New exact solutions of relativistic, dissipative hydrodynamics

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arXiv:2003.08859

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Outline

New, exact solution of relativistic Navier-Stokes (NS) and Israel-Stewart (IS) theory

→ spherically symmetric Hubble-flow: great amount of freedom of dissipative coefficients

Asymptotic behaviour of the new solutions

→ effects of dissipative coefficients in final state measurments?

Applications of the new solutions

→ indirect description of experimental data

→ producing new, non relativistic solutions

Relativistic hydrodynamics (Navier-Stokes)

Local conservation of the four momentum and the particle number:

$$\partial_\mu (n u^\mu) = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

The energy-momentum tensor is:

$$T^{\mu\nu} = (\varepsilon + p)\, u^\mu u^\nu - p g^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu} - \Delta^{\mu\nu} \Pi$$

The heat current (with the heat conductivity λ):

$$q^\mu = \lambda \left(g^{\mu\nu} - u^\mu u^\nu\right)\left(\partial_\nu T - T u^\rho \partial_\rho u_\nu\right)$$

ζ: bulk viscosity
η: shear viscosity

The following terms describes the viscous effects:

$$\pi^{\mu\nu} = \eta \left[\Delta^{\mu\rho}\partial_\rho u^\nu + \Delta^{\nu\rho}\partial_\rho u^\mu\right] - \frac{2}{d}\eta \Delta^{\mu\nu}\partial_\rho u^\rho \qquad \Pi = -\zeta \partial_\rho u^\rho$$

Relativistic hydrodynamics (Israel-Stewart)

Local conservation of the four momentum and the particle number:

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Relativistic hydrodynamics (Israel-Stewart)

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$$\Pi = -\zeta \partial_\rho u^\rho - \tau_\Pi u_\rho \partial^\rho \Pi$$

To close the equation system:

EoS: ε=κp

In this work: *κ=const.*

ζ: bulk viscosity

η: shear viscosity

Hubble-type solutions: scale variable

Hubble-type velocity field: $u^\mu = \frac{x^\mu}{\tau} = \gamma\left(1, \frac{r_x}{t}, \frac{r_y}{t}, \frac{r_z}{t}\right)$

Scale equation: $u^\mu \partial_\mu s = 0$

Directional scale variables: $s_x = \frac{r_x}{t},\ s_y = \frac{r_y}{t},\ s_z = \frac{r_z}{t}$

Satisfy the scale equation separately: $u^\mu \partial_\mu s_i = \partial_\tau s_i = 0$

Hubble-type solutions: equations to solve

Navier-Stokes theory

Continuity equation: $\partial_\tau n + \frac{d}{\tau} n = 0$

Energy conservation $\partial_\tau p + \left(1 + \frac{1}{\kappa}\right) \frac{d}{\tau} p = \frac{d^2}{\tau^2} \frac{\zeta}{\kappa}$

Euler-equation: $p\tau - \zeta d = \phi(\tau)$

Entropy equation: $\partial_\tau \sigma + \frac{d}{\tau} \sigma = \frac{d^2}{\tau^2} \frac{\zeta}{T} \geq 0$

Ansatz for bulk viscosity: $\zeta = \zeta_0 \frac{p}{p_0}$

M. Nagy, M. Csanád, Z. Jiang, T. Csörgő: arXiv:1909.02498

Israel-Stewart theory

Continuity equation: $\partial_\tau n + \frac{d}{\tau} n = 0$

Energy conservation: $\partial_\tau p + \left(1 + \frac{1}{\kappa}\right) \frac{d}{\tau} p = -\frac{d}{\tau} \frac{\Pi}{\kappa}$

Bulk pressure: $\Pi = -\zeta \frac{d}{\tau} - \tau_\Pi \dot{\Pi}$

Euler-equation: $p + \Pi = \Psi(\tau)$

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T. Csörgő, G. K.: arXiv:2003.08859

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$\tau_\Pi \rightarrow 0$, or Π is constant

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T. Csörgő, G. K.: arXiv:2003.08859

Heat conduction and shear viscosity cancelled!

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Analytic solutions of NS equations, with κ=const

The solution of the pressure is: $p(\tau) = p_0 \left(\frac{p_A}{p_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa}\right)}, \quad \frac{p_A}{p_0} = f_{A,0} = \exp\left[\frac{d^2\zeta_0}{\kappa_0 p_0 \tau_0}\right]$

The temperature has a generalized form: $T = T_0 \left(\frac{T_A}{T_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right)$

T. Csörgő, G. K.: arXiv:2003.08859

Conserved charge, μ > 0

$$p = nT$$

$$\frac{T_A}{T_0} = \frac{p_A}{p_0} = f_{0,A} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0}\right)$$

$$n = n_0 \left(\frac{\tau_0}{\tau}\right)^d \mathcal{V}\left(s_x, s_y, s_z\right)$$

No conserved charge, μ = 0

$$p = \frac{T\sigma}{1+\kappa}$$

$$\frac{T_A}{T_0} = \left(\frac{\sigma_A}{\sigma_0}\right)^{\frac{1}{\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0} \frac{1}{1+\kappa_0}\right)$$

$$\sigma = \sigma_0 \left(\frac{\sigma_A}{\sigma_0}\right)^{1-\frac{\tau_0}{\tau}} \left(\frac{\tau_0}{\tau}\right)^d \mathcal{V}\left(s_x, s_y, s_z\right), \quad \frac{\sigma_A}{\sigma_0} = f_{0,A}^{\frac{\kappa_0}{1+\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{p_0 \tau_0} \frac{1}{1+\kappa_0}\right)$$

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No conserved charge, μ = 0

$$\boxed{\mathcal{T}\left(s_x, s_y, s_z\right) = \frac{1}{\mathcal{V}\left(s_x, s_y, s_z\right)}}$$

$$p = \frac{T\sigma}{1+\kappa}$$

$$\frac{T_A}{T_0} = \left(\frac{\sigma_A}{\sigma_0}\right)^{\frac{1}{\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0} \frac{1}{1+\kappa_0}\right)$$

