Big\{ C_{1112}C_{1122}C_{1222} - C_{1111}C_{1222}^2 - C_{1112}^2 C_{2222} + C_{1212}\left(-C_{1122}^2 + 4C_{1111}C_{2222}\right) \geq 0 \\
& \text{or} \quad \Big[\Delta \equiv 3\left(4C_{1111}C_{2222} - C_{1112}C_{1222}\right) + \left(C_{1122} + C_{1212}\right)^2 \geq 0 \\
& \text{and} \quad \frac{3C_{1112}^2}{4C_{1111}} - 2\left(C_{1122} + C_{1212}\right) \leq \sqrt{\Delta} \leq C_{1212} - 2C_{1122} \\
& \text{and} \quad 2\Delta^{3/2} \geq 27\left(C_{1111}C_{1222}^2 + C_{1112}^2 C_{2222}\right) - 9\left(C_{1122} + C_{1212}\right)\left(8C_{1111}C_{2222} + C_{1112}C_{1222}\right) \\
& \qquad\qquad + 2\left(C_{1122} + C_{1212}\right)^3 \Big] \Big\}
\end{aligned}
$$

What if n>2? Either

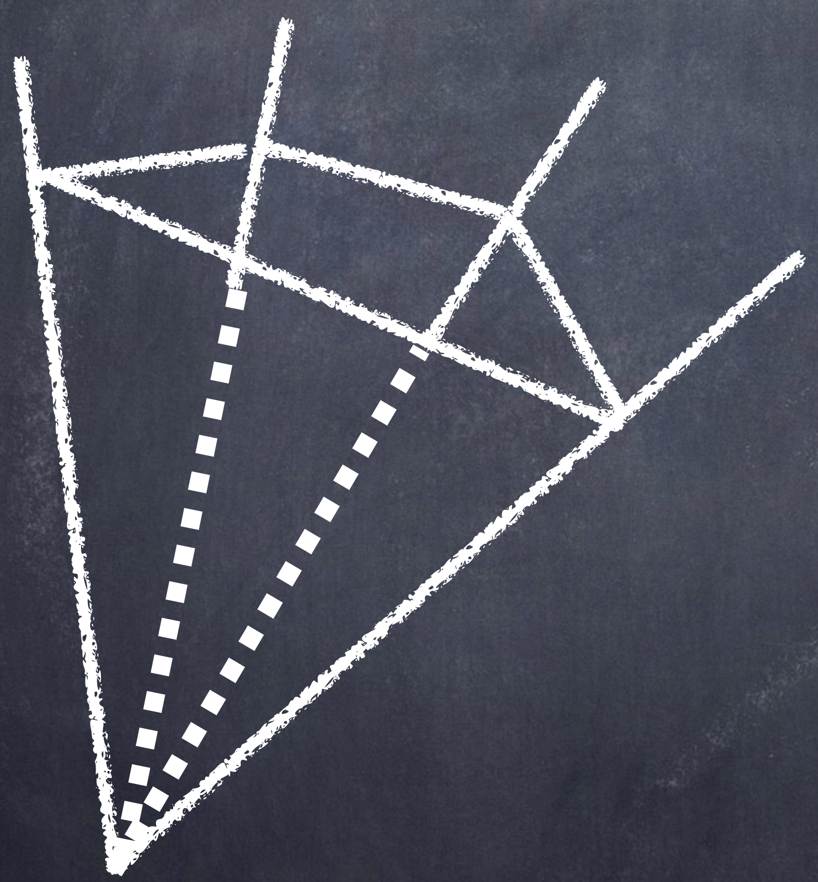
1. Randomly search for ERs. (MC sampling of ERs); or

2. For a given amplitude M, numerically minimize Q.M on the spectrahedron.

   - This is a semidefinite programming (SDP).

The "MC" approach

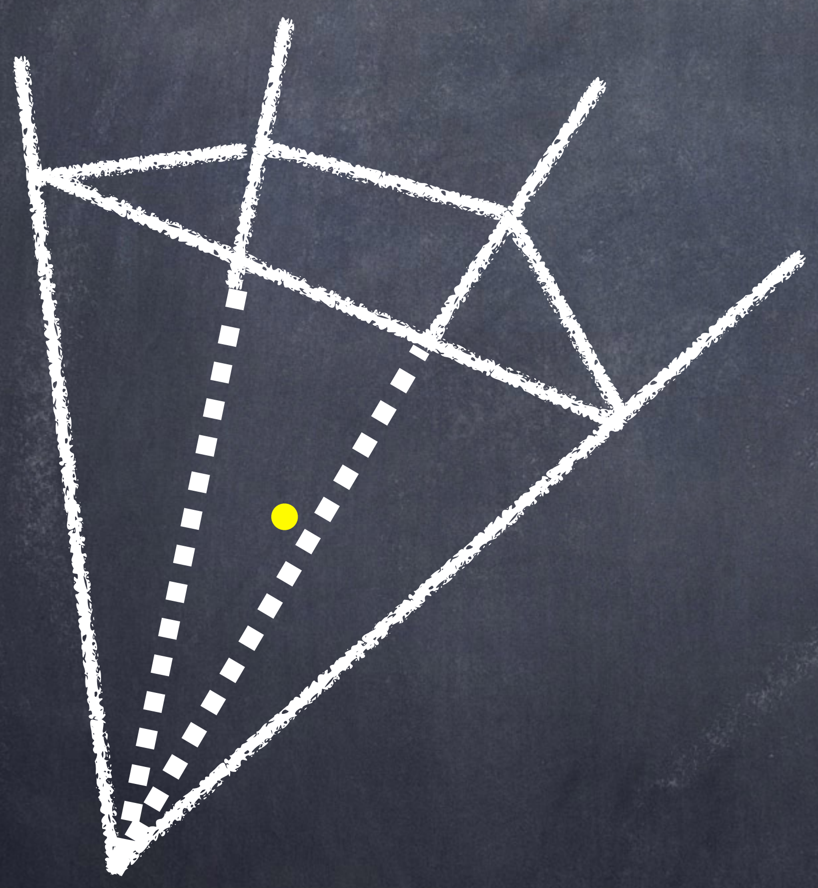
Recall each point x is contained in a unique facet, $F(x)$, determined by $Null(B)$



The "MC" approach

Recall each point x is contained in a unique facet, F(x), determined by Null(B)

- Start with a random point x



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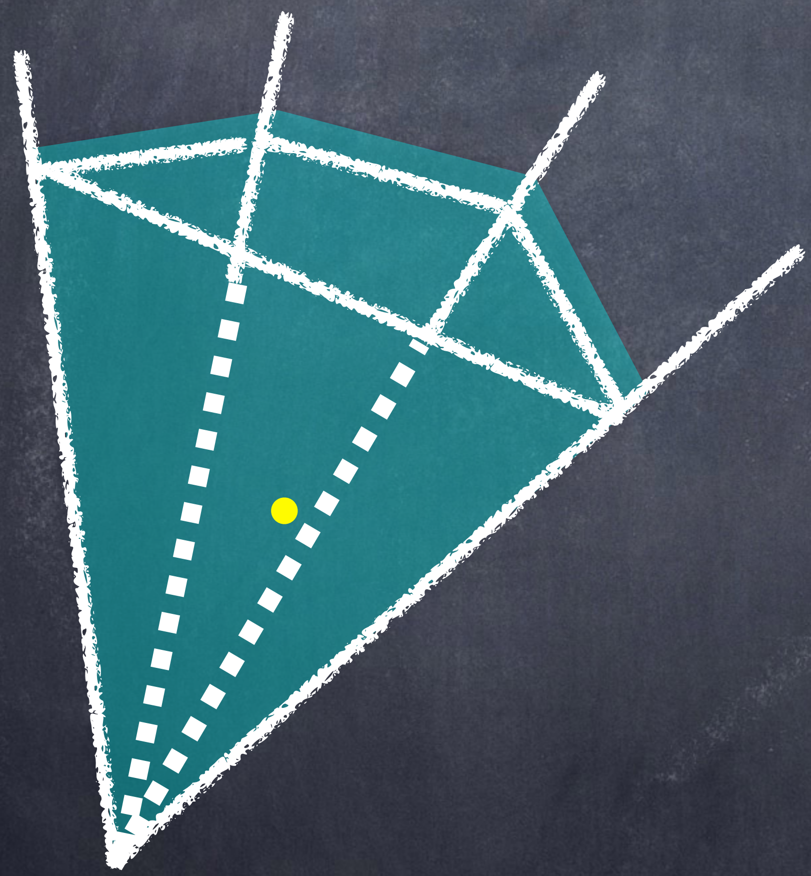
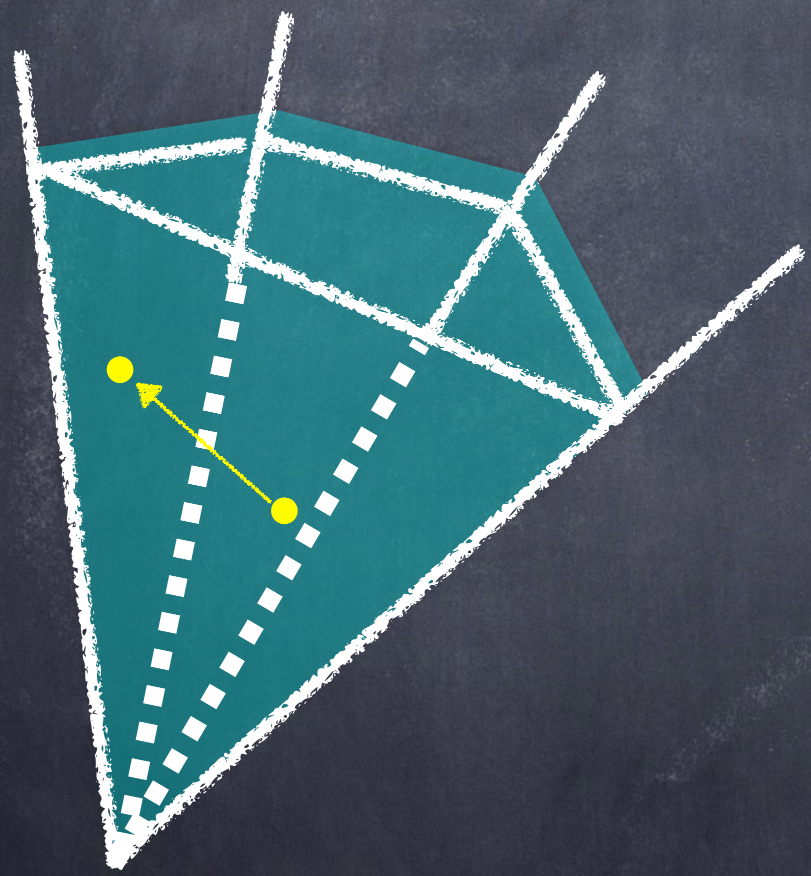


