Local Unitarity: A representation of differential cross-sections that is locally free of infrared singularities at any order

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Why better predictions?

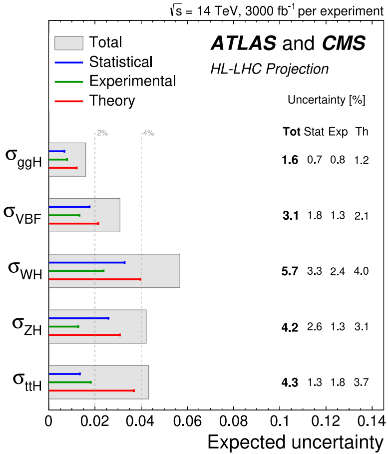
Goal

The goal of this project is to improve the predictive power of Quantum Field Theories

- The Standard Model (SM) is hugely successful
- Higgs boson agrees well between theory and experiment
- SM cannot be the end of the story: does not describe gravity, dark matter, etc
- Beyond SM (BSM) physics could show itself in small deviations

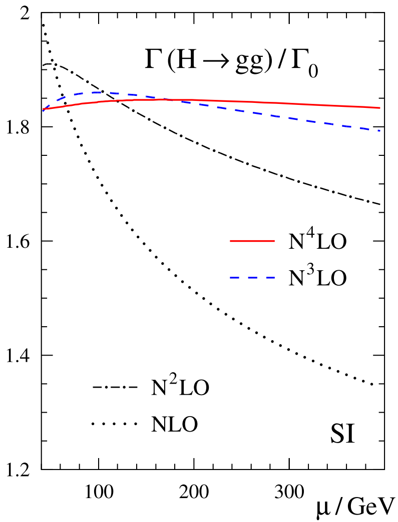
Keeping up with experiment

- High Luminosity LHC is expected to drastically reduce statistical uncertainty
- Theory will be the leading contribution to the uncertainty
- We need to compute more higher-order corrections



The effect of higher-order corrections

- Large reduction of uncertainty
- Red line is an *NNNNLO* computation [Herzog, BR, Ueda, Vermaseren, Vogt '17]
- NLO cross sections are availabe with the push of a button
- Why no automation at NNLO?



Traditional pipeline

Integrate loop amplitudes analytically

- Substitute the Feynman rules in $D = 4 - 2\epsilon$
- Integration by Parts reduction to master integrals
- Use differential equations to solve master integrals

Integrating the cross section numerically

- Square the amplitude
- Construct local IR counterterms
- Add parton distribution functions (PDFs)
- Add observables, do jet clustering, etc.
- Monte Carlo integrate

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Local unitarity

Wishes	Local Unitarity
Differential cross sections	
No restrictions on mass scales	
No reduction to master integrals	
No complicated function evaluations	
No dimensional regularisation	
No IR counterterms for loop degrees of freedom	
No IR counterterms for real degrees of freedom	
Integration in lower dimensions	
Contour deformation only when needed	
Method generic for any process to any order	

Local unitarity

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Differential cross sections	✓
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Integration in lower dimensions	✓
Contour deformation only when needed	✓
Method generic for any process to any order	✓

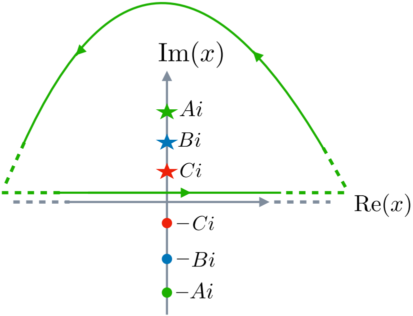
Amplitudes

- First we study the singularities of amplitudes
- Our treatment of amplitudes will guide the extension to cross sections
- Loop-Tree Duality (LTD): integrate out the loop energies analytically

Residue theorem

$$I = \int dx F(x)$$

$$F(x) = \frac{1}{(x-Ai)(x+Ai)} \frac{1}{(x-Bi)(x+Bi)} \frac{1}{(x-Ci)(x+Ci)}$$



Cauchy: $R(x^*) \equiv Res(F, x = x^*)$

$$I = (-2\pi i)\left[R(Ai) + R(Bi) + R(Ci)\right]$$

Inverse propagator notation

$$(k+p_i)^2 - m_i^2 + i\epsilon = (k^0+p_i^0)^2 - (\vec{k}+\vec{p}_i)^2 - m_i^2 + i\epsilon$$

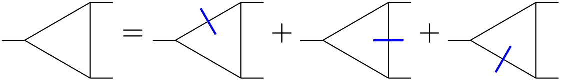
Factorizing:

$$\left(k^0 + p_i^0 + \sqrt{(\vec{k}+\vec{p}_i)^2 + m_i^2 - i\epsilon}\right)\left(k^0 + p_i^0 - \sqrt{(\vec{k}+\vec{p}_i)^2 + m_i^2 - i\epsilon}\right)$$