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Analytic solutions of IS equations, with κ=const

Bulk viscosity:

$$\Pi(\tau) = \Pi_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\tau_\Pi}\frac{\zeta_0}{\Pi_0}} \exp\left(-\frac{\tau - \tau_0}{\tau_\Pi}\right)$$

Pressure:

$$p(\tau) = p_A \left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa}\right)} \left[1 + \frac{p_0 - p_A}{p_A} \cdot \frac{\Gamma\left(B, \frac{\tau}{\tau_\Pi}\right)}{\Gamma\left(B, \frac{\tau_0}{\tau_\Pi}\right)}\right]$$

Constants:

$$p_A = p_0 - \frac{\Pi_0 d}{\kappa} \left(\frac{\tau_0}{\tau_\Pi}\right)^{-B} \exp\left(\frac{\tau_0}{\tau_\Pi}\right) \Gamma\left(B, \frac{\tau_0}{\tau_\Pi}\right)$$

$$B = d\left(1 + \frac{1}{\kappa} - \frac{\zeta_0}{\Pi_0}\frac{1}{\tau_\Pi}\right)$$

T. Csörgő, G. K.: arXiv:2003.08859

Asymptotically perfect fluid solutions

In the $\tau \gg \tau_0$ limit, both the NS and IS cases lead to the same asymptotic perfect fluid temperature profile and pressure:

$$T \sim T_A \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right) \qquad p \sim p_A \left(\frac{\tau_0}{\tau}\right)^{d\left(1+\frac{1}{\kappa_0}\right)}$$

If μ=0 the entropy density asymptotically equals to a perfect fluid form (and if μ≠0 the particle density is unchanged):

$$\sigma \sim \sigma_A \left(\frac{\tau_0}{\tau}\right)^{d} \mathcal{V}\left(s_x, s_y, s_z\right) \qquad \frac{T_A}{T_0} = \left(\frac{\sigma_A}{\sigma_0}\right)^{\frac{1}{\kappa_0}} = \exp\left(\frac{\zeta_0 d^2}{\kappa_0 p_0 \tau_0} \frac{1}{1+\kappa_0}\right)$$

The bulk viscosity is absorbed to the asymptotic normalization constants!

The effect of bulk viscosity is scaled out!

T. Csörgő, L. P. Csernai, Y. Hama, T. Kodama: arXiv:nucl-th/0306004

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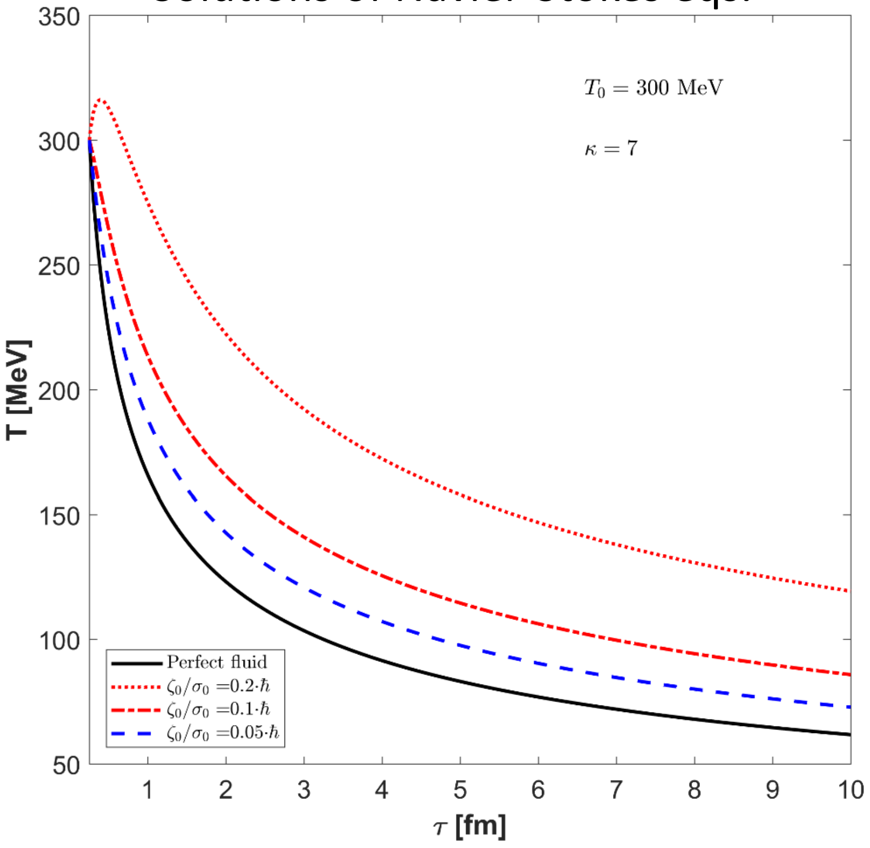
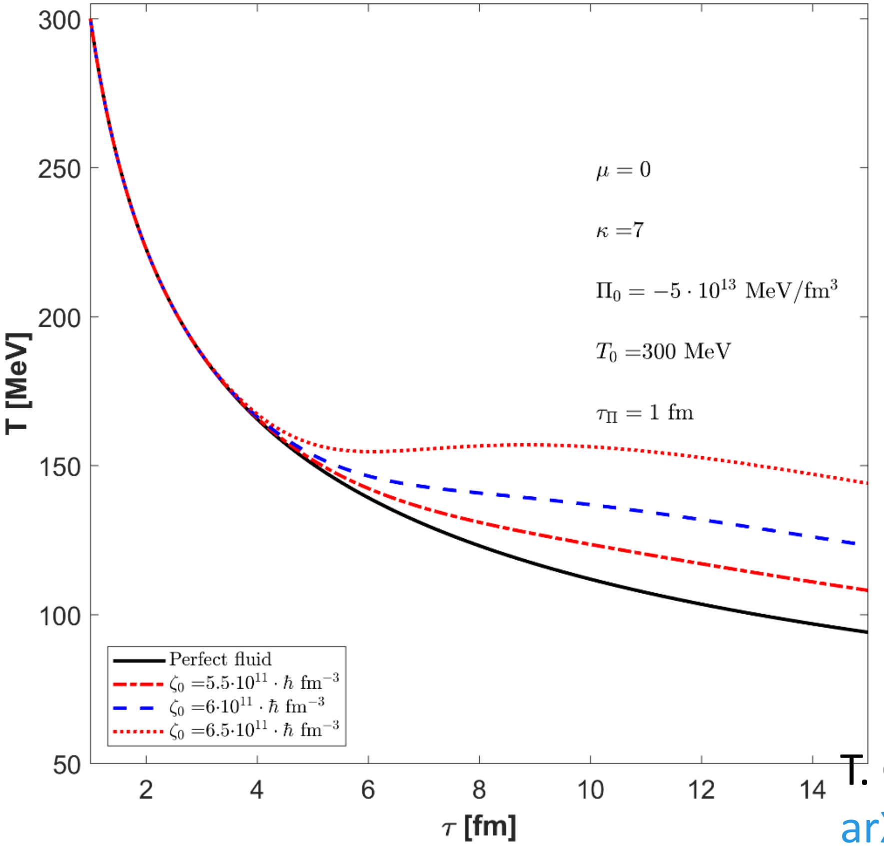
Evolution of the temperature: same initial conditions