- Start with a random point x
- Find the (k-)face F(x)

The "MC" approach

Recall each point x is contained in a unique facet, F(x), determined by Null(B)



- Start with a random point x
- Find the (k-)face F(x)
- Take a random straight-line in F(x) that crosses x. Find its intersection with the boundary of the cone (this is a SDP).

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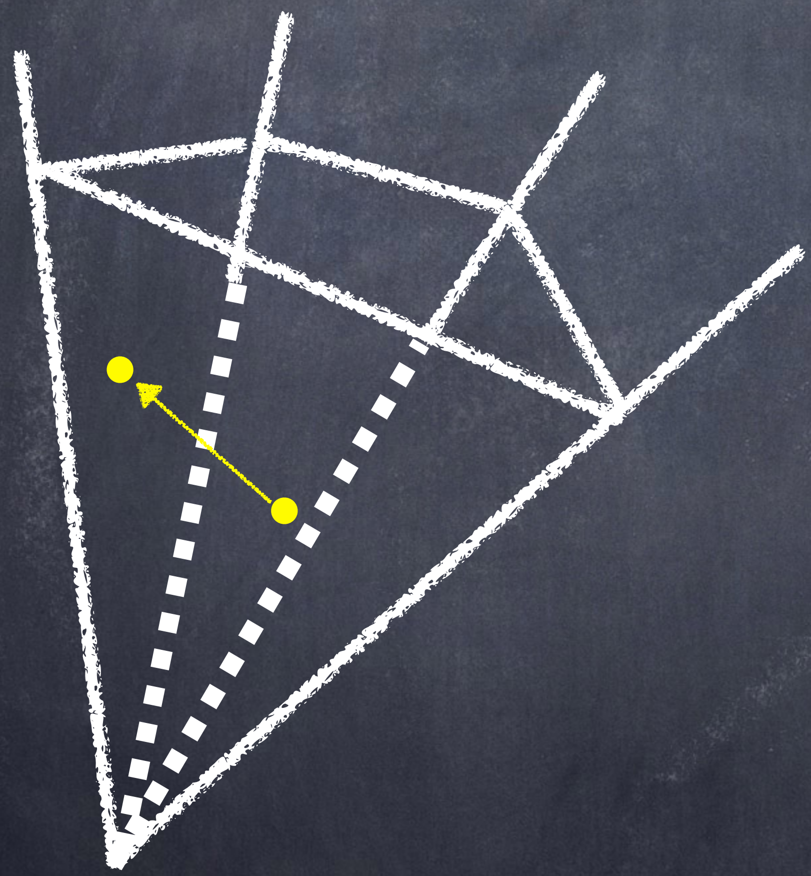

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- Take x to be the intersection point and iterate, until F(x) is dimension 1

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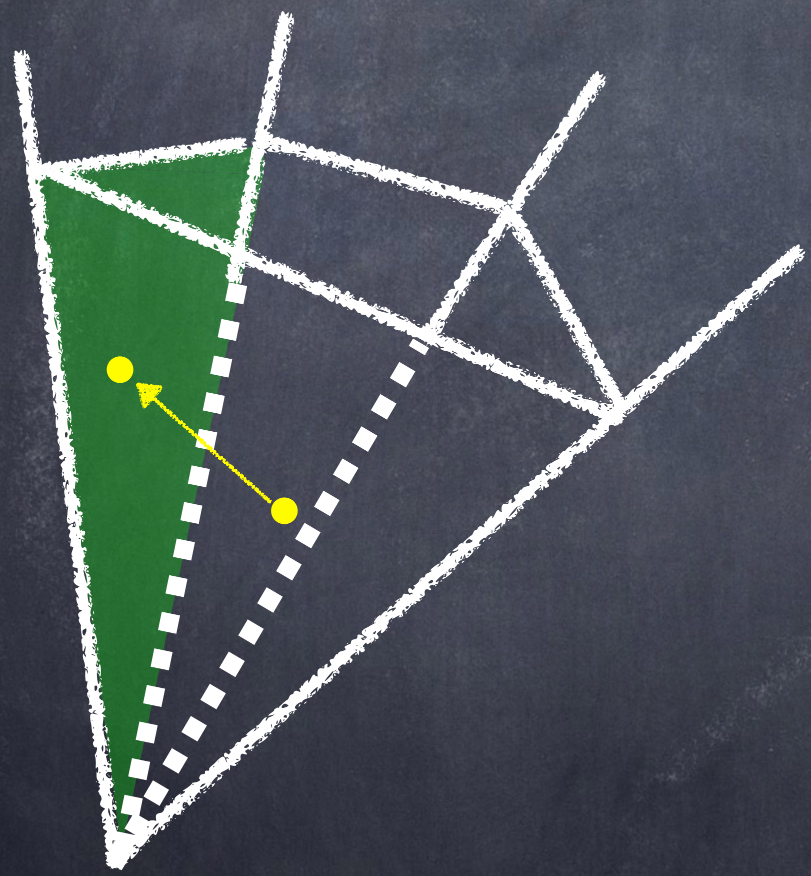

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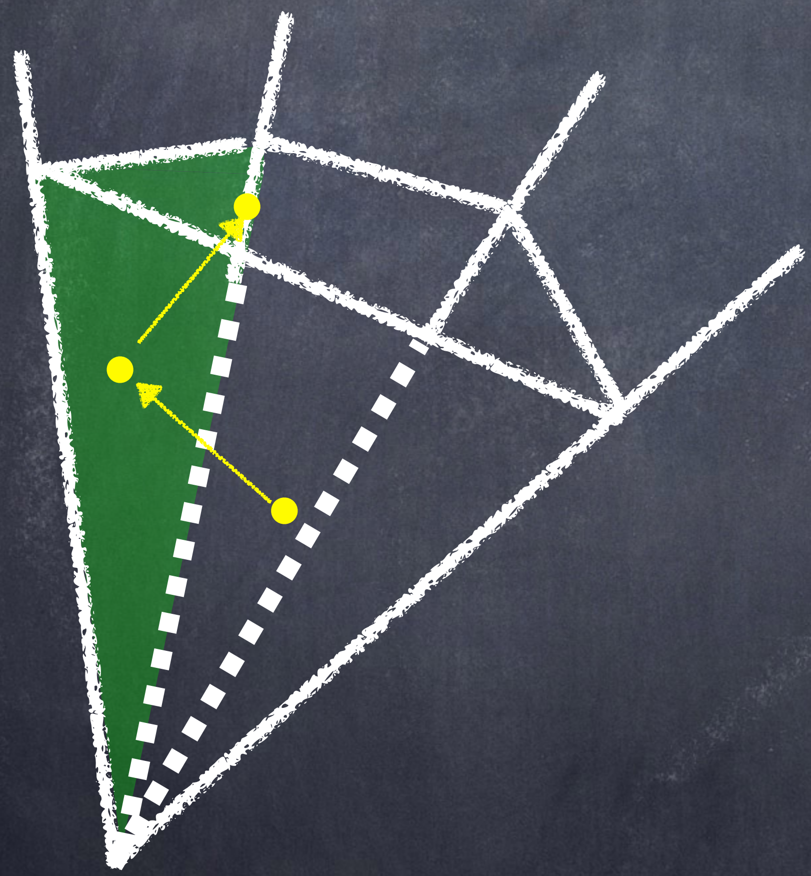