We call the spatial and mass part Δ:

$$\left(k^0 + p_i^0 + \Delta_i\right)\left(k^0 + p_i^0 - \Delta_i\right)$$

Residue theorem for propagators



We take the positive energy solution:

$$\delta(k^0 + p_i^0 - \Delta_i) \to k^0 = \Delta_i - p_i^0$$

Effect of δ on propagator i:

$$\left(k^0 + p_i^0 + \Delta_i\right) \to 2\Delta_i$$

Effect of δ on other propagators j:

$$\left(\Delta_i - p_i^0 + p_j^0 + \Delta_j\right)\left(\Delta_i - p_i^0 + p_j^0 - \Delta_j\right) = E_{ij} H_{ij}$$

Loop Tree Duality (LTD)

- LTD yields a sum of cut diagrams without loops [Catani, Gleisberg, Krauss, Rodrigo, Winter '09]
- Expression at one-loop:

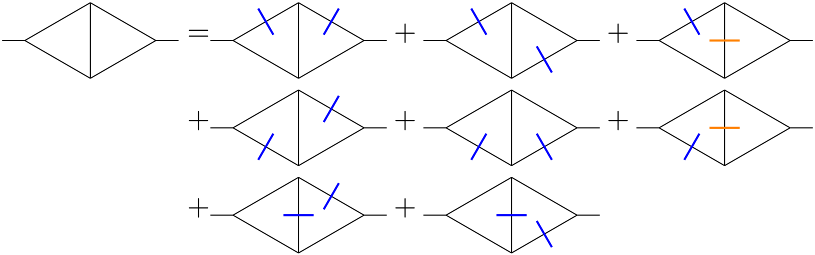
$$I = -\int d^4 k \sum_i^N \tilde{\delta}(q_i^2) \prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{q_j^2 - i0\eta(q_j - q_i)}$$

- Alternatively:

$$I = -\int d^3 k \sum_i^N \frac{1}{2\Delta_i} \prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{\left(\Delta_i - p_i^0 + p_j^0 + \Delta_j\right)\left(\Delta_i - p_i^0 + p_j^0 - \Delta_j\right)}$$

Multi-Loop-Tree Duality

- Integrate the energies from all loop variables one by one [Capatti,Kermanschah,Hirschi,Ruijl '19, Bierenbaum, Catani, Draggiotis, Rodrigo '10]
- Effect: cut all propagator combinations that leave no loops
- Due to the iterative procedure, sometimes the negative energy solution has to be taken (orange)



Singular structure

When is the inverse propagator 0?

$$E_{ij} \equiv \Delta_i - p_i^0 + p_j^0 + \Delta_j = 0$$
$$H_{ij} \equiv \Delta_i - p_i^0 + p_j^0 - \Delta_j = 0$$

- Δ_i is always ≥ 0
- E_{ij} is an ellipsoid
- H_{ij} is a hyperboloid

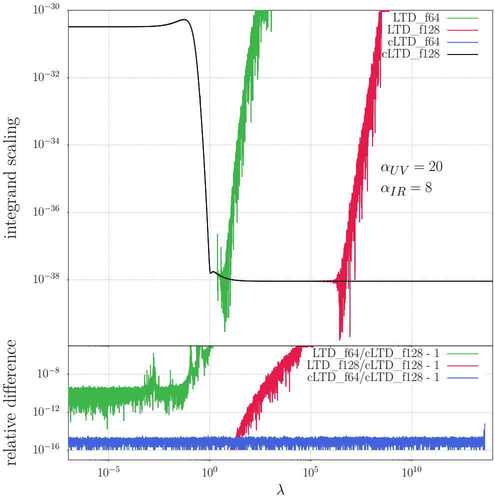
Hyperboloid cancellation

Every hyperboloid appears twice with opposite sign:

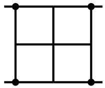
$$H_{ij} = \Delta_i - p_i^0 + p_j^0 - \Delta_j$$
$$H_{ji} = \Delta_j - p_j^0 + p_i^0 - \Delta_i$$

- The first surface appears when cutting i and the second appears in the cut of j
- All hyperboloids cancel!
- All singularities are ellipsoids and are thus bounded!
- This holds for any loops

Manifestly Causal Loop-Tree Duality (cLTD)



- An LTD term of



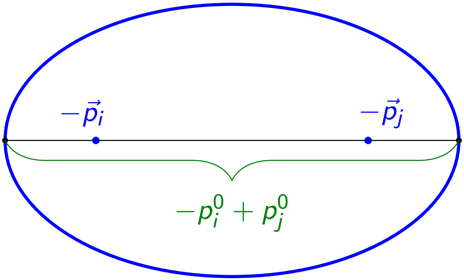
  scales as λ^0 and the sum as λ^{-8}
- The hyperboloids can be canceled analytically [Capatti, Kermanschah, Hirschi, Pelloni, BR '20]
- cLTD gives improved ultraviolet behaviour

Ellipsoids

$$E_{ij} = \sqrt{(\vec{k}+\vec{p}_i)^2 + m_i^2 - i\epsilon} + \sqrt{(\vec{k}+\vec{p}_j)^2 + m_j^2 - i\epsilon} - p_i^0 + p_j^0 = 0$$

