Hydrodynamics with 50 particles. What does it mean and how to think about it?

G.Torrieri

UNICAMP

FAPESP
FUNDAÇÃO DE AMPARO À PESQUISA
DO ESTADO DE SÃO PAULO

Based on 2007.09224 ,shortened version of
https://www.youtube.com/watch?v=oLYouz0YMHM

- The necessity to redefine hydro
  - Small fluids and fluctuations
  - Statistical mechanicists and mathematicians

- A possible answer:
  - Describing equilibrium at the operator level using the Zubarev operator
  - Definining non-equilibrium at the operator level using Crooks theorem

  Relationship to usual hydrodynamics analogous to "Wilson loops" vs "Chiral perturbation" regarding usual QCD

- Discussion, extensions, implementations etc.

Some experimental data warmup (Why the interest in relativistic hydro ?)
(2004) Matter in heavy ion collisions seems to behave as a perfect fluid, characterized by a very rapid thermalization

RHIC Scientists Serve Up 'Perfect' Liquid

New state of matter more remarkable than predicted — raising many new questions

April 18, 2005

TAMPA, FL — The four detector groups conducting research at the Relativistic Heavy Ion Collider (RHIC) — a giant atom "smasher" located at the U.S. Department of Energy's Brookhaven National Laboratory — say they've created a new state of hot, dense matter out of the quarks and gluons that are the basic particles of atomic nuclei, but it is a state quite different and even more remarkable than had been predicted. In peer-reviewed papers summarizing the first three years of RHIC findings, the scientists say that instead of behaving like a gas of free quarks and gluons, as was expected, the matter created in RHIC's heavy ion collisions appears to be more like a *liquid*.

"Once again, the physics research sponsored by the Department of Energy is producing historic results," said Secretary of Energy Samuel Bodman, a trained chemical engineer. "The DOE is the principal federal funder of basic research in the physical sciences, including nuclear and high-energy physics. With today's announcement we see that investment paying off."

"The truly stunning finding at RHIC that the new state of matter created in the collisions of gold ions is more like a liquid than a gas gives us a profound insight into the earliest moments of the universe," said Dr. Raymond L. Orbach, Director of the DOE Office of Science.

Based on v_n fit to low viscosity to ideal hydro

The conventional widsom

Hydrodynamics is an "effective theory", built around coarse-graining and "fast thermalization". Fast w.r.t. Gradients of coarse-grained variables

If thermalization instantaneus, then isotropy,EoS enough to close evolution

$$T_{\mu\nu} = (e + P(e))u_\mu u_\nu + P(e) g_{\mu\nu}$$

In rest-frame at rest w.r.t. u^μ

$$T_{\mu\nu} = \mathrm{Diag}\left(e(p), p, p, p\right)$$

(NB: For simplicity we assume no conserved charges, $\mu_B = 0$)

If thermalization not instantaneus,

$$T_{\mu\nu} = T_{\mu\nu}^{eq} + \Pi_{\mu\nu} \quad , \quad u_\mu \Pi^{\mu\nu} = 0$$

$$\sum_n \tau_{n\Pi} \partial_\tau^n \Pi_{\mu\nu} = -\Pi_{\mu\nu} + \mathcal{O}\left(\partial u\right) + \mathcal{O}\left((\partial u)^2\right) + ...$$

A series whose "small parameter" (Barring phase transitions/critical points/... all of these these same order):

$$K \sim \frac{l_{micro}}{l_{macro}} \sim \frac{\eta}{sT} \nabla u \sim \frac{\mathrm{Det}\Pi_{\mu\nu}}{\mathrm{Det}T_{\mu\nu}} \sim ...$$

and the transport coefficients calculable from asymptotic correlators of microscopic theory

Navier-Stokes $\sim K$, Israel-Stewart $\sim K^2$ etc.

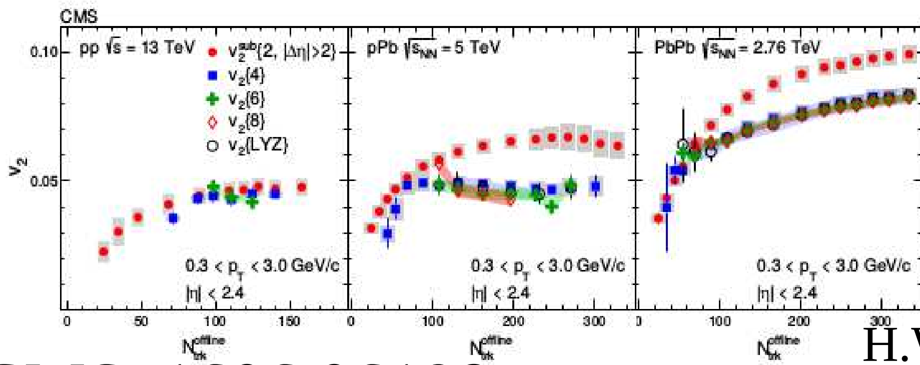
SCIENCE

The LHC Might Have Created The Smallest Drop Of Liquid Ever

A tiny drop could have big implications for our understanding of particle collisions.

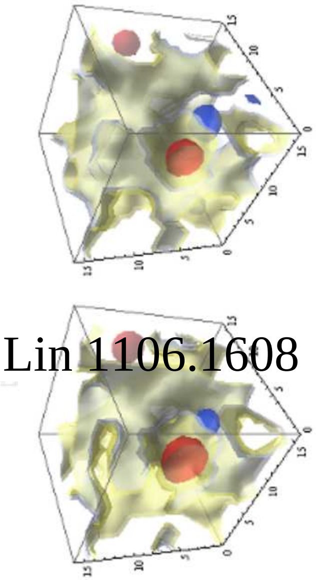
By Shaunacy Ferro *May 8, 2013*

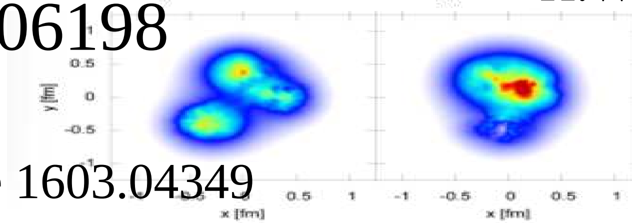
But data surprises us!