Solutions of Navier-Stokes eqs.



Solutions of Isreal-Stewart eqs.



T. Csörgő, G. K.: arXiv:2003.08859

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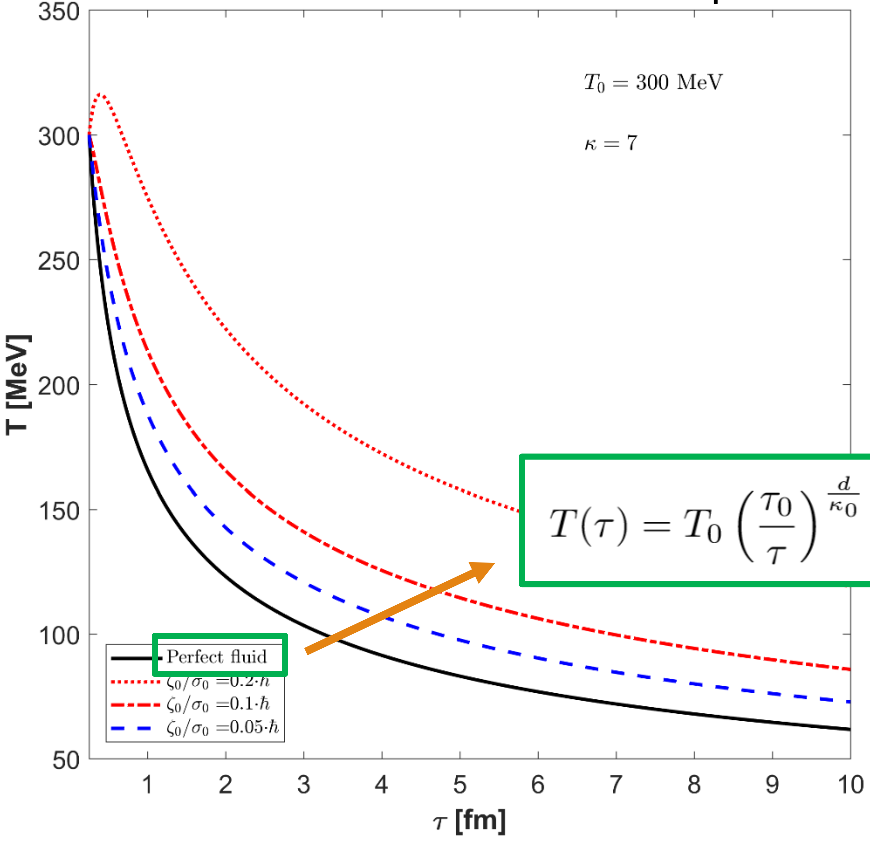