Massless case (and ignoring $i\epsilon$):

$$|\vec{k}+\vec{p}_i| + |\vec{k}+\vec{p}_j| - p_i^0 + p_j^0 = 0$$



Results

- Existence depends on external momenta
- It is possible to have configurations without ellipsoids

[PRL '19, Zeno Capatti, Dario Kermanschah, Valentin Hirschi, Ben Ruijl]

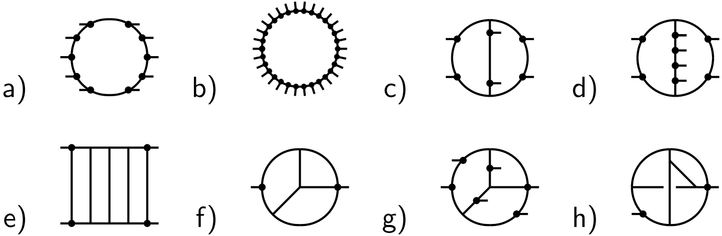
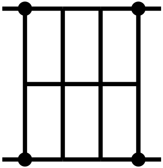


Table: All within 1% of the analytical result

Pushing the boundaries

6-loop four-point function:



- Analytic: $8.4044862640909 \cdot 10^{-19}$ [Basso, Dixon '17]
- Numerical: $8.38533 \cdot 10^{-19} \pm 2.99674 \cdot 10^{-21}$

Ellipsoids II

$$\sqrt{(\vec{k}+\vec{p_i})^2 + m_i^2 - i\epsilon} + \sqrt{(\vec{k}+\vec{p_j})^2 + m_j^2 - i\epsilon} - p_i^0 + p_j^0 = 0$$

- A deformation is only required on the ellipsoid surfaces
- The signature is simply:

$$\frac{-i\epsilon}{\sqrt{(\vec{k}+\vec{p})^2 + m^2}} = -i\epsilon$$

- A valid contour deformation should always yield this signature

Complex contour deformation

Ingredients:

- $k \to k - i\kappa(k)$
- Determine correct deformation direction
- Determine correct deformation magnitude
- Compute Jacobian of deformation

Deformation direction

- Expanding ellipsoids equation to first order:

$$\sqrt{(\vec{k} - i\vec{\kappa} + \vec{p_i})^2 + m_i^2} + \sqrt{(\vec{k} - i\vec{\kappa} + \vec{p_j})^2 + m_j^2} - p_i^0 + p_j^0$$

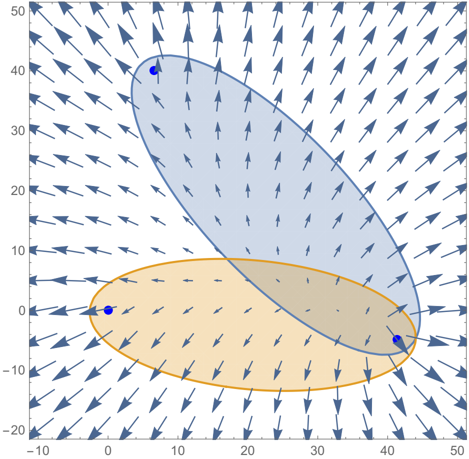
yields:

$$\vec{\kappa} \cdot \underbrace{\left(\frac{\vec{k} + \vec{p_i}}{\Delta_i} + \frac{\vec{k} + \vec{p_j}}{\Delta_j} \right)}_{\text{Normal vector}} > 0$$

- Positive projection on normal $\rightarrow$ point outward

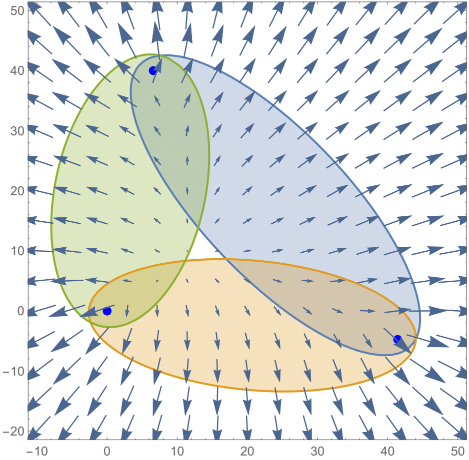
Deformation direction

- The normal is always a good deformation for one ellipsoid
- The sum of normals also works for intersections of two ellipsoids
- Deformation was proposed in [Buchta, Chachamis, Draggiotis, Rodrigo '17]



Normal vectors I

- In general the sum will not work
- Exponential dampening around the ellipsoids does not work without extensive finetuning