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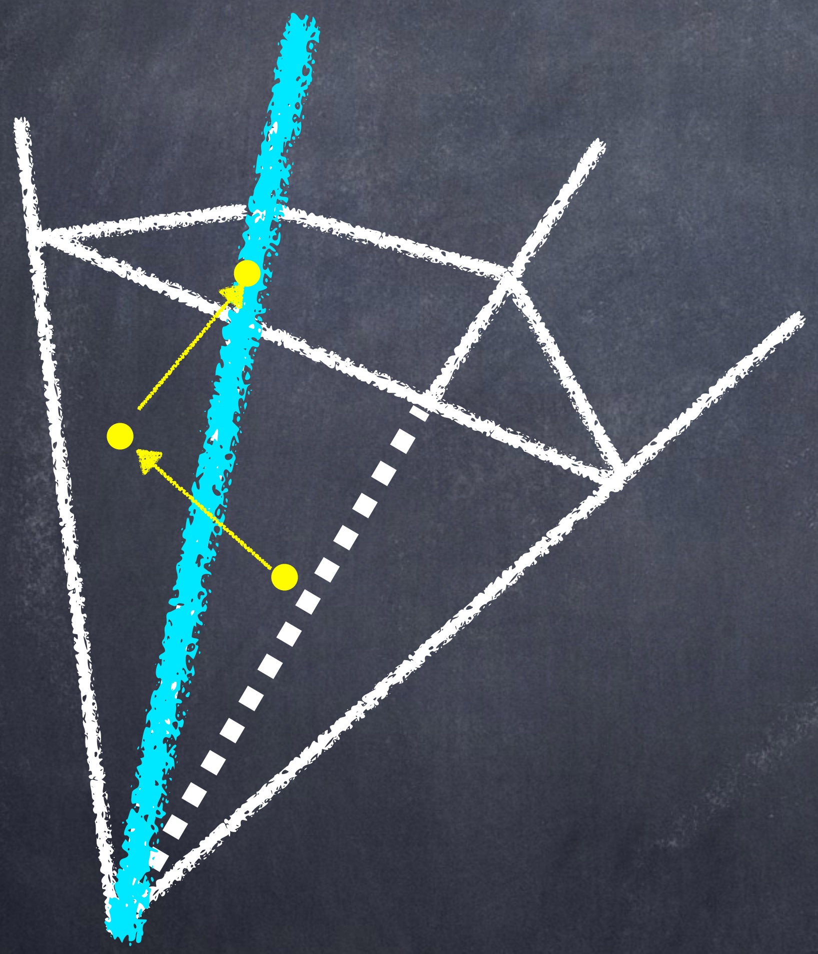


- Start with a random point x
- Find the (k-)face F(x)
- Take a random straight-line in F(x) that crosses x. Find its intersection with the boundary of the cone (this is a SDP).
- Take x to be the intersection point and iterate, until F(x) is dimension 1
- An ER is found.

Works for large problems!

4-Gluon OPs ->

$Q_{G^4}^{(1)}$	$(G^A_{\mu\nu} G^{A\mu\nu})(G^B_{\rho\sigma} G^{B\rho\sigma})$
$Q_{G^4}^{(2)}$	$(G^A_{\mu\nu} \widetilde{G}^{A\mu\nu})(G^B_{\rho\sigma} \widetilde{G}^{B\rho\sigma})$
$Q_{G^4}^{(3)}$	$(G^A_{\mu\nu} G^{B\mu\nu})(G^A_{\rho\sigma} G^{B\rho\sigma})$
$Q_{G^4}^{(4)}$	$(G^A_{\mu\nu} \widetilde{G}^{B\mu\nu})(G^A_{\rho\sigma} \widetilde{G}^{B\rho\sigma})$

$Q_{G^4}^{(7)}$	$d^{ABE} d^{CDE} (G^A_{\mu\nu} G^{B\mu\nu})(G^C_{\rho\sigma} G^{D\rho\sigma})$
$Q_{G^4}^{(8)}$	$d^{ABE} d^{CDE} (G^A_{\mu\nu} \widetilde{G}^{B\mu\nu})(G^C_{\rho\sigma} \widetilde{G}^{D\rho\sigma})$

Plus a $(D6)^2$ term from double insertion

$\vec{x} \cdot \vec{C} \geq 0$ x given by

$[0,0,0,1,0,0,0]$	$[0,0,6,3,7,2,0]$	$[24,0,12,21,15,14,0]$	$[0,0,96,24,64,40,-81]$
$[0,0,1,1,1,0,0]$	$[8,6,1,6,0,2,0]$	$[24,32,24,4,8,0,-27]$	$[40,32,80,4,0,0,-189]$
$[2,0,1,0,0,0,0]$	$[0,6,3,12,5,0,0]$	$[48,36,21,27,25,0,0]$	$[0,0,24,120,40,104,-81]$
$[0,2,0,1,0,0,0]$	$[8,6,1,12,0,0,0]$	$[32,40,4,80,0,0,-27]$	$[0,0,120,24,104,40,-81]$
$[0,0,3,0,2,0,0]$	$[0,6,6,9,10,4,0]$	$[0,48,0,48,0,40,-81]$	$[96,0,144,24,64,40,-81]$
$[0,0,0,3,0,2,0]$	$[0,12,0,14,0,0,-9]$	$[24,0,36,24,16,40,-81]$	$[48,0,96,24,0,40,-243]$
$[1,1,2,2,0,0,0]$	$[0,0,8,8,0,8,-27]$	$[0,0,48,24,32,40,-81]$	$[0,192,168,96,112,120,-405]$
$[6,0,3,0,2,0,0]$	$[12,0,14,0,0,0,-27]$	$[0,0,24,48,16,56,-81]$	$[168,480,168,156,56,160,-729]$
$[4,2,2,1,2,0,0]$	$[6,8,12,1,0,0,-27]$	$[88,32,56,4,40,0,-27]$	$[264,384,156,168,16,200,-729]$
$[0,0,4,0,0,0,-9]$	$[8,16,4,8,0,8,-27]$	$[96,42,27,84,25,0,0]$	$[288,384,216,168,0,200,-891]$
$[6,0,6,0,5,0,0]$	$[0,24,0,12,0,8,-27]$	$[96,66,42,39,50,4,0]$	$[480,384,480,168,160,200,-729]$
$[0,0,3,6,5,4,0]$	$[8,22,1,14,0,10,-27]$	$[120,42,39,42,40,14,0]$	$[336,768,672,216,0,200,-2187]$

7D polyhedral cone with 48 facets!

The SDP approach

Given amplitude $\mathcal{M}^{ijkl}$, how to check if it's allowed by positivity?

$$\mathbf{C}^{n^4} = \left\{ \mathcal{M} \in \overrightarrow{\mathbf{S}}^{n^4} \mid \mathcal{Q} \cdot \mathcal{M} \geq 0 \; \forall \mathcal{Q} \in \mathbf{Q}^{n^4} \right\}$$

The semi-definite programming (SDP) approach:

$$\begin{aligned} &\min \quad \mathcal{Q} \cdot \mathcal{M} \\ &\text{subject to} \quad \mathcal{Q} \in \text{spectrahedron} \end{aligned}$$

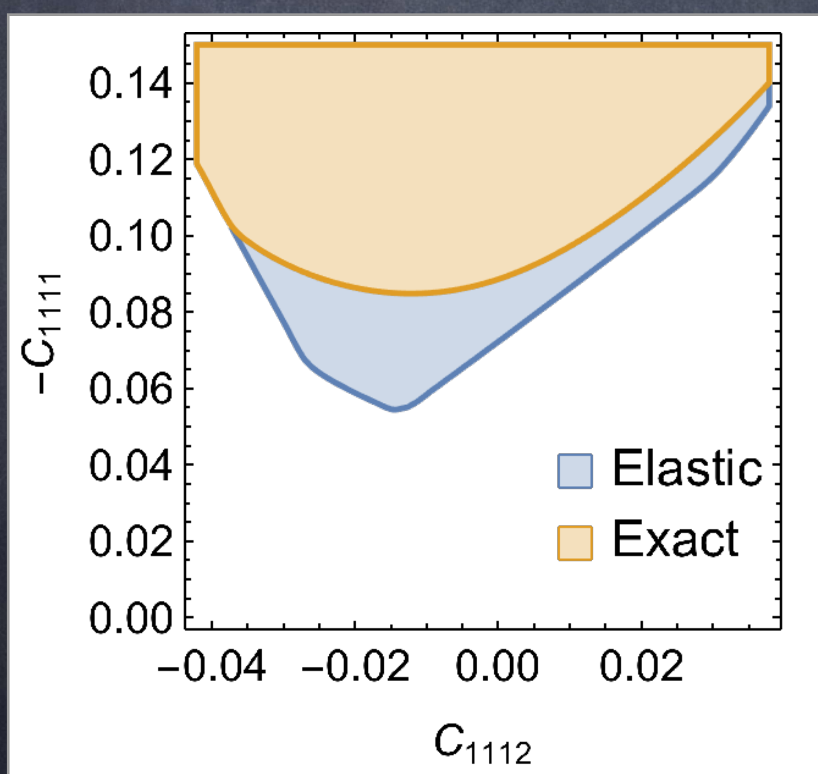
If a solution exists, then M is allowed by positivity.

- Solvable within polynomial complexity.
- In contrast to elastic positivity, which is NP-hard.