CMS 1606.06198

H.W.Lin 1106.1608



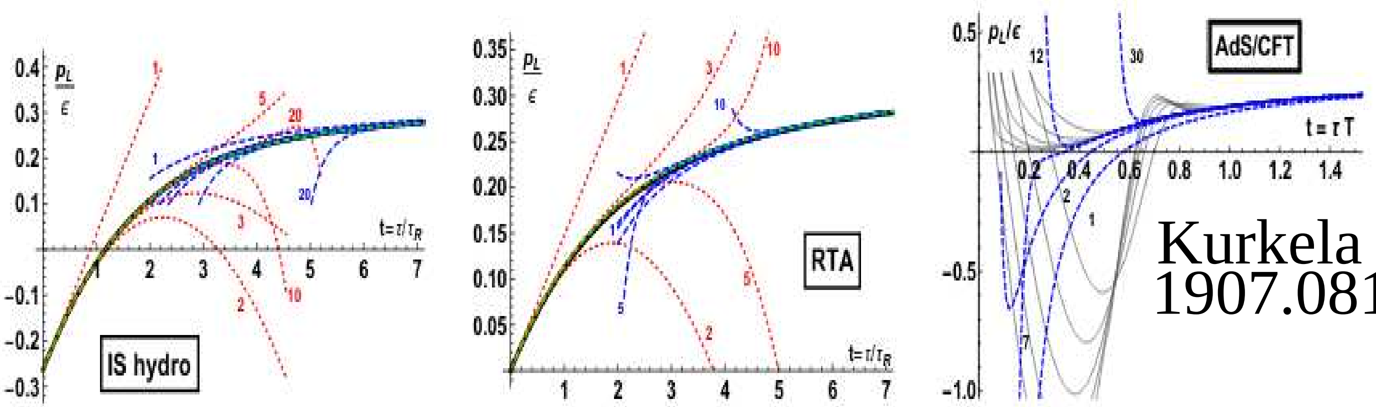


BSchenke 1603.04349

1606.06198 (CMS) : When you consider geometry differences, hydro with $\mathcal{O}(20)$ particles "just as collective" as for 1000. Thermalization scale $\ll$ color domain wall scale.

Little understanding of this in "conventional widsom"

Hydrodynamics in small systems: “hydrodynamization” /”fake equilibrium”
A lot more work in both AdS/CFT and transport theory about ”hydrodynamization” /”Hydrodynamic attractors”



Kurkela et al 1907.08101.

Fluid-like systems far from equilibrium (large gradients)! Usually from 1D solution of Boltzmann and AdS/CFT EoMs! “hydrodynamics converges even at large gradients with no thermal equilibrium”

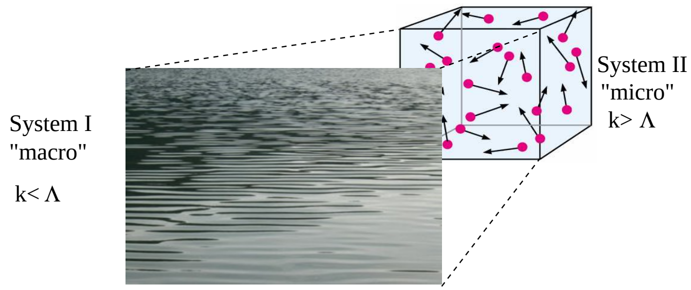
But I have a basic question: ensemble averaging!

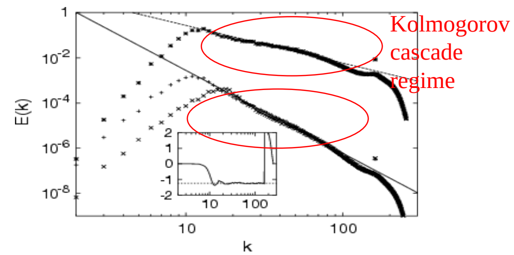
- Ensemble averaging , $\langle F(\{x_i\},t)\rangle \neq F(\{\langle x_i\rangle\},t)$ suspect for any non-linear theory. molecular chaos in Boltzmann, Large N_c in AdS/CFT, all assumed . But for $\mathcal{O}(50)$ particles?!?!

- How do microscopic, macroscopic and quantum corrections talk to eac other? EoS is given by $p = T\ln Z$ but $\partial^2 \ln Z/\partial T^2, dP/dV$??

- How does dissipation work in such a "semi-microscopic system"? If $T_{\mu\nu} \rightarrow \hat{T}_{\mu\nu}$ what is $\hat{\Pi}_{\mu\nu}$ Second law fluctuations? Sometimes because of a fluctuation entropy decreases!

None of these issues depend directly on mean free path, can be seen in Boltzmann or AdS/CFT! "Local equilibrium" ill-defined.





Statistical mechanics: This is a <u>system</u> in <u>global</u> equilibrium, described by a partition function $Z(T, V, \mu)$, derivatives give averages, $\langle E\rangle$,fluctuations $\langle(\Delta E)^2\rangle$ etc.

Fluid dynamics: This is the state of a <u>field</u> in <u>local</u> equilibrium which can be perturbed in an infinity of ways. The perturbations will then interact and dissipate according to the <u>Euler/N-S equations</u>

Reconciling these means a rigorous treatment of fluctuations, which are <u>non</u>-perturbative (vortices! Nicolis et al, 1011.6396)

If you take what you learned in statistical mechanics and perturb it, you get a millenium problem!
Wild solutions, anomalous dissipation etc...

I am a physicist so I care little about the "existence of ethernal solutions" to an approximate equation, Turbulent regime and microscopic local equilibria need to be consistent

The hope...