Normal vectors II

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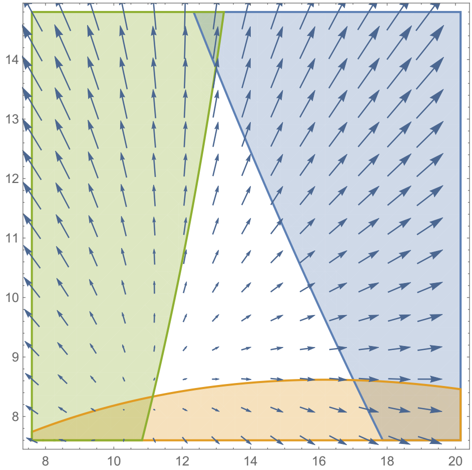
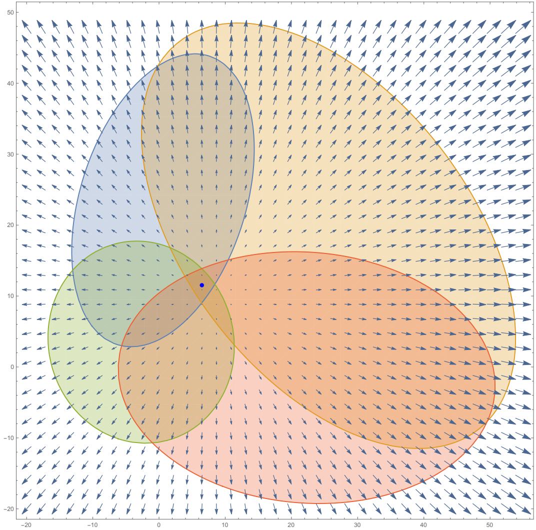


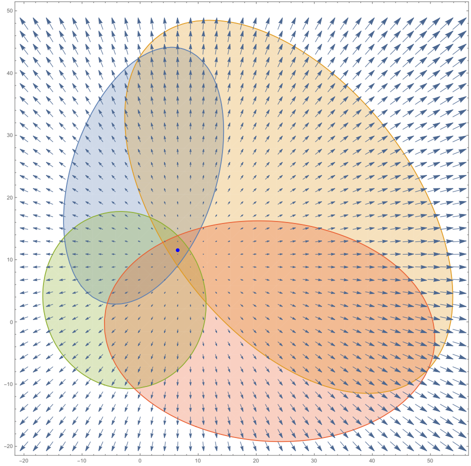
How to deform here?

Deformation sources



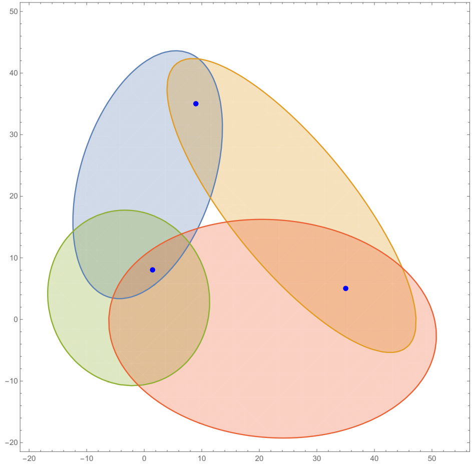
Deformation direction

- Any *source* inside all ellipsoids will produce a good vector!
- Finding a point of overlap is a convex optimization problem
- A solution is always found if it exists



Deformation direction

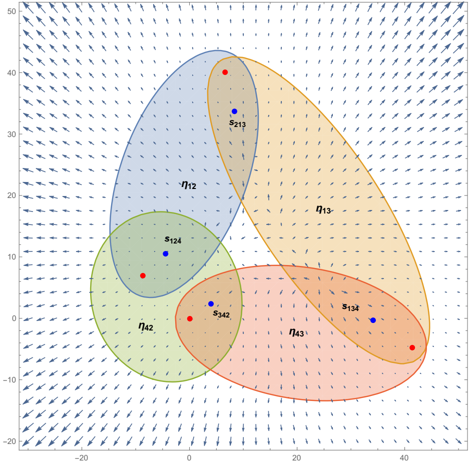
- Finding the maximal overlap structure is NP-hard
- Three sources needed for picture on the right



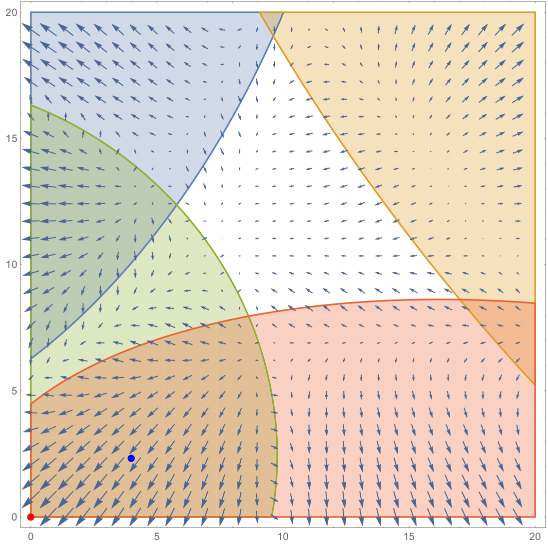
Deformation exclusion

- Deformation field is sum of fields multiplied by function $T(E_{ij})$ that is 0 on a surface E_{ij} and goes to 1 away from it
- Exclusion guarantees correct deformation

$$\vec{\kappa} = (\vec{k} - \vec{s}_{124})T(E_{13})T(E_{43}) + (\vec{k} - \vec{s}_{213})T(E_{42})T(E_{43}) + (\vec{k} - \vec{s}_{134})T(E_{42})T(E_{12}) + (\vec{k} - \vec{s}_{342})T(E_{12})T(E_{13})$$