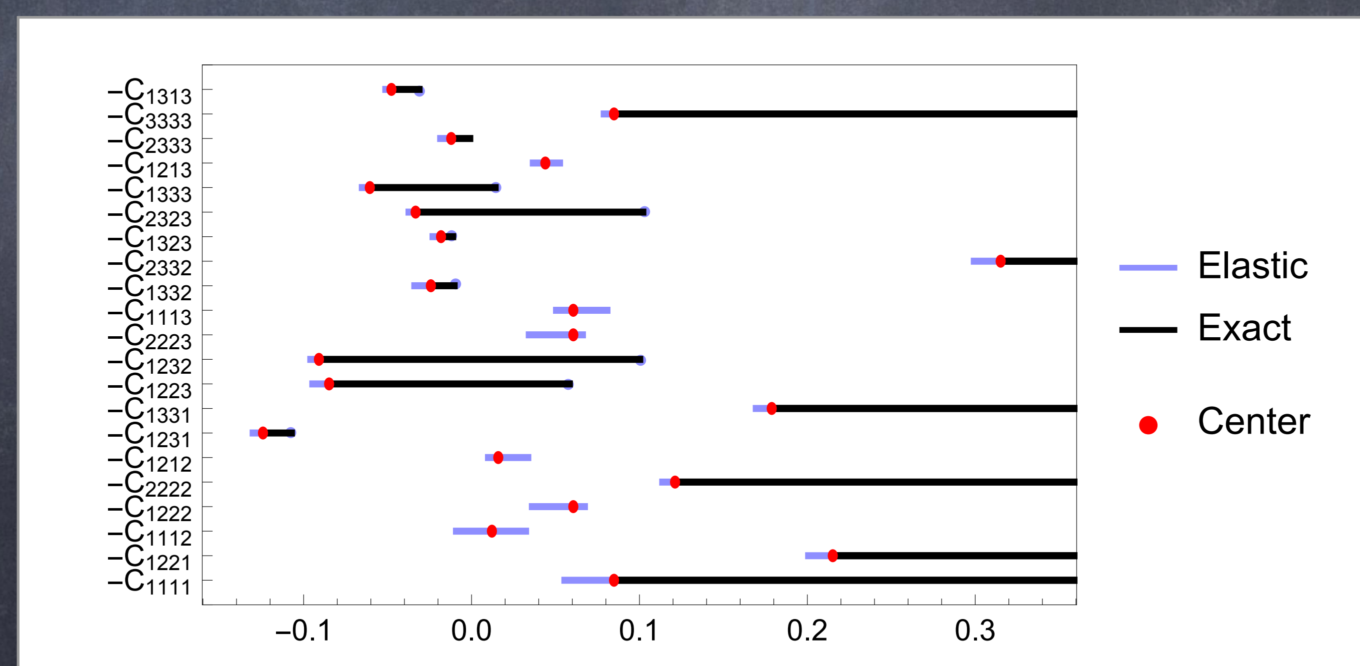
Example: improvements in SMEFT

- SMEFT VVVV (aQGC) operators (W+B, n=12 modes): reproduced bounds by [2009,04490, K. Yamashita, CZ and S.-Y. Zhou] which were obtained by taking O(1000) discrete ERs in the amplitude space.
- SM flavor sector (n=3 fields) [2004.02885, Remmen & Rodd]
    - Flavor violating NP sets lower bounds on flavor conserving ones.

$e^+e^- \to e^+e^-$

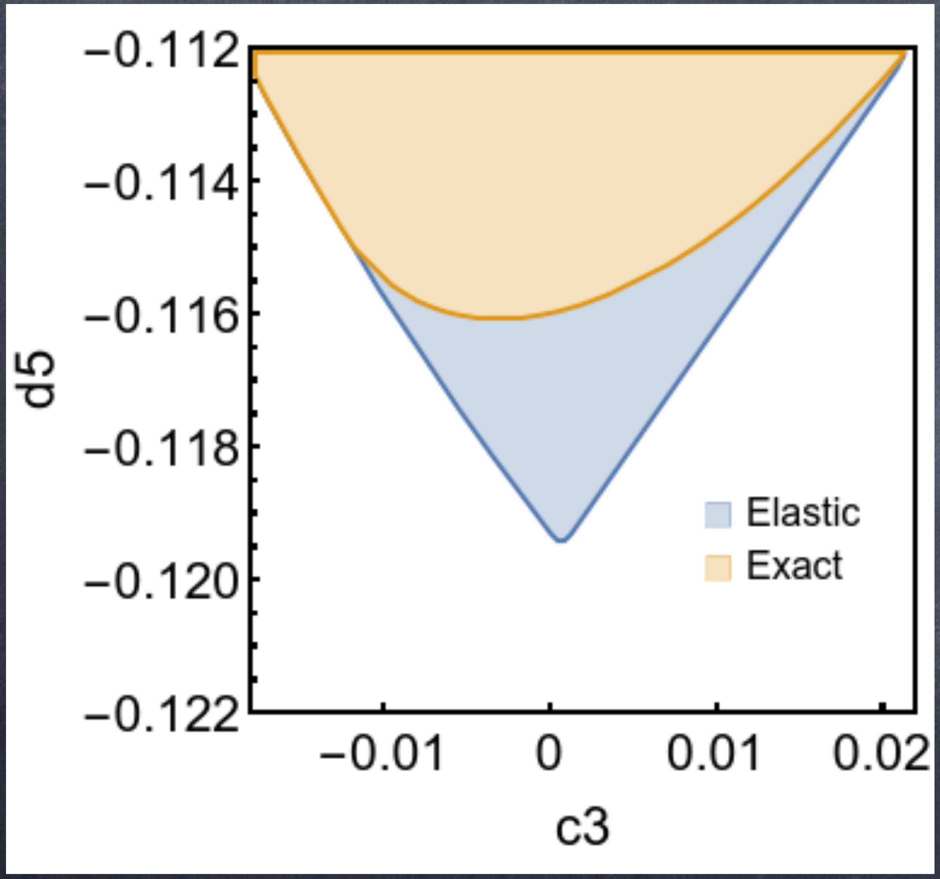


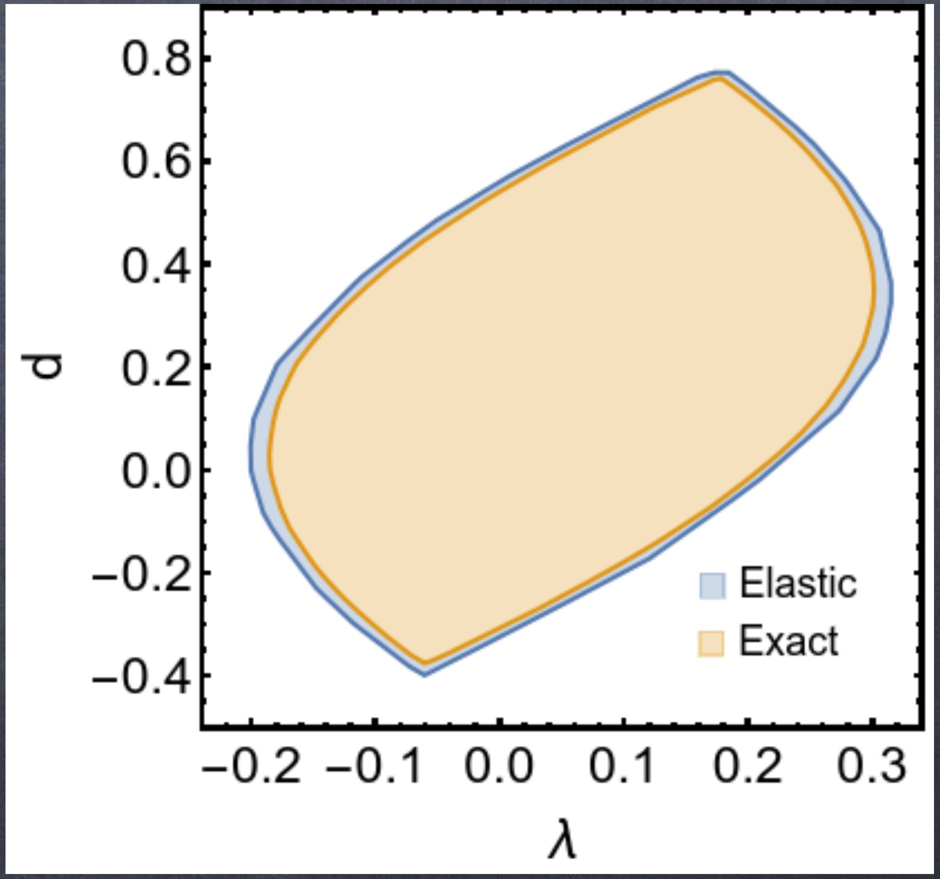
$\mu \to 3e$



Example: improvements in spin-2 EFT

- dRGT massive gravity (n=5): improves slightly the minimum value of d5.
- Z2 bi-field spin-2 EFT (n=10): improving the elastic (superposed) positivity.





Summary

- Positive structures arise at the dim-8 level in EFT coefficient space, as a consequence of axiomatic QFT principles.
- Realistic problems often involve multi-field EFTs, in which a convex geometric perspective helps to understand these structures.
- We convert the problem of finding bounds to a geometric problem: finding the ERs of a spectrahedron.
    - For small n, can be solved analytically.
    - For large n, can be solved as a semi-definite programming problem.
- Improved some previous results.

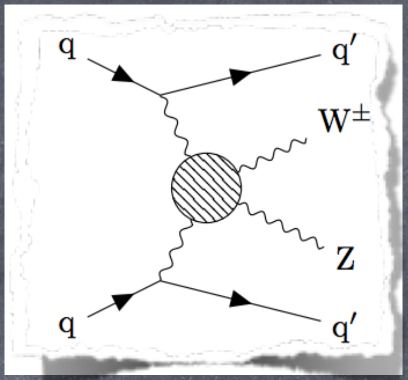
Backups

$$\mathcal{Q}_{ex} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 2 & 3 & 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 3 \end{bmatrix}$$

Anomalous quartic-gauge boson couplings

[K. Yamashita, **CZ**, S.-Y. Zhou 2009.04490]