Considering fluctuations non-perturbatively will conceptually stabilize hydrodynamics, perhaps fluctuations and instabilities help local thermalization!

Our proposal

Every statistical theory needs a "state space" and an "evolution dynamics"
The ingredients

State space:Zubarev hydrodynamics Mixes micro and macro DoFs

Dynamics: Crooks fluctuation theorem provides the dynamics via a definition of $\Pi_{\mu\nu}$ from fluctuations

$\hat{T}^{\mu\nu}$ is an operator, so any decomposition, such as $\hat{T}_0^{\mu\nu} + \hat{\Pi}^{\mu\nu}$ must be too!

Zubarev partition function for local equilibrium: think of Eigenstate thermalization...

Let us generalize the GC ensemble to a co-moving frame $E/T \to \beta_\mu T^\mu_\nu$

$$\hat{\rho}(T_0^{\mu\nu}(x), \Sigma_\mu, \beta_\mu) = \frac{1}{Z(\Sigma_\mu, \beta_\mu)} \exp\left[- \int_{\Sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}_0^{\mu\nu} \right]$$

Z is a partition function with a field of Lagrange multiplies β_μ , with microscopic and quantum fluctuations included.

Effective action from $\ln[Z]$. Correction to Lagrangian picture?

All normalizations diverge but hey, it's QFT! (Later we resolve this!)

This is perfect global equilibrium. What about imperfect local?

- Dynamics is not clear. Becattini et al, 1902.01089: Gradient expansion in β_μ . Reproduces Euler and Navier-Stokes, but...
  - 2nd order Gradient expansion (Navier stokes) non-causal perhaps...
  - Use Israel-Stewart, $\Pi_{\mu\nu}$ arbitrary perhaps...
  - Foliation $d\Sigma_\mu$ arbitrary but not clear how to link to Arbitrary $\Pi_{\mu\nu}$
- What about fluctuations? Coarse-graining and fluctuations mix? How does one truncate?

An operator formulation

$$\hat{T}^{\mu\nu} = \hat{T}_0^{\mu\nu} + \hat{\Pi}_{\mu\nu}$$

and $\hat{T}_0^{\mu\nu}$ truly in equilibrium!

$$\hat{\rho}(T_0^{\mu\nu}(x), \Sigma_\mu, \beta_\mu) = \frac{1}{Z(\Sigma_\mu, \beta_\mu)} \exp\left[- \int_{\Sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}_0^{\mu\nu} \right]$$

describes all cumulants and probabilities

$$\langle T_0^{\mu\nu}(x_1) T_0^{\mu\nu}(x_2) ... T_0^{\mu\nu}(x_n) \rangle = \prod_i \frac{\delta^n}{\delta \beta_\mu(x_i)} \ln Z$$

and also the full energy-momentum tensor

$$\langle T^{\mu\nu}(x_1)T^{\mu\nu}(x_2)...T^{\mu\nu}(x_n)\rangle = \prod_i \frac{\delta^n}{\delta g_{\mu\nu}(x_i)} \ln Z$$

What this means

- Equilibrium at "probabilistic" level

$$\hat{T}^{\mu\nu} = \hat{T}_0^{\mu\nu} + \hat{\Pi}^{\mu\nu}$$

- KMS Condition obeyed by "part of density matrix" in equilibrium, "expand" around that! An operator constrained by KMS condition is still an operator! $\equiv$ time dependence in interaction picture

Does this make sense

$$\hat{T}_0^{\mu\nu} + \hat{\Pi}^{\mu\nu} \qquad , \qquad \hat{\rho}_{T_{\mu\nu}} = \frac{\hat{\rho}_{T_0} + \hat{\rho}_{\Pi_0}}{\mathrm{Tr}\left(\hat{\rho}_{T_0} + \hat{\rho}_{\Pi_0}\right)} \simeq \hat{\rho}_{T_0}\left(1 + \delta\hat{\rho}\right)$$

For <u>any</u> flow field β_μ and <u>lagrangian</u> we can <u>define</u>

$$Z_{T_0}(J(y)) = \int \mathcal{D}\phi \exp\left[-\int_0^{T^{-1}(x_i^\mu)} d\tau' \int d^3x \left(L(\phi) + J(y)\phi\right)\right] \propto$$

$$\propto \exp\left[-\beta^0 \hat{T}_{00}\right]\Big|_{\beta_\mu = (T^{-1}(x,t),\vec{0})}$$

E.g. Nishioka, 1801.10352 $\langle x|\,\rho\,|x'\rangle =$

$$= \frac{1}{Z}\int_{\tau=-\infty}^{\tau=\infty}\int \left[\mathcal{D}\phi, \mathcal{D}y(\tau)\,\mathcal{D}y'(\tau)\right]\, e^{-iS(\phi y, y')} \cdot \underbrace{\delta\left[y(0^+)-x'\right]\,\delta\left[y'(0^-)-x\right]}_{\frac{\delta J_i(y(0^+))}{\delta J_i(x')}\frac{\delta J_j(y(0^-))}{\delta J_j(x)}}$$

$$\Rightarrow \frac{\delta^2}{\delta J_i(x)\delta J_j(x')}\ln\left[Z_{T_0}(T^{\mu\nu}, J)\times Z_{\Pi}(J)\right]_{J=J_1(x)+J_2(x')}$$

$J_1(x)+J_2(x')$ chosen to respect Matsubara conditions!

Any ρ can be separated like this for <u>any</u> β_μ . The question is, is this a good approximation? "Close enough to equilibrium"

The source J related to the smearing in "weak solutions". Pure maths angle?