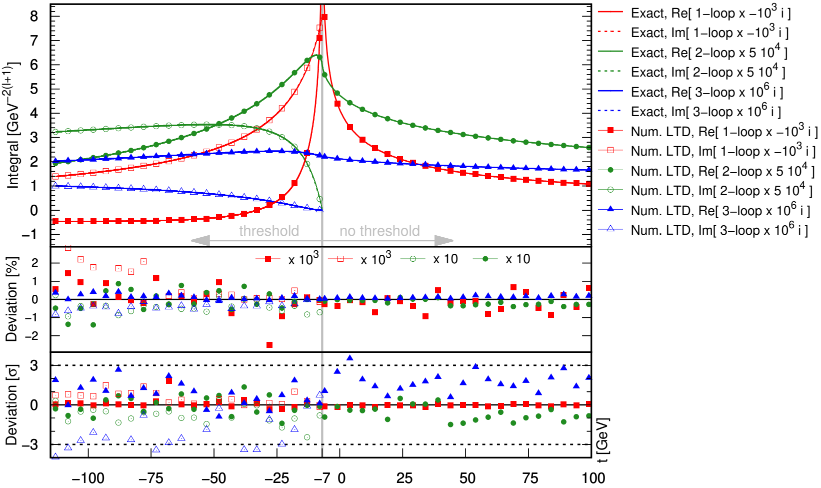
Deformation direction



Deformation magnitude

- $k \to k - i\lambda(k)\kappa(k)$
- Deformation needs to be as large as possible to be furthest from the divergence for numerical stability
- It can also not be too large:
  - Cannot cross any branch cuts of the square root
  - Needs to satisfy expansion condition
  - Cannot cross poles on the complex plane

A scan for multi-loop box topologies



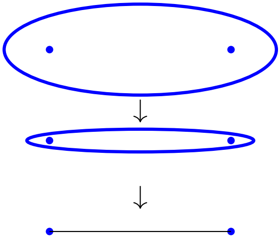
Pinched ellipsoids

- An ellipsoid is *pinched* when it has no interior
- No deformation possible
- *All* IR singularities are pinched ellipsoids
- Pinching not possible if there are masses
- Occurs when external momenta are on-shell:

$$|\vec{k}+\vec{p}|+|\vec{k}|-p^0=0 \to$$
$$|\vec{k}+\vec{p}|+|\vec{k}|-|\vec{p}|=0$$

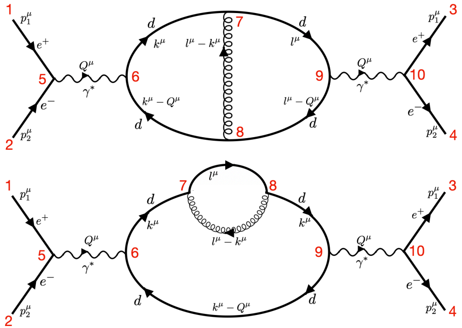
when $\vec{k}=x\vec{p}$ with $0 \leq x \leq 1$

- Collinear singularity with soft singularities on the focal points

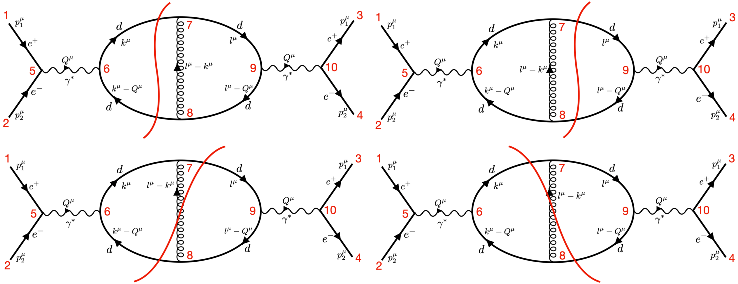


Optical theorem

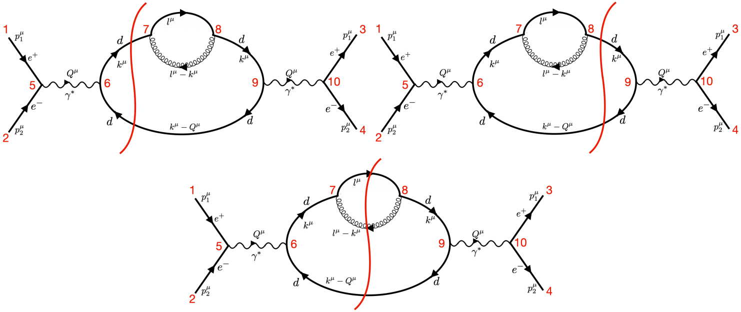
- The cross section is given by the sum of all Cutkosky cuts over all *supergraphs*
- Each cut supergraph is IR finite
- For NLO $e^+e^- \to d\bar{d}$ we have 2 supergraphs:



Double-triangle cuts

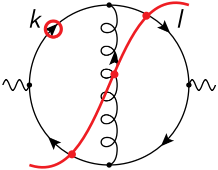


Bubble cuts


1
p_1^μ
e^+
5
Q^μ
γ^*
e^-
2
p_2^μ
6
d
k^μ
7
l^μ
8
$l^\mu - k^\mu$
d
k^μ
9
$k^\mu - Q^\mu$
d
d
Q^μ
γ^*
10
e^+
p_1^μ
3
e^-
p_2^μ
4

Cancelling singularities I

- Consider Cutkosky cuts with additional on-shell propagators:



- $k^0 \pm |\vec{k}| \to (|\vec{l}| + |\vec{l} - \vec{k}| + |\vec{k}|)\ (|\vec{l}| + |\vec{l} - \vec{k}| - |\vec{k}|) \xrightarrow{\lim} 2|k|$

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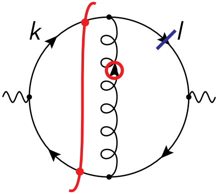

- $l^0 - k^0 \pm |\vec{l} - \vec{k}| \to$
  $(|\vec{l}| - |\vec{k}| + |\vec{l} - \vec{k}|)\,(|\vec{l}| - |\vec{k}| - |\vec{l} - \vec{k}|) \xrightarrow{\lim} -2|\vec{l} - \vec{k}|$

Cancelling singularities I

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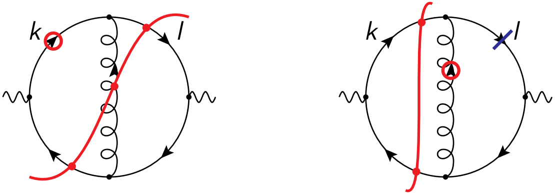
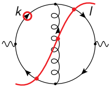


- $k^0 \pm |\vec{k}| \to (|\vec{l}|+|\vec{l}-\vec{k}|+|\vec{k}|)\,(|\vec{l}|+|\vec{l}-\vec{k}|-|\vec{k}|) \xrightarrow{\lim} 2|k|$
- $l^0 - k^0 \pm |\vec{l}-\vec{k}| \to$
  $(|\vec{l}|-|\vec{k}|+|\vec{l}-\vec{k}|)\,(|\vec{l}|-|\vec{k}|-|\vec{l}-\vec{k}|) \xrightarrow{\lim} -2|\vec{l}-\vec{k}|$
- Exactly the same on-shell propagators (and $2\Delta_i$s), but with a relative sign

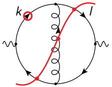
Cancelling singularities: observable functions



$$= N_3 \delta(p^0 - |\vec{k} - \vec{p}| - |\vec{l} - \vec{k}| - |\vec{l}|) \mathcal{O}_3(-k + p, -l + k, l)$$

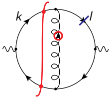
$$\xrightarrow{\lim} N \delta(p^0 - |\vec{k} - \vec{p}| - |\vec{k}|) \; \mathcal{O}_3(-k + p, -l + k, l)\big|_{k^0 = |\vec{k}|}$$

Cancelling singularities: observable functions



$$= N_3\delta(p^0 - |\vec{k} - \vec{p}| - |\vec{l} - \vec{k}| - |\vec{l}|)\mathcal{O}_3(-k + p, -l + k, l)$$

$$\xrightarrow{\lim} N\delta(p^0 - |\vec{k} - \vec{p}| - |\vec{k}|)\ \mathcal{O}_3(-k + p, -l + k, l)\big|_{k^0=|\vec{k}|}$$

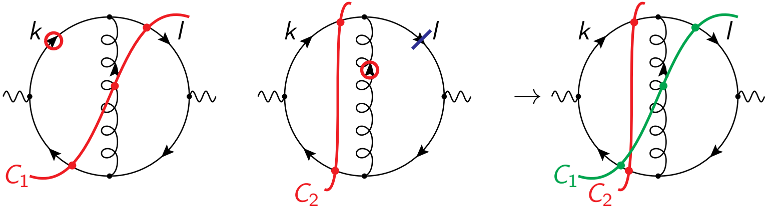


$$= N_2\delta(p^0 - |\vec{k} - \vec{p}| - |\vec{k}|)\mathcal{O}_2(k, -k + p)$$

$$\xrightarrow{\lim} -N\delta(p^0 - |\vec{k} - \vec{p}| - |\vec{k}|)\ \mathcal{O}_2(k, -k + p)\big|_{l^0-k^0=-|\vec{l}-\vec{k}|}$$

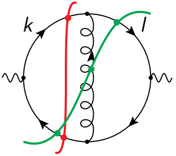
- Energy-conserving δs are the same on pinches
- Only cancels for IR-safe observables: $\mathcal{O}_3|_{\text{on pinch}} = \mathcal{O}_2|_{\text{on pinch}}$