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right)$$

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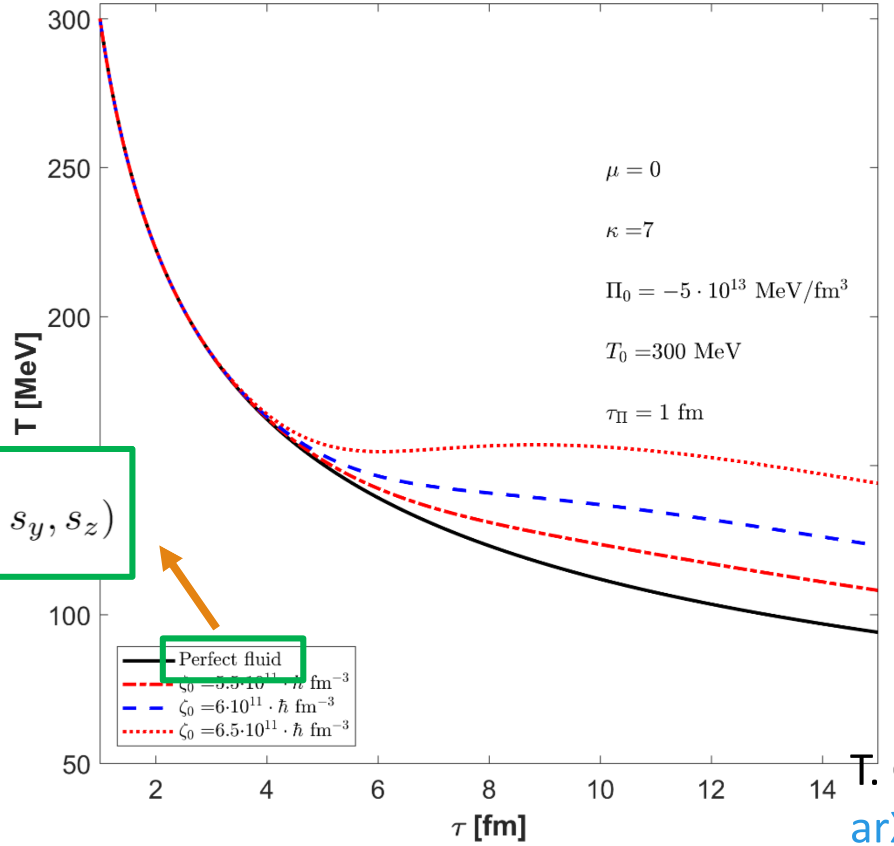
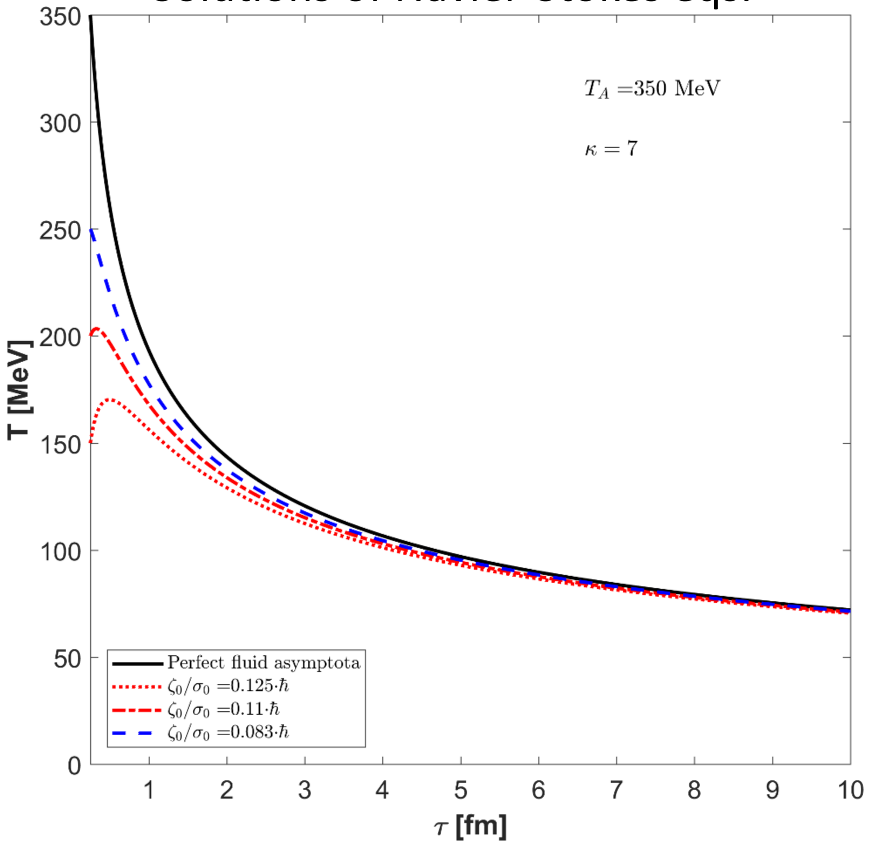


T. Csörgő, G. K.: arXiv:2003.08859

Evolution of the temperature: same attractor

Solutions of Navier-Stokes eqs.



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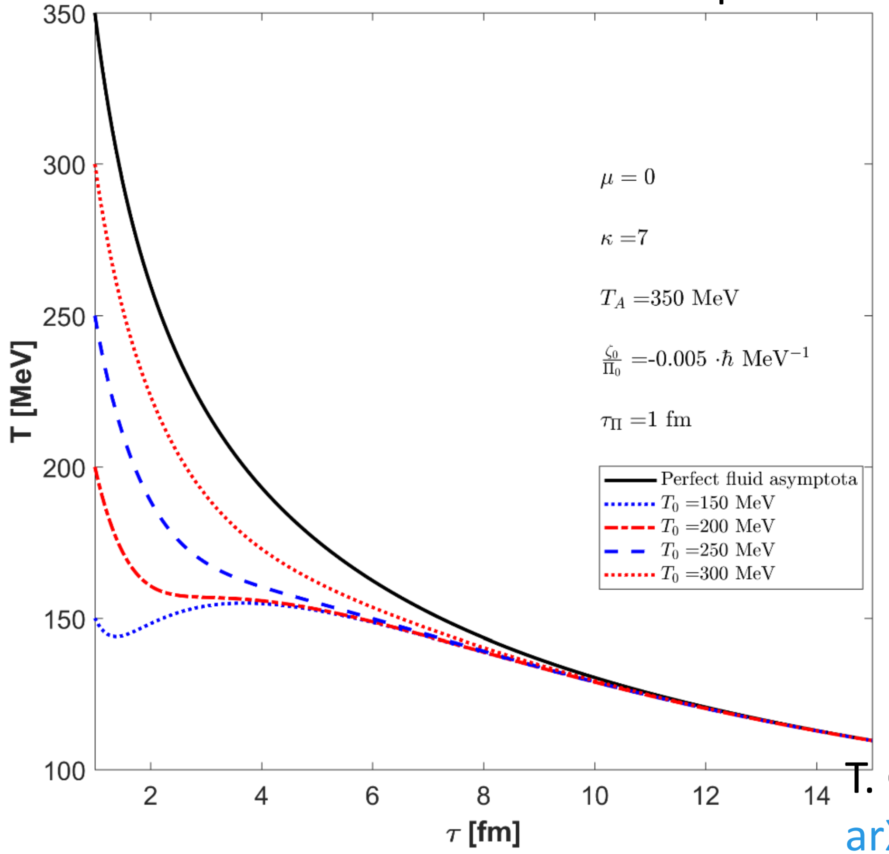