Entropy/Deviations from equilibrium

- In quantum mechanics Entropy function of density matrix

$$s = Tr(\hat{\rho} \ln \hat{\rho}) = \frac{d}{dT} (T \ln Z)$$

Conserved in quantum evolution, not coarse-graining/gradient expansion

- In IS entropy function of the dissipative part of E-M tensor

$$n^{\nu} \partial_{\nu} (s u^{\mu}) = n^{\mu} \frac{\Pi^{\alpha\beta}}{T} \partial_{\alpha} \beta_{\beta} \quad , \quad \geq 0$$

$n_{\mu} = d\Sigma_{\mu}/|d\Sigma_{\mu}|, \Pi_{\mu\nu}$ arbitrary. How to combine coarse-graining? if vorticity non-zero $n_{\mu} u^{\mu} \neq 0$

What about fluctuations

$$n^\nu \partial_\nu \left(s u^\mu \right) = n^\mu \frac{\Pi^{\alpha\beta}}{T} \partial_\alpha \beta_\beta \quad , \qquad \geq 0$$

- If n_μ arbitrary cannot be true for "any" choice
- 2nd law is true for "averages" anyways, sometimes entropy can decrease

We need a fluctuating formulation!

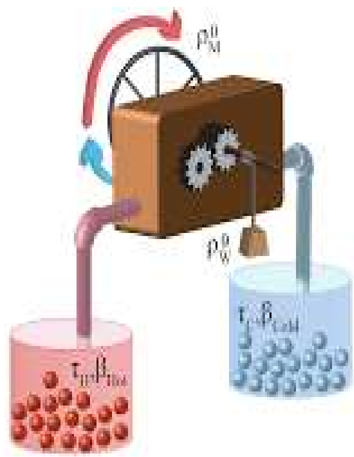
- "Statistical" (probability depends on "local microstates")
- Dynamics with fluctuations, time evolution of β_μ distribution

So we need

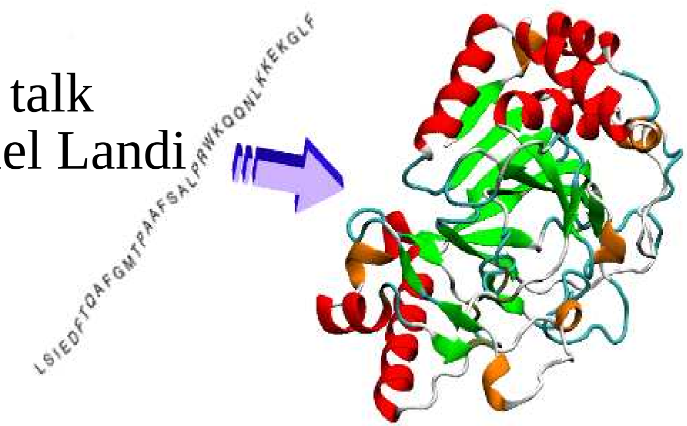
- a similarly probabilistic definition of $\hat{\Pi}^{\mu\nu} = \hat{T}^{\mu\nu} - \hat{T}_0^{\mu\nu}$ as an <u>operator</u>!!

- Probabilistic dynamics, to update $\hat{\Pi}_{\mu\nu}, \hat{T}_{\mu\nu}$!

Crooks fluctuation theorem!



From talk
Gabriel Landi



Relates fluctuations, entropy in <u>small</u> <u>fluctuating</u> systems (Nano,proteins)

Crooks fluctuation theorem!

$$P(W)/P(-W) = \exp[\Delta S]$$

P(W) Probability of a system doing some work in its usual thermal evolution

P(-W) Probability of the same system "running in reverse" and decreasing entropy due to a thermal fluctuation

ΔS Entropy produced by $P(W)$

Looks obvious but...

Is valid for systems very far from equilibrium (nano-machines, protein folding and so on)

Proven for Markovian processes and fluctuating systems in contact with thermal bath

Leads to irreducible fluctuation/dissipation: TUR (more later!)

Applying it to locally equilibrium systems within Zubarev's formalism is straight-forward . Since ratios of probabilities, divergences are resolved!

How is Crooks theorem useful for what we did? Guarnieri et al, arXiv:1901.10428 (PRX) derive Thermodynamic uncertainity relations from

$$\hat{\rho}_{ness} \simeq \hat{\rho}_{les}(\lambda) e^{\hat{\Sigma}} \frac{Z_{les}}{Z_{ness}} \quad , \quad \hat{\rho}_{les} = \frac{1}{Z_{les}} \exp\left[-\frac{\hat{H}}{T} \right]$$

$\hat{\rho}_{les}$ is Zubarev operator while Σ is calculated with a Kubo-like formula

$$\hat{\Sigma} = \delta_\beta \Delta \hat{H}_+ \quad , \quad \hat{H}_+ = \lim_{\epsilon \to 0^+} \epsilon \int dt e^{\epsilon t} e^{-\hat{H} t} \Delta \hat{H} e^{\hat{H} t}$$

Relies on

$$\lim_{w \to 0} \left\langle \left[\hat{\Sigma}, \hat{H} \right] \right\rangle \to 0 \equiv \lim_{t \to \infty} \left\langle \left[\Sigma \hat{(t)}, \hat{H}(0) \right] \right\rangle \to 0$$

This "infinite" is "small" w.r.t. hydro gradients. $\equiv$ Markovian as in Hydro with $l_{mfp} \to \partial$ but with operators$\to$ carries all fluctuations with it!

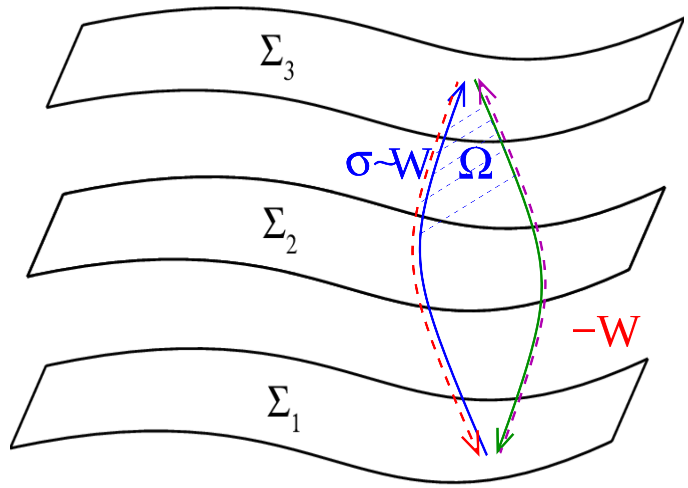
$$P(W)/P(-W) = \exp\left[\Delta S\right] \quad \mathrm{Vs} \quad S_{eff} = \ln Z$$