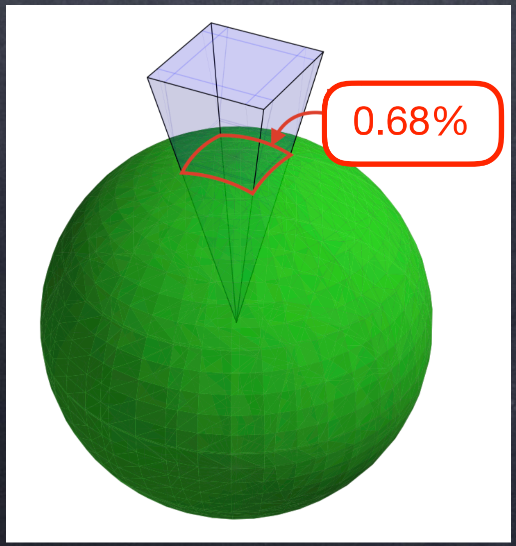
$$O_{T,0} = \mathrm{Tr}[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}]\mathrm{Tr}[\hat{W}_{\alpha\beta}\hat{W}^{\alpha\beta}] \qquad O_{T,1} = \mathrm{Tr}[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}]\mathrm{Tr}[\hat{W}_{\mu\beta}\hat{W}^{\alpha\nu}]$$
$$O_{T,2} = \mathrm{Tr}[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}]\mathrm{Tr}[\hat{W}_{\beta\nu}\hat{W}^{\nu\alpha}] \qquad O_{T,10} = \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\mathrm{Tr}[\hat{W}_{\alpha\beta}\tilde{W}^{\alpha\beta}]$$
$$O_{T,5} = \mathrm{Tr}[\hat{W}_{\mu\nu}\hat{W}^{\mu\nu}]\hat{B}_{\alpha\beta}\hat{B}^{\alpha\beta} \qquad O_{T,6} = \mathrm{Tr}[\hat{W}_{\alpha\nu}\hat{W}^{\mu\beta}]\hat{B}_{\mu\beta}\hat{B}^{\alpha\nu}$$
$$O_{T,7} = \mathrm{Tr}[\hat{W}_{\alpha\mu}\hat{W}^{\mu\beta}]\hat{B}_{\beta\nu}\hat{B}^{\nu\alpha} \qquad O_{T,11} = \mathrm{Tr}[\hat{W}_{\mu\nu}\tilde{W}^{\mu\nu}]\hat{B}_{\alpha\beta}\tilde{B}^{\alpha\beta}$$
$$O_{T,8} = \hat{B}_{\mu\nu}\hat{B}^{\mu\nu}\hat{B}_{\alpha\beta}\hat{B}^{\alpha\beta} \qquad O_{T,9} = \hat{B}_{\alpha\mu}\hat{B}^{\mu\beta}\hat{B}_{\beta\nu}\hat{B}^{\nu\alpha}$$



Infinite number of ERs!

Linear:

$$F_{T,2} \geq 0$$
$$4F_{T,1} + F_{T,2} \geq 0$$
$$F_{T,2} + 8F_{T,10} \geq 0$$
$$8F_{T,0} + 4F_{T,1} + 3F_{T,2} \geq 0$$
$$12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10} \geq 0$$
$$4F_{T,0} + 4F_{T,1} + 3F_{T,2} + 12F_{T,10} \geq 0$$
$$4F_{T,6} + F_{T,7} \geq 0$$
$$F_{T,7} \geq 0$$
$$2F_{T,8} + F_{T,9} \geq 0$$
$$F_{T,9} \geq 0$$



Quadratic:

$$F_{T,9}\left(F_{T,2} + 4F_{T,10}\right) \geq F_{T,11}^2$$
$$16\left(2\left(F_{T,0} + F_{T,1}\right) + F_{T,2}\right)\left(2F_{T,8} + F_{T,9}\right) \geq \left(4F_{T,5} + F_{T,7}\right)^2$$
$$32\left(2F_{T,8} + F_{T,9}\right)\left(3F_{T,0} + F_{T,1} + 2F_{T,2} + 4F_{T,10}\right) \geq 3\left(4F_{T,5} + F_{T,7}\right)^2$$
$$2\sqrt{2}\sqrt{F_{T,9}\left(F_{T,2} + 8F_{T,10}\right)} \geq \max\left(4F_{T,6} + F_{T,7} - 4F_{T,11}, F_{T,7} + 4F_{T,11}\right)$$
$$4\sqrt{\left(8F_{T,0} + 4F_{T,1} + 3F_{T,2}\right)\left(2F_{T,8} + F_{T,9}\right)} \geq \max\left(-8F_{T,5} - F_{T,7}, 8F_{T,5} + 4F_{T,6} + 3F_{T,7}\right)$$
$$4\sqrt{F_{T,9}\left(12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10}\right)} \geq \max\left(4F_{T,6} + F_{T,7} - 4F_{T,11}, F_{T,7} + 4F_{T,11}\right)$$
$$4\sqrt{6}\sqrt{\left(2F_{T,8} + F_{T,9}\right)\left(12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10}\right)} \geq \max\left[-3\left(8F_{T,5} + F_{T,7}\right), 3\left(8F_{T,5} + 4F_{T,6} + 3F_{T,7}\right)\right]$$
$$\sqrt{6}\sqrt{\left(4F_{T,8} + 3F_{T,9}\right)\left(6F_{T,0} + 2F_{T,1} + 3F_{T,2} + 6F_{T,10}\right)} \geq \max\left[-3\left(2F_{T,5} + F_{T,11}\right), 3\left(2F_{T,5} + F_{T,7} + F_{T,11}\right)\right]$$
$$2\sqrt{\left(12F_{T,8} + 7F_{T,9}\right)\left(12F_{T,0} + 4F_{T,1} + 5F_{T,2} + 4F_{T,10}\right)} \geq \max\left(-12F_{T,5} - F_{T,7} - 2F_{T,11}, -12F_{T,5} + 4F_{T,6} - F_{T,7} - 2F_{T,11}, -12F_{T,5} - F_{T,7} + 2F_{T,11}, 12F_{T,5} + 4F_{T,6} + 5F_{T,7} + 2F_{T,11}\right)$$

To formulate this approach, **symmetries of the system help**
(will also discuss cases without symmetries)

- Make use of symmetries of the problem (SM symmetries, helicities)
- Dispersion relation: $M^{ijkl} = \sum_X{}' \int_{(\epsilon\Lambda)^2}^{\infty} \frac{d\mu \, {m_X}^{ij} {m_X}^{kl}}{\pi(\mu - \frac{1}{2}M^2)^3} + (j \leftrightarrow l)$

Dynamics

- Becomes: $M^{ijkl} = \sum_{X\in r}{}' \int_{(\epsilon\Lambda)^2}^{\infty} d\mu \frac{|<X|M|r>|^2}{\pi\left(\mu - \frac{1}{2}M^2\right)^3} P_r^{i(j|k|l)}$

Symmetry

i(j|k|l): j,l symmetrized

- P_r^{ijkl} is the projective operator of an irrep r, obtained by CG coefficients.

$$P_r^{ijkl} = \sum_\alpha C_{i,j}^{r,\alpha} \left(C_{k,l}^{r,\alpha}\right)^*$$

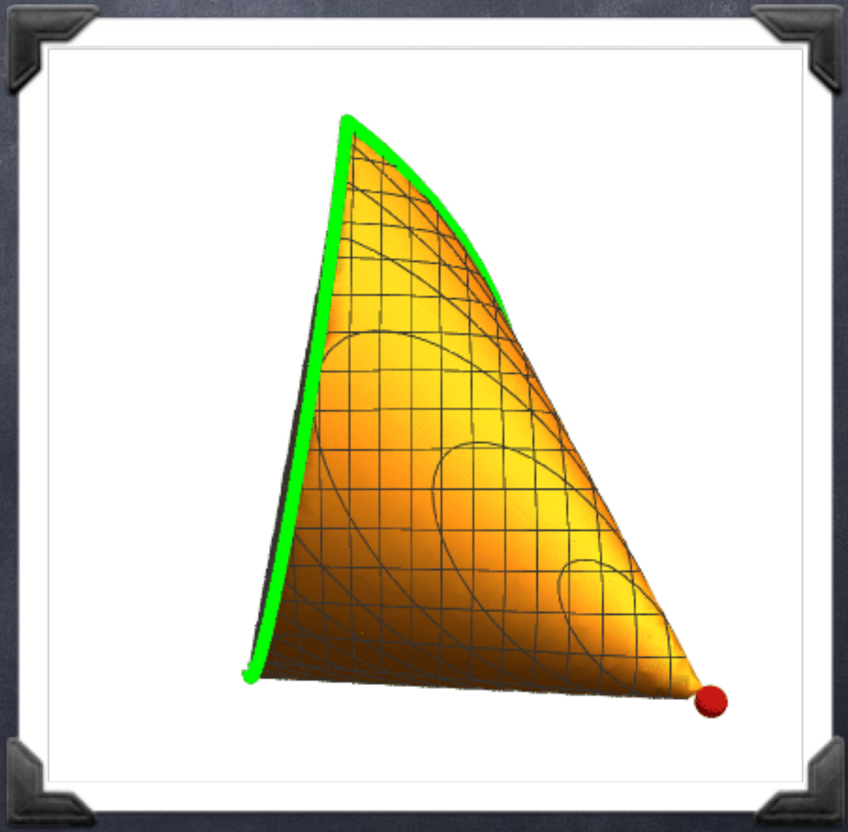
- The generators are simply (subset of) $P_r^{i(j|k|l)}$

- At least for simple cases, the ext(G) can be found by inspection.
- E.g. simplest case:
  n=2, with some Z2 symmetry, e=f=0, T -> $\begin{pmatrix} a & b & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & d & b \\ 0 & 0 & b & d \end{pmatrix}$
- There are two kinds of ERs
  - ER1: a=b=c=0, d=1
  - ER2: ac=b², d=|b|, a,c>0
- To get bounds, write the amplitude as

$$M^{ijkl} = \begin{pmatrix} C_1 & C_2 & 0 & 0 \\ C_2 & C_3 & 0 & 0 \\ 0 & 0 & C_4 & C_2 \\ 0 & 0 & C_2 & C_4 \end{pmatrix}$$

$$C_1, C_3, C_4 \geq 0 \text{ and } \sqrt{C_1 C_3} \geq \pm 2C_2 - C_4$$

A 3D cross section
of the 4D cone (a,b,c,d)