Cancellations between Cutkosky cuts

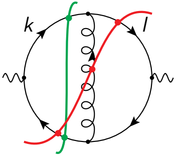


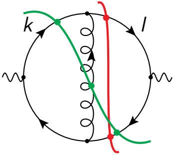
- Singularities lie on intersections of C_1 and C_2
- The intersection point of C_1 and C_2 is finite in $C_1 + C_2$
- The cancellations are pairwise (as with hyperboloids)

Cancellations between Cutkosky cuts

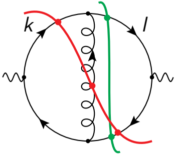


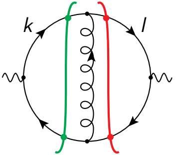
cancels



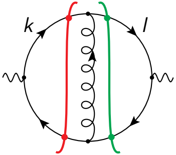


cancels



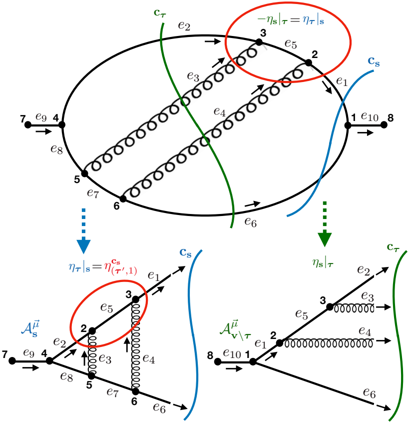


cancels



Non-pinched ellipsoid cancellation if the observable is 1!

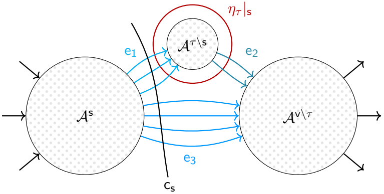
Higher-loop example



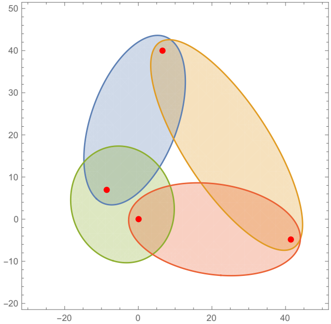
- Singularity when $\vec{p}_1 // \vec{p}_2 // \vec{p}_3 // \vec{p}_4$
- Real-emission triple-collinear phase-space singularity
- Cancels with the pinched surface in the two-loop amplitude
- Cut c_τ with e_1 on-shell cancels with cut c_s with LTD cut e_3, e_4 and e_2 on-shell

Cancellation pattern

Pairwise cancellation pattern holds for **any threshold** at **any order**

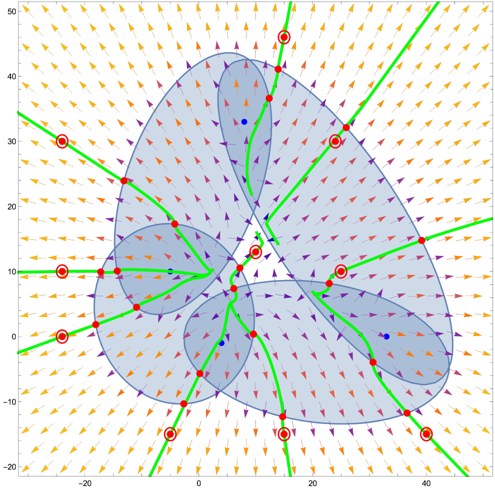


Sampling on ellipsoids



- The energy delta functions are ellipsoids, e.g. $\delta(p^0 - |\vec{k}| - |\vec{k} - \vec{p}|)$
- Map integration variables $[0,1]^{3L}$ onto surfaces
- Make sure every point on an ellipsoid can be sampled
- Use a global map (not per Cutkosky cut) so that both cuts are sampled on their intersection

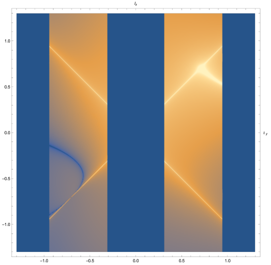
Sampling on ellipsoids



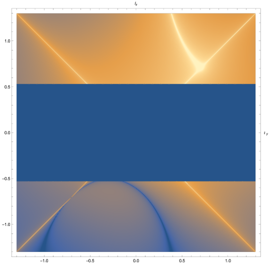
- The deformation field gives us this map!
- Simply walk back along the vector field
- Can be solved using an ODE solver

Results for Double-Triangle I

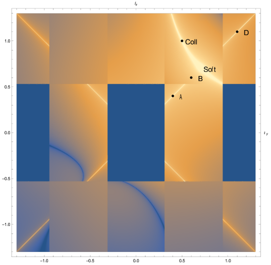
Semi-inclusive cross section for $(\vec{k}, \vec{l}) = ((0, k_y, \frac{1}{\sqrt{2}}), (0, \frac{1}{\sqrt{2}}, l_z))$