KMS condition reduces the functional integral to a Metropolis type weighting, $\equiv$ periodic time at rest with β_μ

Markovian systems exhibit Crooks theorem, two adjacent cells interaction outcome probability proportional to number of ways of reaching outcome . The normalization divergence is resolved since <u>ratios</u> of probabilities are used . "instant decoherence/thermalization" within each step

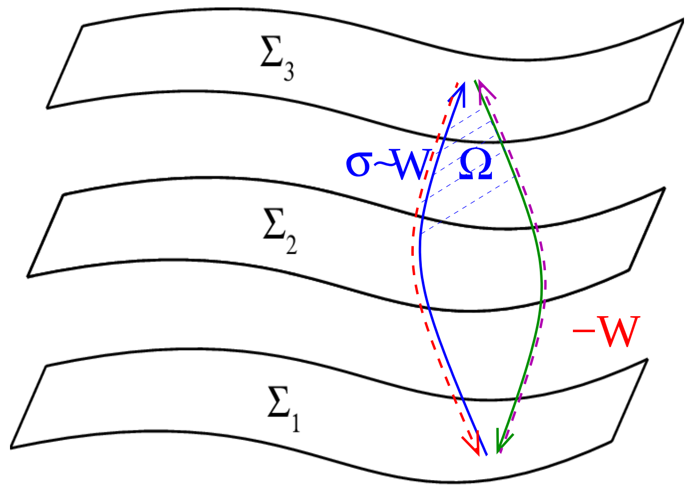
Relationship to gradient expansion similar to relationship between Wilson loop coarse-graining (Jarzynski's theorem, used on lattice ,Caselle et al, 1604.05544) with hadronic EFTs

Applying Crooks theorem to Zubarev hydrodynamics: Stokes theorem



$$-\int_{\Sigma(\tau_0)} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu\right) = -\int_{\Sigma(\tau')} \mathrm{d}\Sigma_\mu \left(\widehat{T}^{\mu\nu}\beta_\nu\right) + \int_\Omega \mathrm{d}\Omega \left(\widehat{T}^{\mu\nu}\nabla_\mu\beta_\nu\right),$$

true for “any” fluctuating configuration.



Let us now invert one foliation so it goes "backwards in time" assuming Crooks theorem means

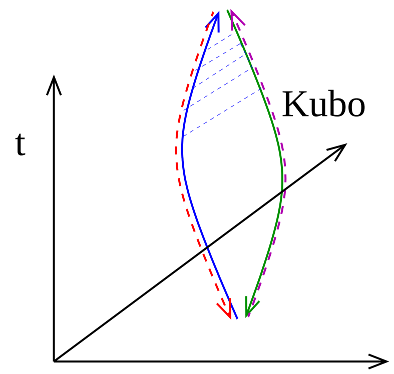
$$\frac{\exp\left[-\int_{\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu}\right]}{\exp\left[-\int_{-\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu}\right]} = \exp\left[\frac{1}{2}\int_\Omega d\Omega_\mu^\mu \left[\frac{\hat{\Pi}^{\alpha\beta}}{T}\right] \partial_\beta \beta_\alpha\right]$$

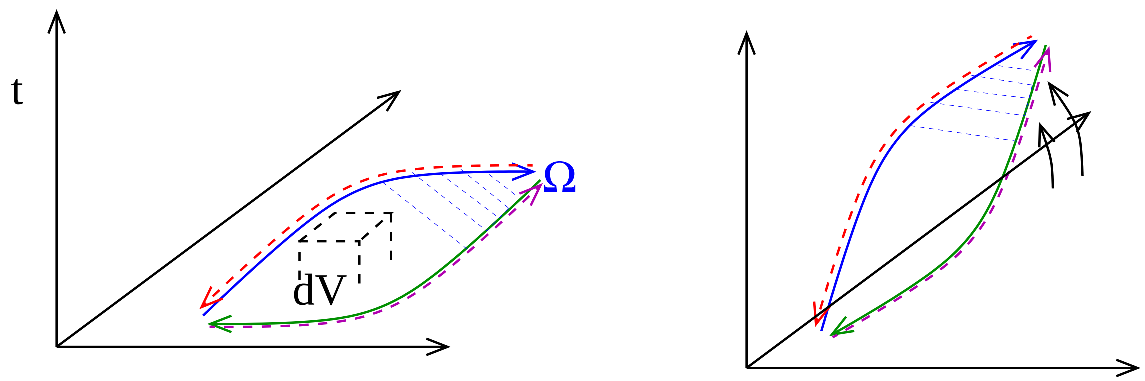
Small loop limit $\left\langle \exp\left[\oint d\Sigma_\mu \omega^{\mu\nu}\beta^\alpha \hat{T}_{\alpha\nu}\right]\right\rangle = \left\langle \exp\left[\int \frac{1}{2} d\Sigma_\mu \beta^\mu \hat{\Pi}^{\alpha\beta}\partial_\alpha \beta_\beta\right]\right\rangle$

A non-perturbative operator equation,divergences cancel out...

$$\left.\frac{\hat{\Pi}^{\mu\nu}}{T}\right|_\sigma = \left(\frac{1}{\partial_\mu \beta_\nu}\right)\frac{\delta}{\delta\sigma}\left[\int_{\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu} - \int_{-\sigma(\tau)} d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu}\right]$$