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$$\begin{pmatrix} a & b & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & d & b \\ 0 & 0 & b & d \end{pmatrix}$$

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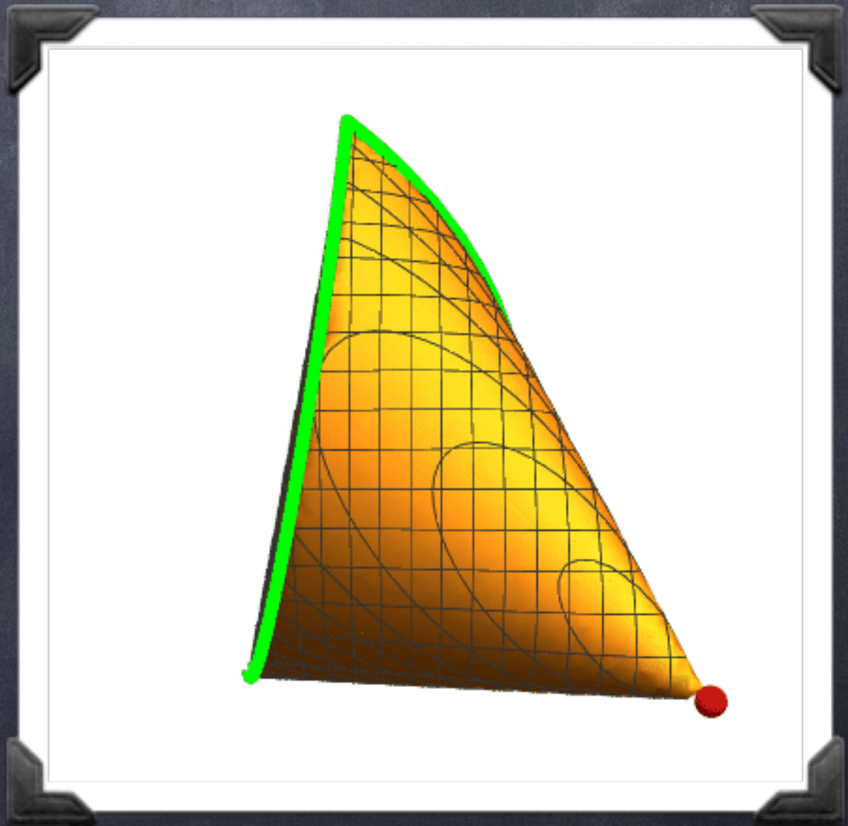
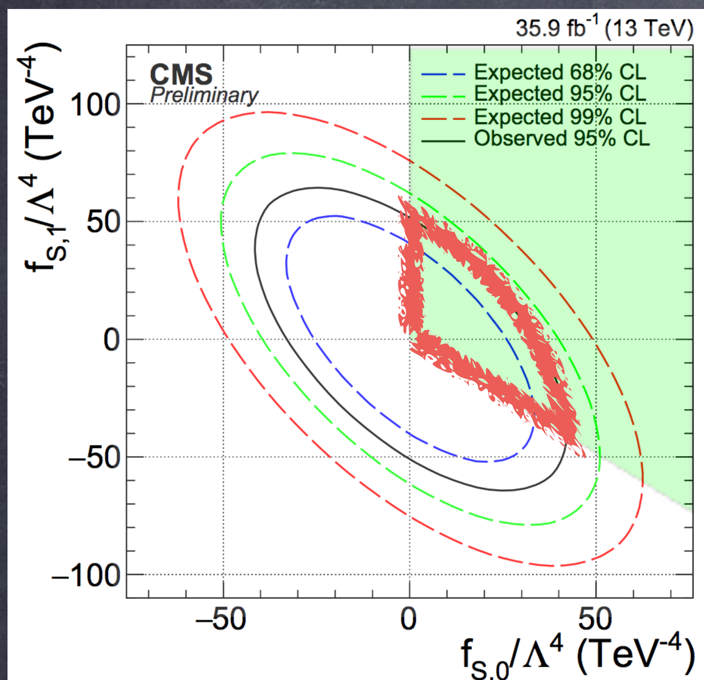
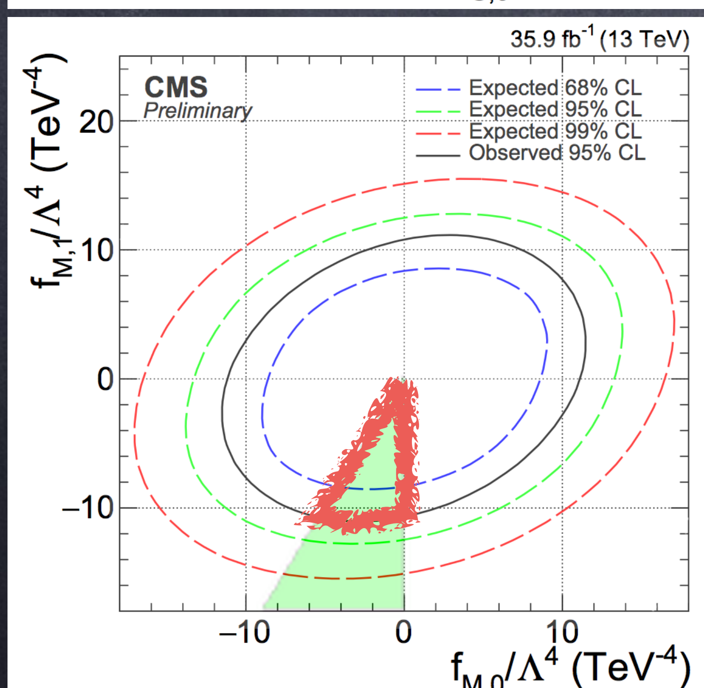


Pheno applications...

- May change the interpretation of measurements.





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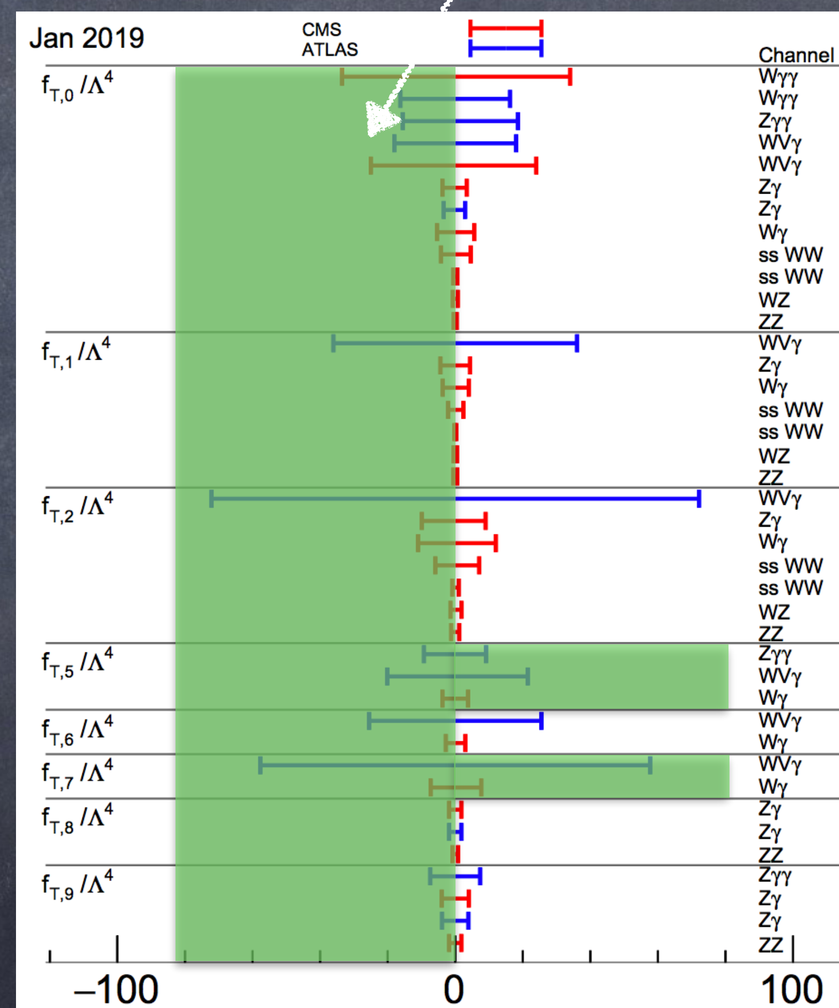
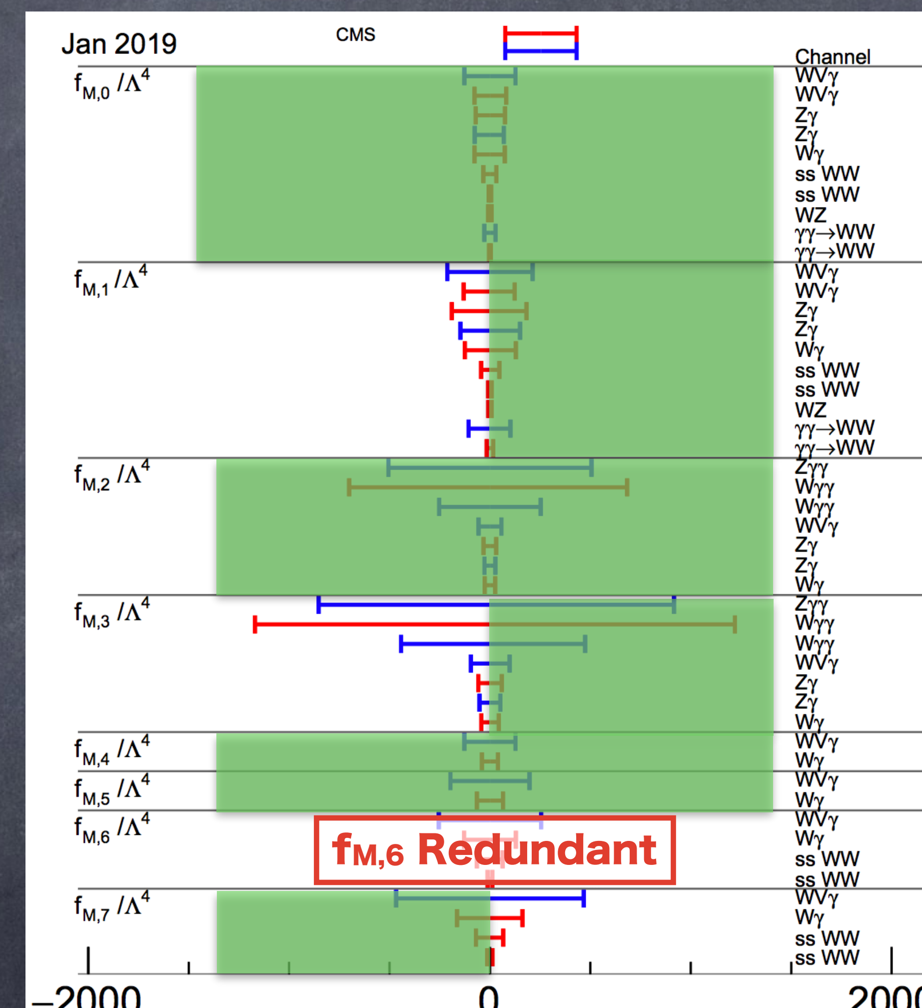