(a) Virtual cut on the left

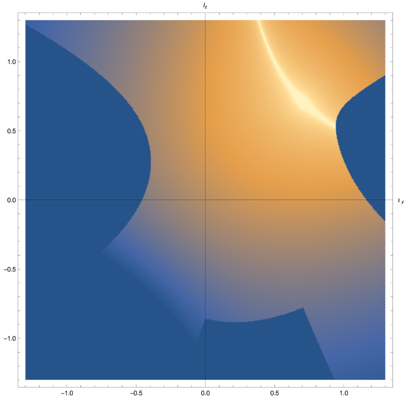


(b) Virtual cut on the right

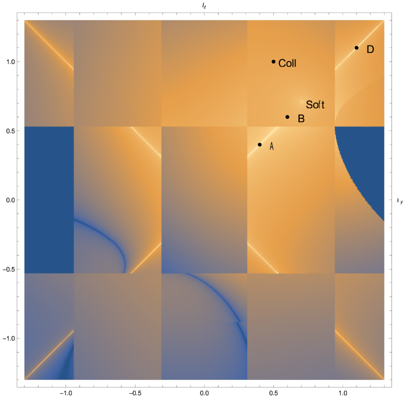


(c) Sum of virtual cuts

Results for Double-Triangle II

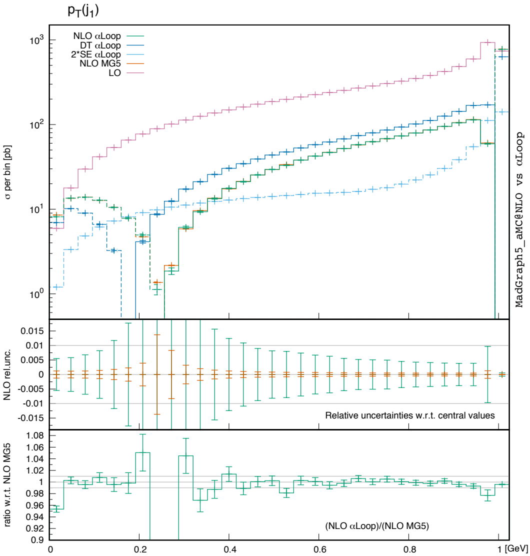


(a) Sum of 2 real cuts

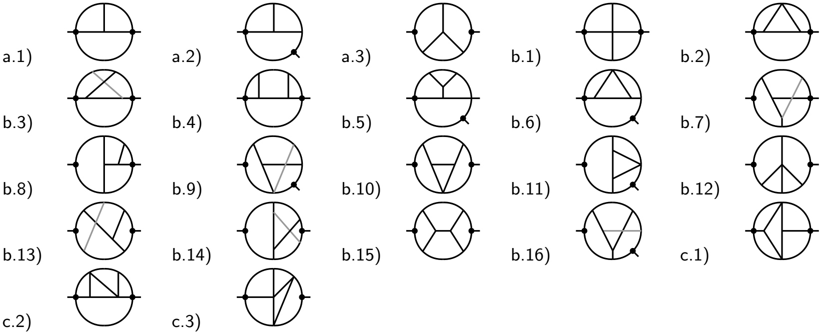


(b) Sum of all cuts

Results for NLO correction to $e^+e^- \to d\bar{d}$

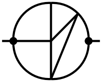


Results for scalar topologies



Results for scalar topologies

- Scalar diagrams have the worst IR divergences
- Example: [Forcer+R*, Herzog,Ruijl,Ueda,Vermaseren '18]

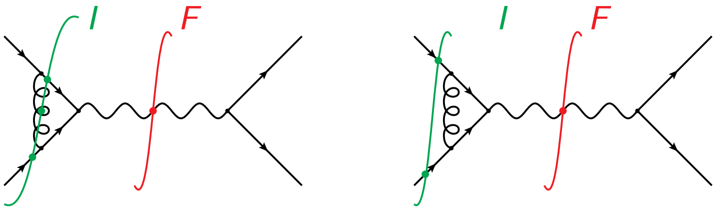


$$\xrightarrow{\text{optical theorem}} \frac{5\pi}{(16\pi)^5}\frac{441}{40}\zeta_7 = 1.77832 \cdot 10^{-9}$$

- LU with 1M samples:

$$I = 1.7797(33) \cdot 10^{-9}$$
$$\Delta_\sigma = 0.42\sigma$$
$$\Delta_\% = 0.077\%$$

Initial-state radiation



- Initial-state infrared singularity cancellations to be studied
- Suggests extra particles in the initial state (as in the final state)
- Fixing the number of initial-state particles is a non-IR-safe observable
- Find connection to classical PDFs and PDF counterterms

Conclusion

- In LU final-state infrared singularities cancel among real and virtual contributions for any process and at any order
- In LU all bottlenecks of traditional methods are avoided
- Initial-state radiation remains to be studied

The end-goal

The Local Unitarity approach will offer a novel way to compute higher-order corrections to any process with the push of a button