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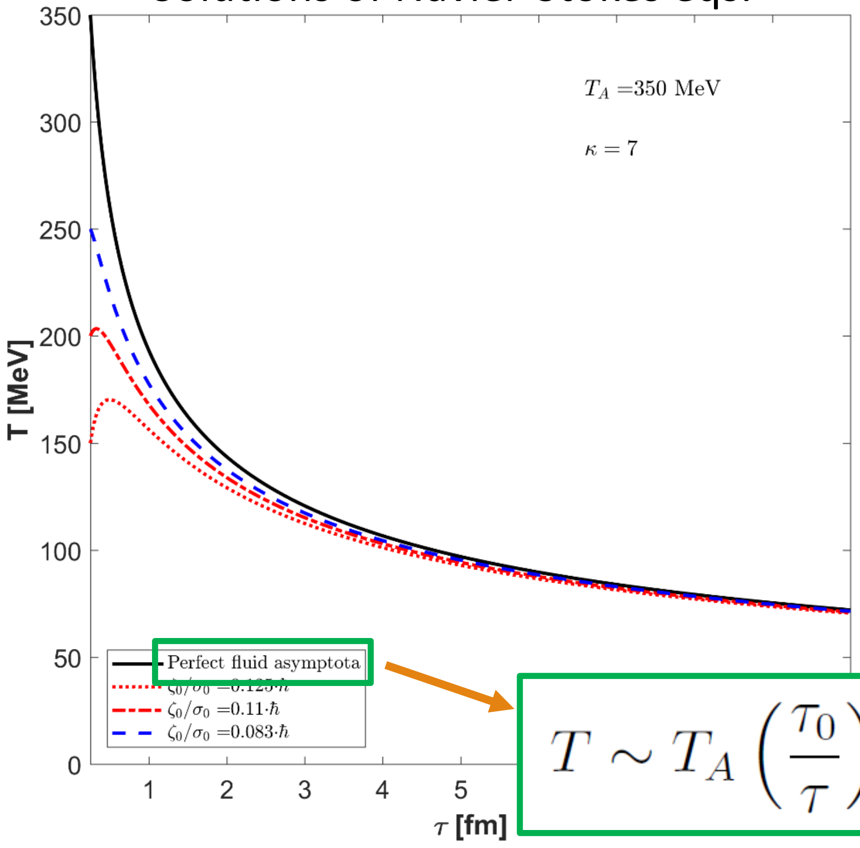

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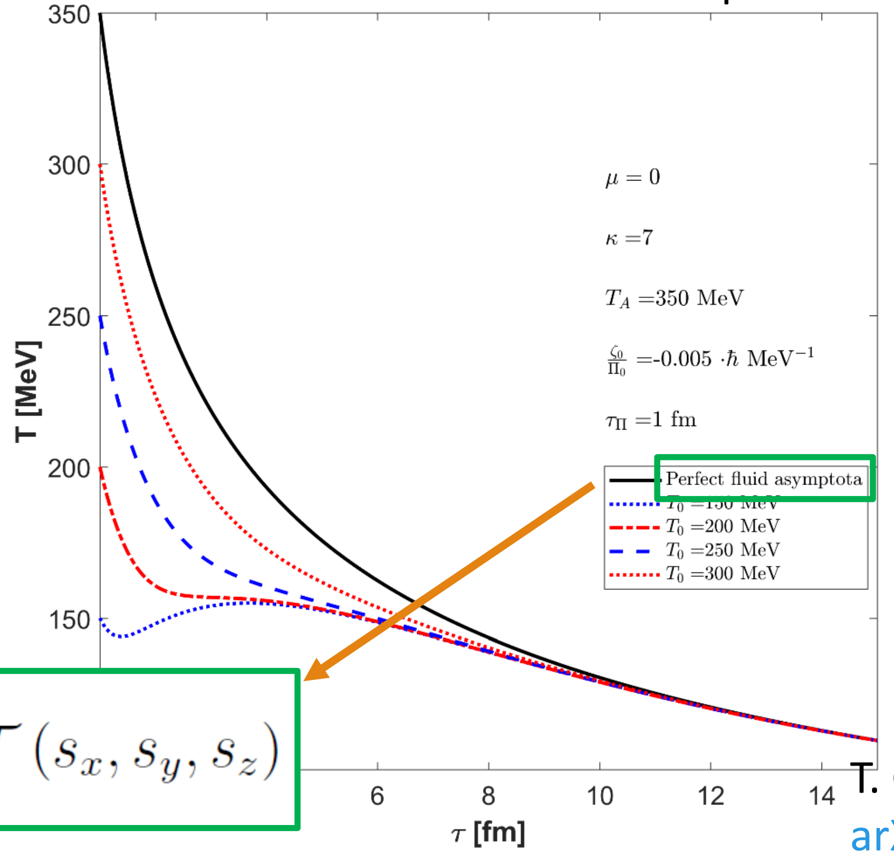

$$\boxed{T \sim T_A \left(\frac{\tau_0}{\tau}\right)^{\frac{d}{\kappa_0}} \mathcal{T}\left(s_x, s_y, s_z\right)}$$