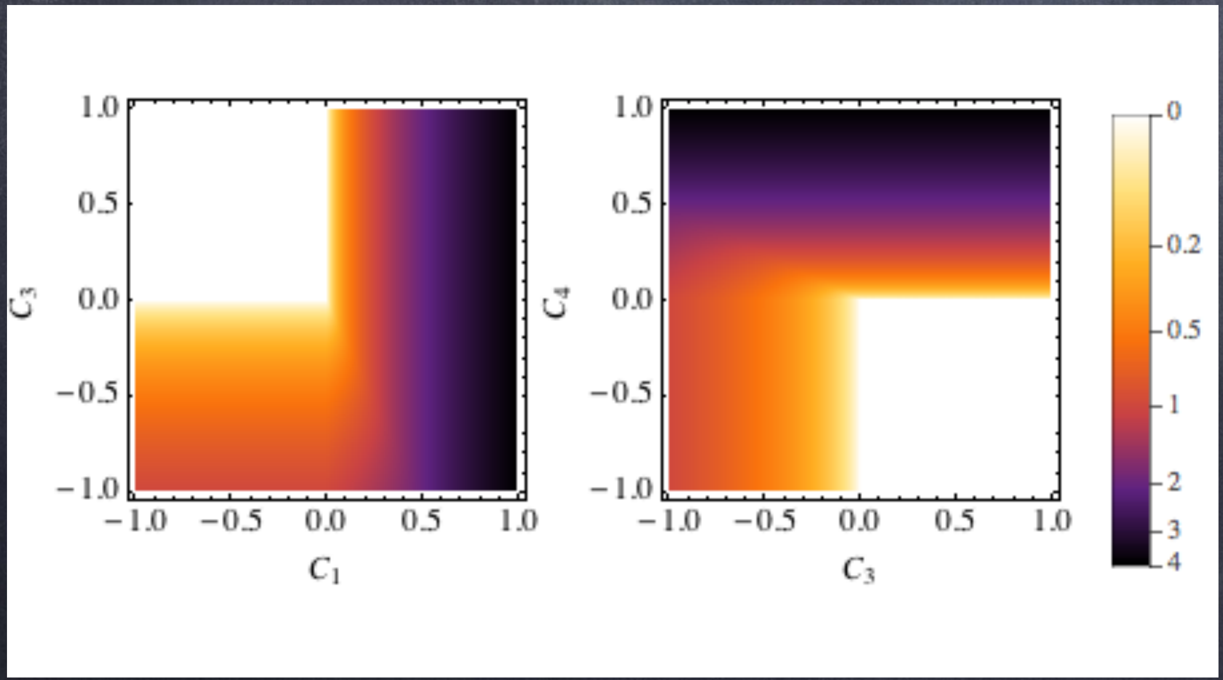
https://twiki.cern.ch/twiki/bin/view/CMSPublic/PhysicsResultsSMPaTGC#aQGC_Results

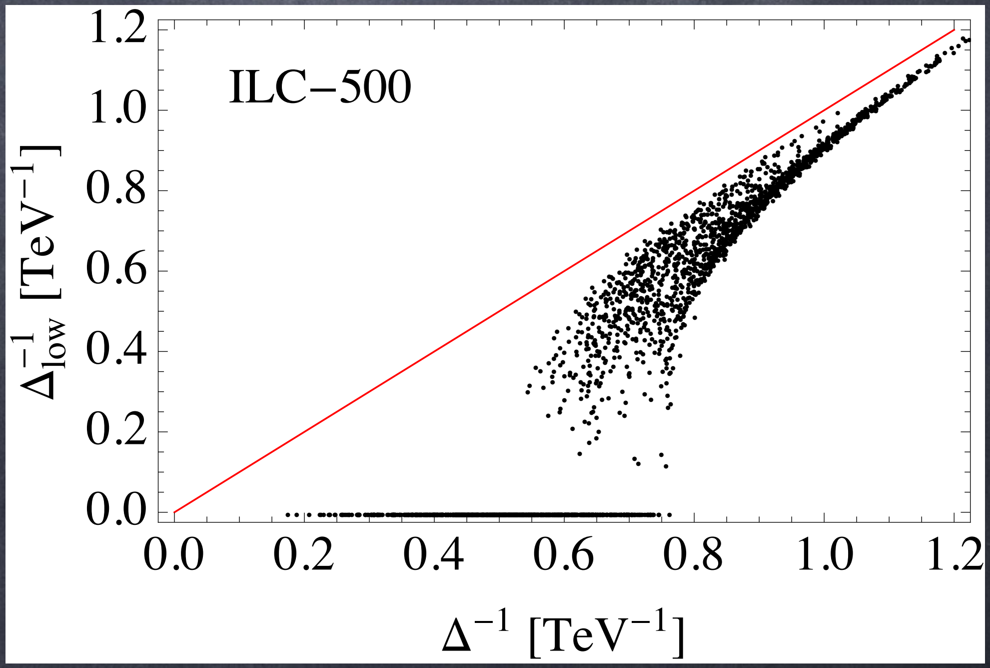
- Test QFT principles: Non-local UV completions violating polynomial boundedness, violation of Lorentz invariance, or even SMEFT expansion not valid…

- A similar study for future ee colliders: measure the "scale of violation"

$$-\Delta^{-4} \equiv \min\left[\min_{\epsilon_1,\epsilon_2} \frac{1}{2}\frac{\mathrm{d}^2 M(s,t=0)(\epsilon_1,\epsilon_2)}{\mathrm{d}s^2}, 0\right]$$

$\Delta^{-1} \in [\Delta^{-1}_{\text{low}}, \Delta^{-1}_{\text{high}}]$, due to exp error