Note that a time-like contour produces a Kubo-formula

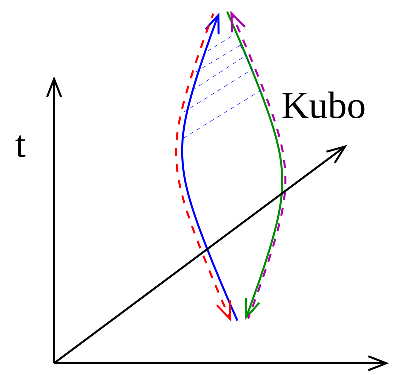




A sanity check: For a an equilibrium spacelike $d\Sigma_\mu = (dV, \vec{0})$ (left-panel) we recover Boltzmann's

$$\Pi^{\mu\nu} \Rightarrow \Delta S = \frac{dQ}{T} = \ln\left(\frac{N_1}{N_2}\right)$$

A sanity check



When $\eta \to 0$ *and* $s^{-1/3} \to 0$ (the first two terms in the hierarchy), Crooks fluctuation theorem gives $P(W) \to 1$ $P(-W) \to 0$ $\Delta S \to \infty$ so Crooks theorem reduces to δ-functions of the entropy current

$$\delta\left(d\Sigma_\mu\left(su^\mu\right)\right) \Rightarrow n^\mu \partial_\mu\left(su^\mu\right) = 0$$

We therefore recover conservation equations for the entropy current, a.k.a. ideal hydro

Crooks theorem: thermodynamic uncertainity relations

Andr M. Timpanaro, Giacomo Guarnieri, John Goold, and Gabriel T. Landi
Phys. Rev. Lett. 123, 090604

$$\frac{\left\langle (\Delta Q)^2 \right\rangle}{\langle Q \rangle^2} \geq \frac{2}{\Delta S(W)}$$

Valid locally in time!

$$\frac{d}{d\tau}\Delta S \geq \frac{1}{2}\frac{d}{d\tau}\frac{\langle Q \rangle^2}{\langle (\Delta Q)^2 \rangle}$$

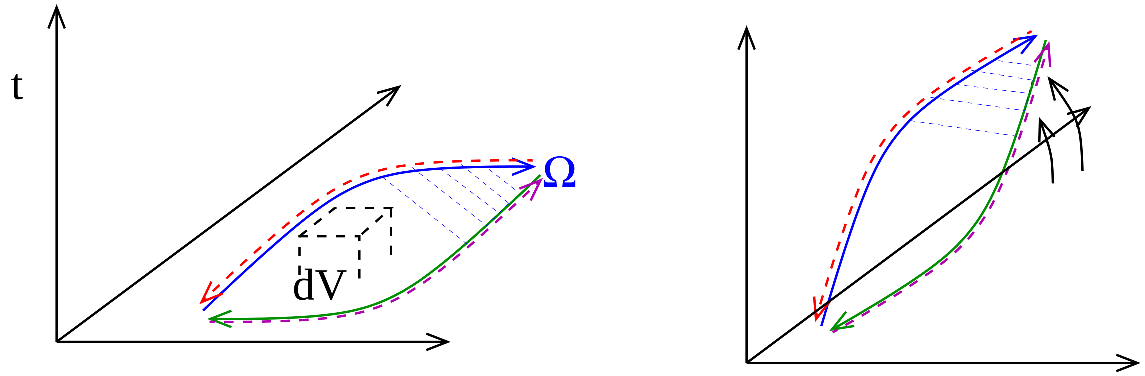
Relates thermal fluctuations and dissipation, producing an irreducible uncertainity. Non-dissipative nano-engines fluctuate like crazy, produces "dissipation" anyway

COnsequences: Hydro-TUR? Separate flow into potential and vortical part

$$\beta_\mu = \partial_\mu \phi + \zeta_\mu \quad , \quad n_\mu \to T \partial_\mu \phi \quad , \quad \omega_{\mu\nu} = g_{\mu\nu}|_{comoving}$$

A likely TUR is

$$\frac{\langle [T_{\mu\gamma}, T_\nu^\gamma] \rangle}{\langle T^{\mu\nu} \rangle^2} \geq \frac{\mathcal{C} \epsilon_{\mu\gamma\kappa} \langle T^{\gamma\kappa} \rangle \beta^\mu}{\Pi^{\alpha\beta} \partial_\beta \zeta_\alpha} \quad , \quad \mathcal{C} \sim \mathcal{O}(1)$$

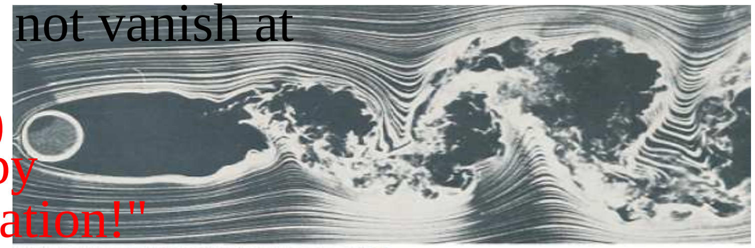


Deform the equilibrium contour and get Kubo formula! (right panel)

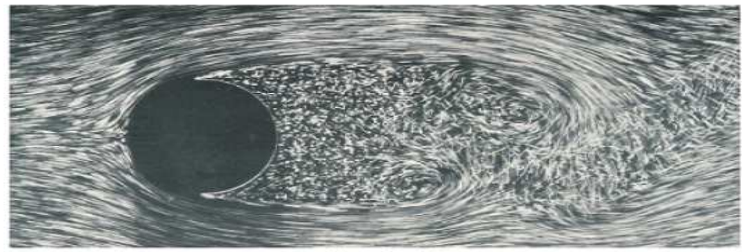
$$\mathcal{C} = \lim_{w\to 0} \frac{\mathrm{Re}\left[F(w)\right]}{\mathrm{Im}\left[F(w)\right]} \quad , \quad F(w) = \int d^3x dt \left\langle T^{xy}(x) T^{xy}(0) \right\rangle e^{i(kx-wt)}$$

−dissipation does not vanish at zero viscosity

Vlad Vicol (talk) "will be proven by a different generation!"