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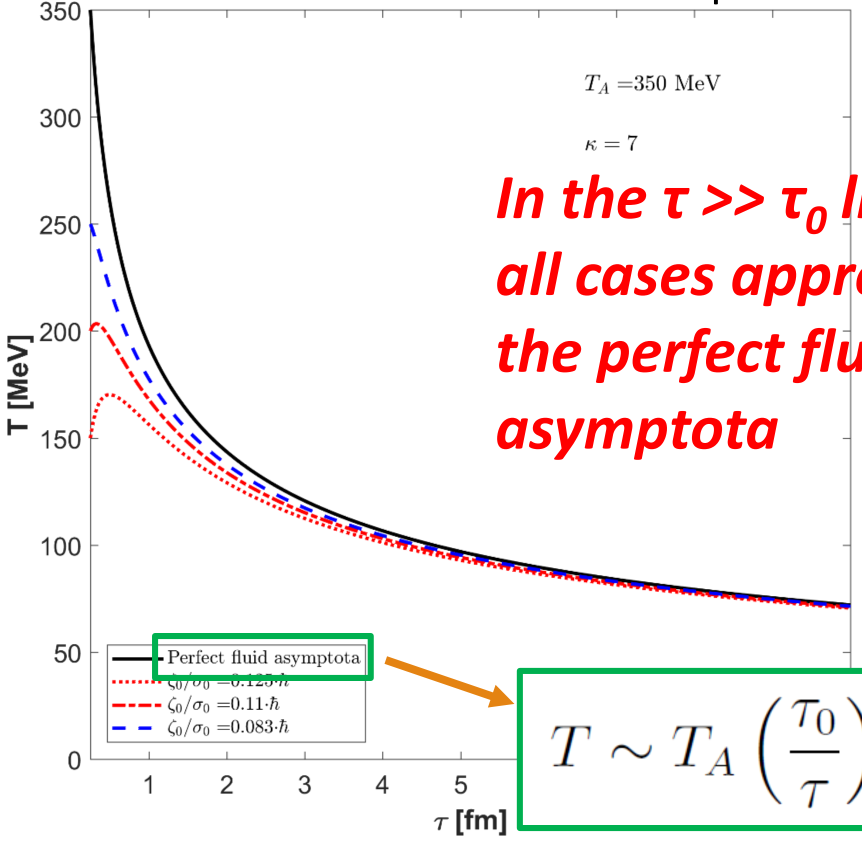

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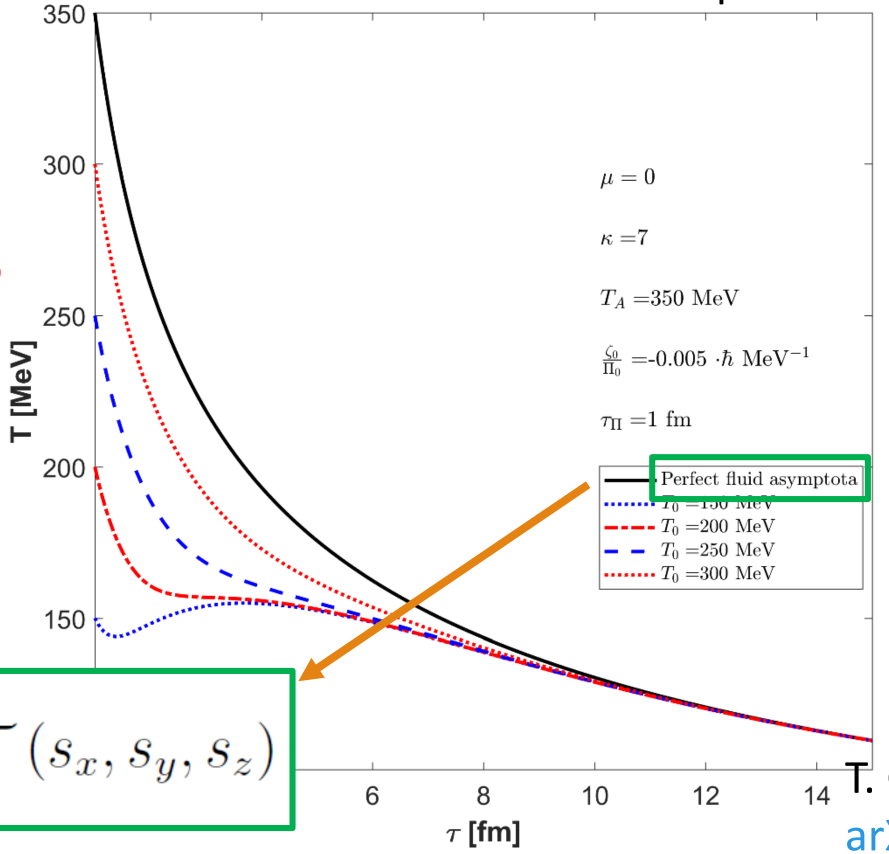

In the $\tau >> \tau_0$ limit, all cases approach the perfect fluid asymptota

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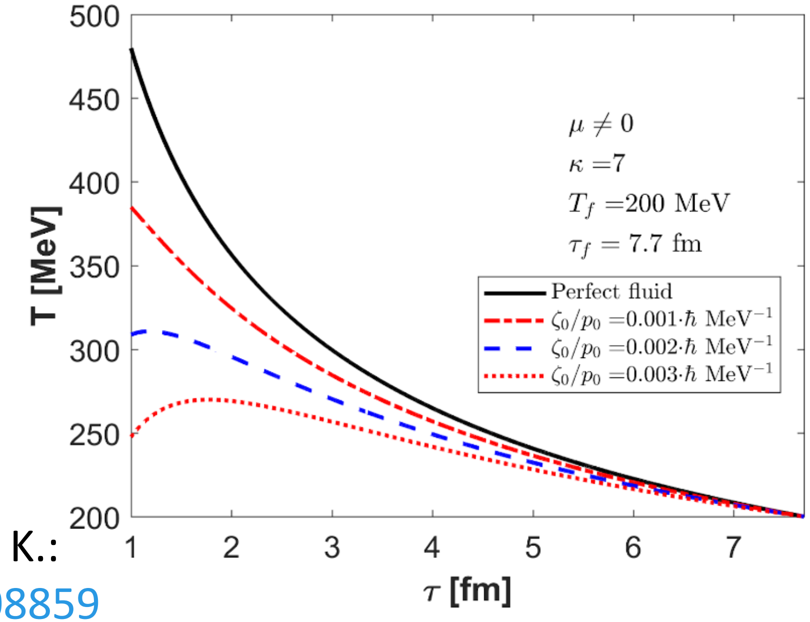
T. Csörgő, G. K.: arXiv:2003.08859