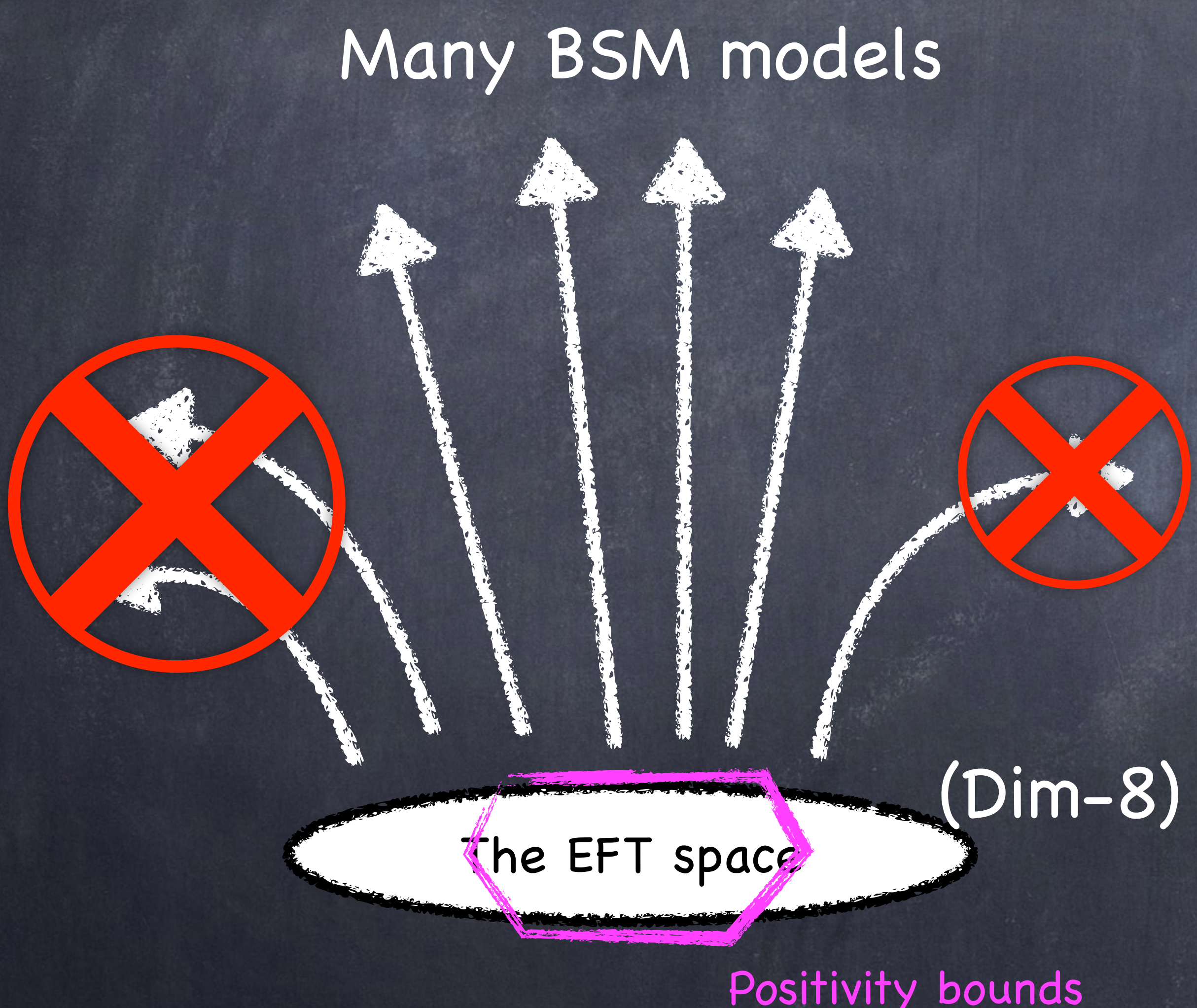


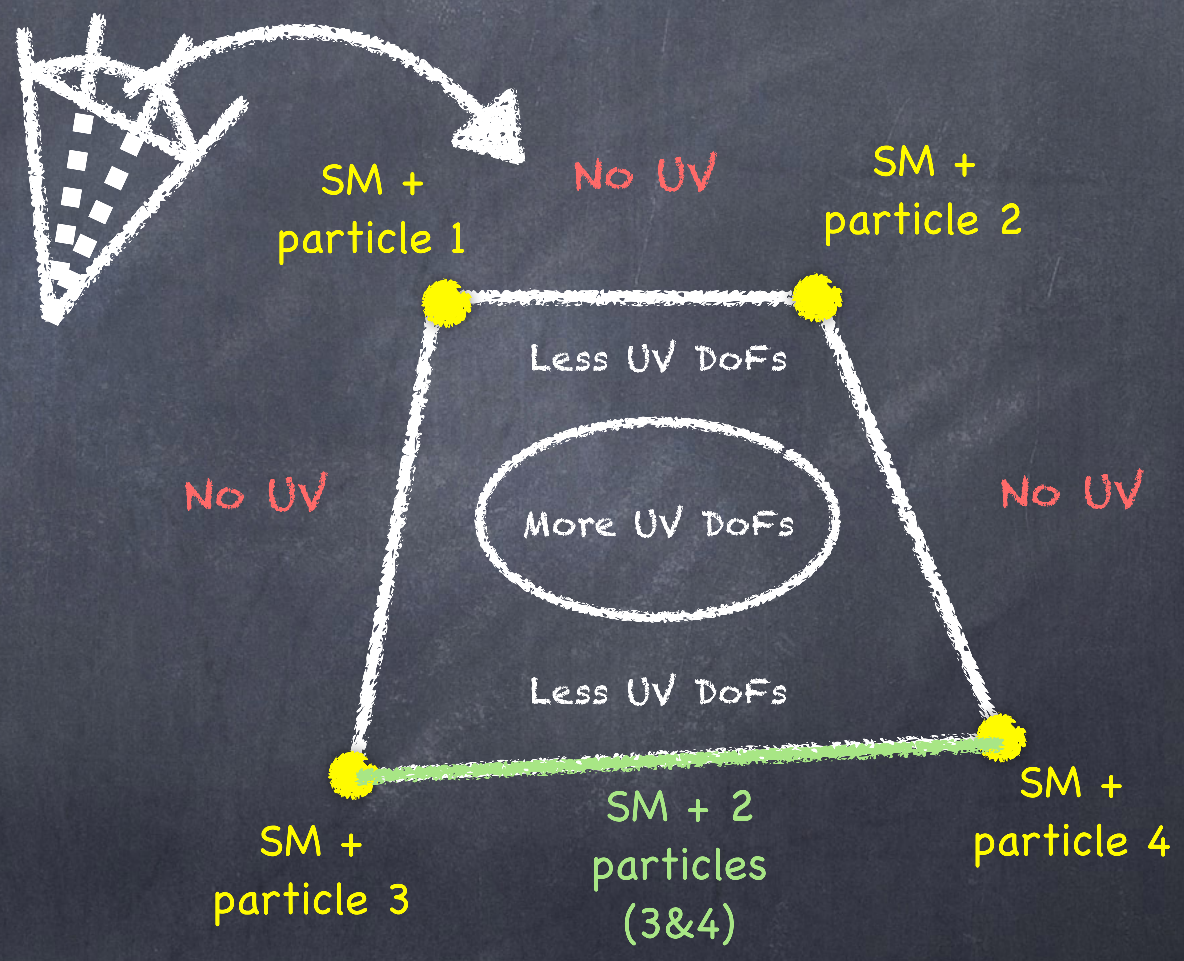


[2009.02212 B. Fuks, Y. Liu, CZ, S.-Y. Zhou]

- Infer UV model from EFT measurements

Inverse problem: Given the measured values of the operator coefficients around the electroweak scale, to what extend can we possibly determine the nature of the new physics beyond the SM? [Gu, Wang, 2008.07551]
see also [S. Dawson et al. 2007.01296] [N. Arkani-Hamed et al. hep-ph/0512190]





[CZ and S.-Y. Zhou 2005.03047]
[2009.02212 B. Fuks, Y. Liu, CZ, S.-Y. Zhou]

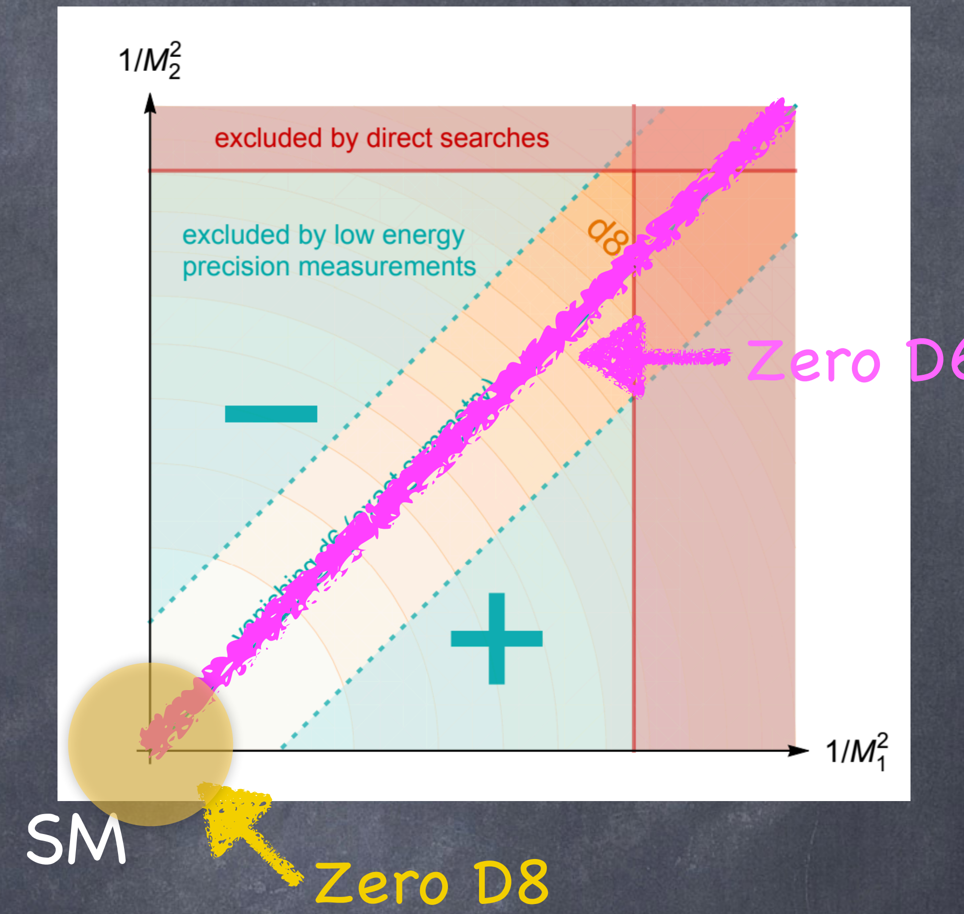
- Testing and confirming the SM: Null result of measurements at dim-6 does not exclude all BSM, but does at dim-8 by using positivity bounds

- Dim-6: no positivity, different states may cancel each other's effects.
  - E.g., scalar and vector generate 4-fermion operators with opposite signs.
  - No UV particle can be absolutely excluded.

- Dim-8: with positivity, different states are not allowed to cancel.
  - All states can be exclude to some absolute scale. (by using posi. bound)
  - Unlike dim-6 cannot lift this limit by adding more and more BSM states.
  - A robust confirmation of the SM.

[2009.02212 B. Fuks, Y. Liu, CZ, S.-Y. Zhou]

[Gu, Wang, 2008.07551]



Need for complete bounds:

- For most of these applications, its very useful to identify the **EXACT positivity bounds**.
    - To test violation of QFT principles, conservative bounds will degrade the sensitivity to the amount of violation.
    - To infer UV models/states, need to locate exactly the "vertices", or "edges", or more strictly the Extremal Rays in the positivity cone.
    - To exclude all BSM models, at least need positivity bounds in all directions. (So that the positivity cone cannot contain a straight line).