(a) Visualizing turbulent cylinder wake at Re = 10000
[Courtesy: Thomas Corke and Hassan Nagib; *from An Album of Fluid Motion by van Dyke (1982)*]



(b) a closer look at Re = 2000 - patterns are identical as in (a)
[Courtesy: ONERA pic. Werle & Gallon (1972) *from An Album of Fluid Motion by van Dyke (1982)*]

$$\frac{\langle [T_{\mu\gamma}, T_\nu^\gamma] \rangle}{\langle T^{\mu\nu} \rangle^2} \geq \frac{\mathcal{O}(1)\, \epsilon_{\mu\gamma\kappa} \langle T^{\gamma\kappa} \rangle \beta^\mu}{\Pi^{\alpha\beta} \partial_\beta \zeta_\alpha}$$

Fluctuations+Low viscosity $\Rightarrow$ Turbulence $\Rightarrow$ high vorticity $\Rightarrow$ dissipation!
(usually mathematicians consider incompressible fluids, non-relativistic!)

Towards equations: Gravitational Ward identity!

$$\partial^{\alpha} \left\{ \left\langle \left[\hat{T}_{\mu\nu}(x), \hat{T}_{\alpha\beta}(x') \right] \right\rangle - \right.$$

$$\left. -\delta(x-x') \left(g_{\beta\mu} \left\langle \hat{T}_{\alpha\nu}(x') \right\rangle + g_{\beta\nu} \left\langle \hat{T}_{\alpha\mu}(x') \right\rangle - g_{\beta\alpha} \left\langle \hat{T}_{\mu\nu}(x') \right\rangle \right) \right\} = 0$$

Small change in $T_{\mu\nu}$ related to infinitesimal shift! Conservation of momentum!

Can be used to fix one component of $\beta_{\mu} = u_{\mu}/T$, so $u_{\mu}u^{\mu} = -1$ and $(\beta_{\mu}\beta^{\mu})^{-1/2} = T$ weights $\hat{\Pi}^{\mu\nu}$ in a way that conserves $\hat{\Pi}^{\mu\nu} + \hat{T}_0^{\mu\nu}$

Putting everything together: Dynamics at Z level

$$\langle T_{\mu\nu}\rangle = \frac{2}{\sqrt{-g}}\frac{\delta \ln Z}{\delta g^{\mu\nu}} = \langle T_0\rangle^{\mu\nu} + \Pi^{\mu\nu}$$

$$\langle T_0^{\mu\nu}\rangle = \frac{\delta^2 \ln Z}{\delta \beta_\mu d n_\nu} \qquad , \qquad \langle \Pi^{\mu\nu}\rangle = \frac{1}{\partial_\mu \beta_\nu}\partial_\gamma \frac{d}{d\ln(\beta_\alpha\beta^\alpha)}\left[\beta^\gamma \ln Z\right]$$

$$\partial_\alpha\left[\frac{2}{\sqrt{-g}}\frac{\delta^2 \ln Z}{\delta g_{\mu\nu}\delta g_{\alpha\beta}} - \delta(x-x')\frac{2}{\sqrt{-g}}\left(g_{\beta\mu}\frac{\delta \ln Z}{\delta g_{\alpha\nu}} + g_{\beta\nu}\frac{\delta \ln Z}{\delta g_{\alpha\mu}} - g_{\beta\alpha}\frac{\delta \ln Z}{\delta g_{\nu\mu}}\right)\right] = 0$$

and, finally, Crook's theorem

$$\frac{\delta^2}{\delta g^{\mu\nu}\delta g^{\alpha\beta}}\ln Z = \frac{\sqrt{-g}}{2}\frac{\beta_\kappa}{2\omega^{\mu\nu}\beta^\alpha}\partial_\beta n^\kappa \partial_\gamma \frac{d}{d\ln(\beta_\alpha\beta^\alpha)}\left[\beta^\gamma \ln Z\right]$$

Ito process

$$\hat{T}_{\mu\nu}(t) = \hat{T}_{\mu\nu}(t_0) + \int \Delta^{\alpha\beta}\left[\hat{T}_{\mu\alpha}\hat{T}_{\beta\nu}\right] + \int \frac{1}{2} d\Sigma_\mu \beta_\nu \hat{\Pi}_{\alpha\beta}\partial^\alpha \beta^\beta$$

$$\ln Z|_{t+dt} = \int \mathcal{D}g_{\mu\nu}(x) T^{\mu\nu}|_{t+dt} \quad , \quad \beta_\mu|_{t+dt} = \frac{\delta \ln Z|_{t+dt}}{\delta T_{\mu\nu}} n_\nu$$

At every point in a foliation, dynamics is regulated by a stochastic term and a dissipation term. Can be done numerically with montecarlo with an ensemble of configurations at every point in time...

Need: <u>Euclidean correlator in equilibrium</u> $\langle T_{\mu\nu}(x) T_{\mu\nu}\rangle (x')$

A numerical formulation

Define a field β_μ field and n_μ

Generate an ensemble of

$$\ln Z|_{t+dt} = \int \mathcal{D}g_{\mu\nu}(x) T^{\mu\nu}|_{t+dt} \quad , \quad \beta_\mu|_{t+dt} = \frac{\delta \ln Z|_{t+dt}}{\delta T_{\mu\nu}} n_\nu$$

According to a Metropolis algorithm ran via Crooks theorem

Reconstruct the new β and $\Pi_{\mu\nu}$. The Ward identity will make sure $\beta_\mu \beta^\mu = -1/T^2$

Computationally intensive (an ensemble at every timestep), but who knows?