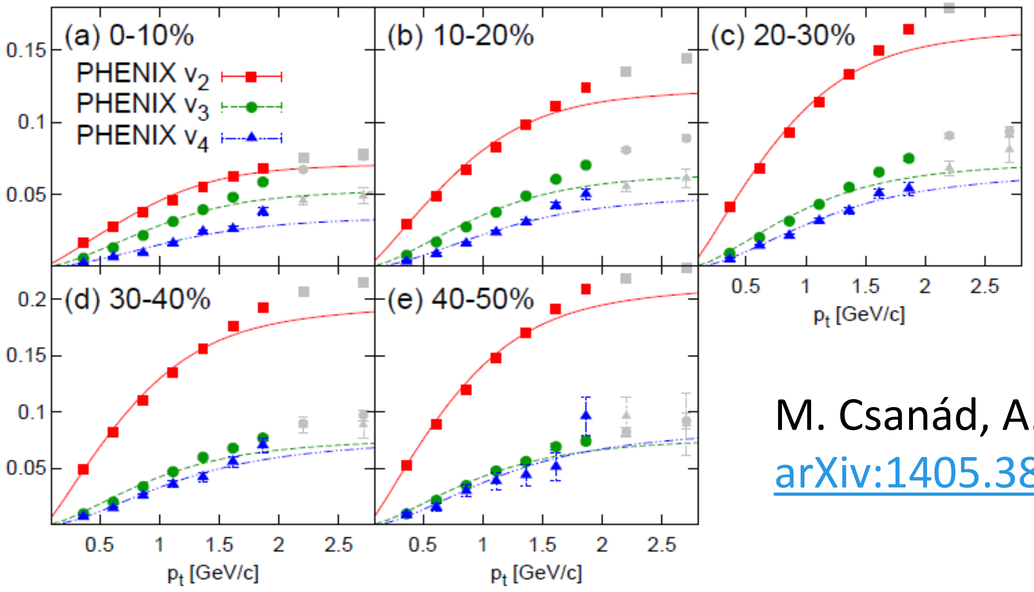
1st application of the solutions of NS eqs.

In arXiv:1405.3877: v_2, v_3 and v_4 were reproduced for $s_{NN}^{1/2}$ = 200 GeV Au+Au collisions with τ_f=7.7 fm/c and T_f=200 MeV final state parameters

We co-varied the initial conditions so that exactly the same freeze-out parameters are obtained



T. Csörgő, G. K.: arXiv:2003.08859



M. Csanád, A. Szabó: arXiv:1405.3877

Further applications of the solutions of NS eqs.

Producing new, dissipative solutions of non relativistic hydro

1st step: Non relativistic limit of the relativistic solution → spherically symmetric solution of non relativistic, dissipative hydro

2nd step: Ellipsoidal generalization

3rd step: Add rotation to the velocity field → $v = v_{Hubble} + v_{rot}$

Result: ***Ellipsoidally symmetric, rotating, dissipative fireball solution of non relativistic hydro***

T. Csörgő, G. K.:
arXiv:2003.08859

Further applications:
Ellipsoidally symmetric, rotating, dissipative fireball solution

Velocity field: $\vec{v} = \vec{v}_H + \vec{v}_{rot}$

$$v_H(\vec{r},t) = \begin{pmatrix} \left(\frac{\dot{X}}{X}\cos^2\vartheta + \frac{\dot{Z}}{Z}\sin^2\vartheta\right) r_x \\ \frac{\dot{Y}}{Y} r_y \\ \left(\frac{\dot{X}}{X}\sin^2\vartheta + \frac{\dot{Z}}{Z}\cos^2\vartheta\right) r_z \end{pmatrix} + \left(\frac{\dot{Z}}{Z} - \frac{\dot{X}}{X}\right)\frac{\sin(2\vartheta)}{2}\begin{pmatrix} r_z \\ 0 \\ r_x \end{pmatrix}$$

$$v_{rot}(\vec{r},t) = \dot{\vartheta}\begin{pmatrix} r_z \\ 0 \\ -r_x \end{pmatrix} + \dot{\vartheta}\begin{pmatrix} \left(\frac{X}{Z}\cos^2\vartheta + \frac{Z}{X}\sin^2\vartheta\right) r_z \\ 0 \\ -\left(\frac{X}{Z}\sin^2\vartheta + \frac{Z}{X}\cos^2\vartheta\right) r_x \end{pmatrix} + \dot{\vartheta}\left(\frac{X}{Z} - \frac{Z}{X}\right)\frac{\sin(2\vartheta)}{2}\begin{pmatrix} r_x \\ 0 \\ -r_z \end{pmatrix}$$

$$\dot{\vartheta} = \frac{\omega(t)}{2} = \frac{\omega_0}{2}\frac{R_0^2}{R(t)^2}$$

$$R = \frac{X+Z}{2}$$

M. I. Nagy, T. Csörgő: arXiv:1309.4390
M. I. Nagy, T. Csörgő: arXiv:1606.09160
T. Csörgő, M. I. Nagy, I. F. Barna: arXiv:1511.02593

Further applications:
Ellipsoidally symmetric, rotating, dissipative fireball solution

Velocity field: $\vec{v} = \vec{v}_H + \vec{v}_{rot}$

Particle density and scale variable:

$$n(\vec{r},t) = n_0 \frac{X_0 Y_0 Z_0}{XYZ} \mathcal{V}(s) = n_0 \frac{V_0}{V} \mathcal{V}(s) \qquad s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2} + \left(\frac{1}{Z^2} - \frac{1}{X^2}\right)\left[\left(r_x^2 - r_z^2\right)\sin^2\vartheta + r_x r_z \sin 2\vartheta\right]$$