A semiclassical limit?

$$\partial_\mu \left\langle \hat{T}^{\mu\nu} \right\rangle = 0 \quad , \quad \partial_\mu \left\langle \hat{T}_0^{\mu\nu} \right\rangle = -\partial_\mu \left\langle \hat{\Pi}^{\mu\nu} \right\rangle$$

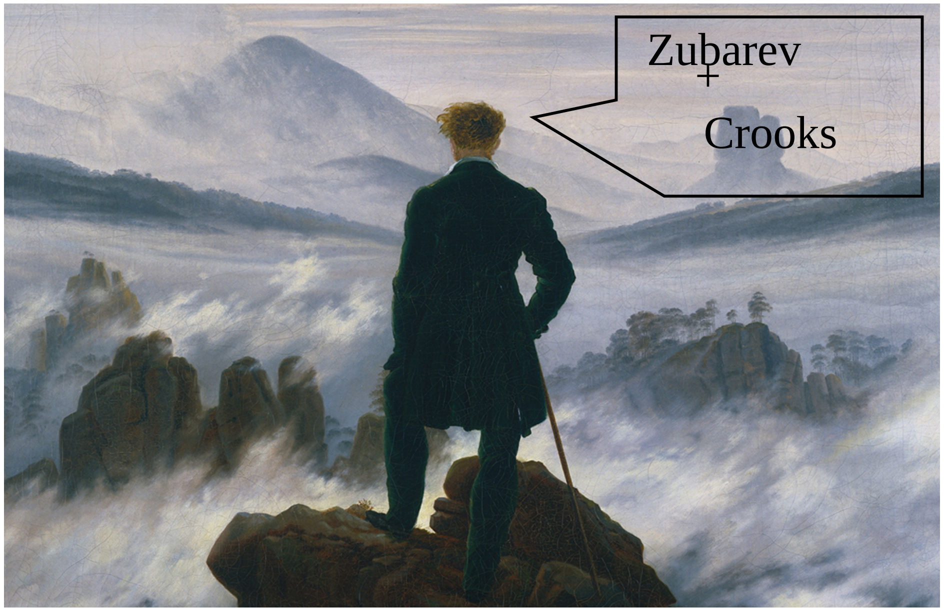
Integrating by parts the second term over a time scale of many $\Delta_{\mu\nu}$ gives, in a frame comoving with $d\Sigma_\mu$

$$\int_0^\tau d\tau' \left\langle \hat{\Pi}_{\mu\nu} \right\rangle \partial^\mu \beta^\nu \sim \beta^\mu \partial_\mu \left\langle \hat{\Pi}_{\mu\nu} \right\rangle + \left\langle \hat{\Pi}_{\mu\nu} \right\rangle = F(\partial^{n\geq 1}\beta_\mu, ...)$$

where $F(\beta_\mu)$ is independent of $\Pi_{\mu\nu}$. (Because local entropy is maximized at vanishing viscosity $F()$ depends on gradients. Israel-Stewart

However , results of, e.g., Gavassino 2006.09843 and Shokri 2002.04719 suggest that fluctuations with decreasing entropy have a role at first order in gradient!

What next?



A 1D example

β, Π are numbers and there is no vorticity so no Σ either!

$$T_0^{\mu\nu} = U^{-1} \begin{pmatrix} e & 0 \\ 0 & -p(E) \end{pmatrix} U \quad , \quad U = (1-\beta^2)^{-1} \begin{pmatrix} 1 & \beta \\ \beta & -1 \end{pmatrix}$$

$$\Pi^{\mu\nu} = \begin{pmatrix} 0 & \Pi \\ \Pi & 0 \end{pmatrix} \quad , \quad \Sigma^\mu \propto \beta^\mu$$

Random matrix distribution of $\{\beta, e\} \leftrightarrow \{\pi\}$

$$\frac{P(\{e\} + d\beta\beta w)}{P(\{e\} - d\beta\beta w)} \propto \exp\left[\{\Pi\}\,\beta^{-1}\,\{\partial\beta\}\right] \delta\left(Ward[e,\beta,\pi]\right)$$

Ward identity can fix $\{\beta\}$ from $\{\pi\}$, rest is Markov chain

Polarization,Chemical potential, rotations,accellerations,...

$$\beta_\mu T^{\mu\nu} \to \beta_\mu T^{\mu\nu} + \mu N^\mu + \mathcal{W}\mathcal{J}^\mu$$

F.Becattini et al, 2007.08249, Prokhorov et. al. 1911.04545: Global equilibrium under general "passive" non-inertial transformation

A paradox: State in "Global equilibrium" (Maximum entropy) but generally does not obey KMS conditions Stationarity/stability!

Global/local equilibrium not the same.

2nd law of thermodynamics defined locally, "entropy" frame dependent in non-inertial fluctuating system

How do you translate all this to dynamics?

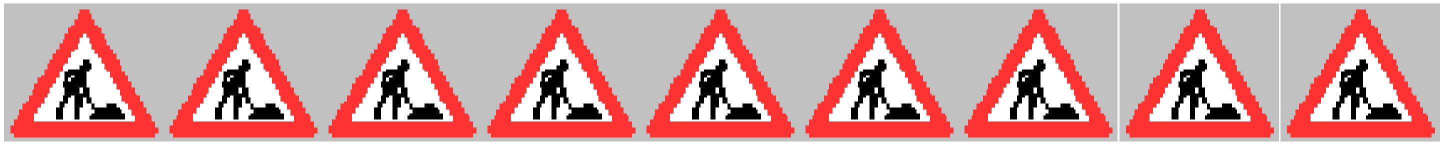
Polarization,Chemical potential, rotations,accellerations,...

$$\beta_\mu T^{\mu\nu} \to \beta_\mu T^{\mu\nu} + \mu N^\mu + \mathcal{W}\mathcal{J}^\mu$$

Crooks Approach allows us to resolve these ambiguities straight-forwardly.

- System evolves to a state where KMS condition obeyed by <u>proper time in the local foliation</u>, ensemble <u>foliation-independent</u>
- Gauge potentials will lead to non-local correlations that never equilibrate, $N^\mu \to N^\mu + U\partial^\mu U$

GT, 1810.12468



Conclusions

- The observation of hydrodynamic behavior with 50 particles forces us to consider a non-perturbative contribution of fluctuation to dynamics. Perhaps fluctuations and turbulence help thermalization?

- A possible definition for such a non-perturbative fluctuating hydrodynamics is...
  - Zubarev hydrodynamics for the "ideal part"
  - Crooks fluctuation theorem for the rest