Temperature and energy conservation:

$$T = T_0 \left(\frac{X_0 Y_0 Z_0}{XYZ}\right)^{\frac{1}{\kappa}} g_T(t) \mathcal{T}(s), \qquad \mathcal{V}(s) = \frac{1}{\mathcal{T}(s)} \exp\left(-\frac{C_E}{2} \int_0^s \frac{du}{\mathcal{T}(u)}\right)$$

$$\kappa \partial_t [\ln(g_T)] = \frac{\zeta_0}{p_0}\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right)^2 + \frac{2\eta_0}{p_0}\left[\frac{\dot{X}^2}{X^2} + \frac{\dot{Y}^2}{Y^2} + \frac{\dot{Z}^2}{Z^2} - \frac{1}{3}\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right)^2\right] + \frac{\eta_0 {\omega_0}^2}{4p_0} \frac{(X_0 + Z_0)^4}{(X+Z)^4}\left(\frac{X}{Z} - \frac{Z}{X}\right)^2$$

Euler equation and angular velocity:

$$X\left(\ddot{X} - R\omega^2\right) = Y\ddot{Y} = Z\left(\ddot{Z} - R\omega^2\right) = C_E \frac{T}{m}$$

$$R = \frac{X+Z}{2}, \qquad \omega = \omega_0 \frac{R_0^2}{R^2}$$

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Velocity field: $\vec{v} = \vec{v}_H + \vec{v}_{rot}$

Particle density and scale variable:

$$n(\vec{r},t) = n_0 \frac{X_0 Y_0 Z_0}{XYZ} \mathcal{V}(s) = n_0 \frac{V_0}{V} \mathcal{V}(s) \qquad s = \frac{r_x^2}{X^2} + \frac{r_y^2}{Y^2} + \frac{r_z^2}{Z^2} + \left(\frac{1}{Z^2} - \frac{1}{X^2}\right)\left[\left(r_x^2 - r_z^2\right)\sin^2\vartheta + r_x r_z \sin 2\vartheta\right]$$

Temperature and energy conservation:

$$T = T_0 \left(\frac{X_0 Y_0 Z_0}{XYZ}\right)^{\frac{1}{\kappa}} g_T(t)\mathcal{T}(s), \qquad \mathcal{V}(s) = \frac{1}{\mathcal{T}(s)} \exp\left(-\frac{C_E}{2}\int_0^s \frac{du}{\mathcal{T}(u)}\right)$$

$$\kappa\partial_t[\ln(g_T)] = \boxed{\frac{\zeta_0}{p_0}\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right)^2} + \boxed{\frac{2\eta_0}{p_0}\left[\frac{\dot{X}^2}{X^2} + \frac{\dot{Y}^2}{Y^2} + \frac{\dot{Z}^2}{Z^2} - \frac{1}{3}\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right)^2\right]} + \boxed{\frac{\eta_0 {\omega_0}^2}{4p_0}\frac{(X_0+Z_0)^4}{(X+Z)^4}\left(\frac{X}{Z} - \frac{Z}{X}\right)^2}$$

effect of bulk

effect of shear without rotation

effect of shear with rotation

Euler equation and angular velocity:

$$X\left(\ddot{X} - R\omega^2\right) = Y\ddot{Y} = Z\left(\ddot{Z} - R\omega^2\right) = C_E \frac{T}{m}$$

$$R = \frac{X+Z}{2}, \qquad \omega = \omega_0 \frac{R_0^2}{R^2}$$

Further application: Asymptotic behaviour (for C_E=0)

If C_E=0, analytic solutions of X, Y and Z scales can be given, and their asymptotic limit are simple:

$$\begin{aligned} X &\propto X_a + \dot{X}_a t, \\ Y &\propto Y_a + \dot{Y}_a t, \\ Z &\propto Z_a + \dot{Z}_a t, \\ \omega &\propto \omega_0 \left[\frac{X_0 + Z_0}{X_a + Z_a + \left(\dot{X}_a + \dot{Z}_a \right) t} \right]^2 \end{aligned}$$

where

$$\begin{aligned} \dot{X}_a &= \frac{1}{2}\left[\dot{X}_0 - \dot{Z}_0 + \sqrt{(\dot{X}_0 + \dot{Z}_0)^2 + (X_0 + Z_0)^2 \omega_0^2} \right] \\ \dot{Y}_a &= \dot{Y}_0 \\ \dot{Z}_a &= \frac{1}{2}\left[\dot{Z}_0 - \dot{X}_0 + \sqrt{(\dot{X}_0 + \dot{Z}_0)^2 + (X_0 + Z_0)^2 \omega_0^2} \right] \end{aligned}$$

At very late times the constant offsets becomes negligible:

$$\begin{aligned} X &\propto \dot{X}_a t \\ Y &\propto \dot{Y}_a t \\ Z &\propto \dot{Z}_a t \end{aligned}$$

In this asymptotic limit, the rotating and dissipative, ellipsoidal fireball tends to a known, perfect fluid relativistic solution: Csörgő, Csernai, Hama, Kodama: arXiv:nucl-th/0306004

A spherical and irrotational Hubble flow is an asymptotic attractor for rotating and ellipsoidal, finite fireballs

The effects of rotation, shear and bulk viscosity are scaled out from asymptotia

Summary

New, analytic, exact solutions of relativistic Navier-Stokes and Israel-Stewart equations with spherically symmetric Hubble-flow

The effect of shear viscosity cancel because of the velocity field

The solutions are causal and asymptotically perfect (the effect of bulk viscosity cancels for late times), both for a finite and vanishing μ

These exact solutions tend to the Csörgő-Csernai-Hama-Kodama perfect fluid solution

Cannot decide from final state measurements that the medium evolved as a perfect fluid with higher initial temperature (T_A) or as a viscous fluid with lower initial temperature (T_0)

We were able to reproduce the experimental data in $s_{NN}^{1/2}$ = 200 GeV Au+Au collisions on v_2, v_3 and v_4

Non relativistic limit: new solutions of non relativistic Navier-Stokes theory

Thank you for your attention!