Toward waveform modeling for gravitational waves from black hole encounters

Gungwon Kang (Chung-Ang University)
Outline

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III. Waveform modeling with general eccentricity
IV. Conclusion
I. Motivation

✓ BBHs are important sources for GW observations

• 50 observations so far
  - O1 (2015/09/12~2016/01/13): 3 BBH
  - O3a (2019/04/01~2019/10/01): 36 BBH +3 others

• NR simulations and waveforms modeling have mostly focused on the last stage of coalescences, e.g., quasi-circular orbits.
• In general,

\[ m_1 \rightarrow (E, L) \rightarrow m_2 \]

\[ e: 0 \sim 1 \]

• “Circularization”:

Peters (1964): Evolution of eccentricity (‘2.5’PN)

Cao & Han (2017)
• Higher-order (3.5) PN calculation: Kocsis & Levin (‘12)

• So, eccentric BBH mergers (e.g., $e \neq 0$) might be relevant, and we may need to develop waveform models with a finite eccentricity.

• BBH waveforms with eccentricity for inspiral phase only:
  ➤ TaylorF2ecc ($e: 0.0001 \sim 0.2$) by C. Kim, J. Kim, H. Lee +, ...

• What kind of waveforms in general for two body encounters?
Gold & Brugmann ('13):

\[ P = P_{QC}, \; \Theta = 50^\circ \]

\[ P = 5P_{QC}, \; \Theta = 14.15^\circ \]

\[ P = P_{QC}, \; \Theta = 60^\circ \]

\[ P = P_{QC}, \; \Theta = 48^\circ \]

\[ P = 5P_{QC}, \; \Theta = 14.20^\circ \]

\[ P = P_{QC}, \; \Theta = 42^\circ \]

\[ P = 5P_{QC}, \; \Theta = 14.30^\circ \]
How weak?: w/ J. Hansen, P. Diener, F. Loebler & H. Kim ('13)

Psi4 (Real part of \( l=2, m=2 \) @ \( r=100 \))

\[ \sim 7 \times 10^{-4} \]

\[ \sim 3 \times 10^{-4} \]

\[ \sim 2.7 \times 10^{-4} \]

\[ \sim 1.3 \times 10^{-4} \]

\[ \sim 5 \times 10^{-6} \]

\[ \Delta t \sim 100M \sim 0.5 \text{msec} \]

\[ M = 0.5 \times 10^{-5} \frac{M}{M_\odot \text{sec}} \]
Masses in BBHs:

- No BBH of stellar origin $> 50M_{\odot}$?
- Similarly, no BBH $< 5M_{\odot}$?

⇒ “Mass Gap” (?):

Belczynski+2011, Sathya+2019

Abbott+19 (arXiv:1811.12940)
Effective spins of binary objects:

\[ \chi_{\text{eff}} = \frac{(m_1 \chi_1 + m_2 \chi_2) \cdot \hat{L}_N}{M} \]
• So, some GWs from BH encounters could be detected in the future detectors, e.g., cosmic explorer or Einstein telescope!
  ➔ We need to prepare waveform models for highly eccentric BBH mergers.
• What are the origins of non-stellar binary black holes?

➔ What is the whole life of a two-body system and the evolution of the waveform associated?
The whole life of a BBH system:

- "Inspiral-Merger-Ringdown" is just a tiny part at the last moment of binary coalescences!

Formation of binary:  
Unbound $\rightarrow$ Bound  
(Hyperbolic $\rightarrow$ Elliptic)

Encounters

Precessions

Coalescence-IMR

"Construct a waveform model covering all of it, in particular, highly eccentric phases!"
Binary formation through gravitational radiation capture

- Formation of compact binaries:
  - Primordial binaries (Postnov & Yungelson 06)
  - Three-body interactions (Aarseth & Heggie ‘76, Bae et al. 14)
- Gravitational radiation capturing processes

- Gravitational radiation captures:
  - Hansen ‘72, Quinlan & Shapiro ‘87, ‘89; Lee ‘95, O’Leary et al. 09, Hong & Lee 15
  - All in the context of Post-Newtonian theory

- How good are the Post-Newtonian results? When do they break down?

Numerical studies at the level of full general relativity
• How often such captures occur?
  – Direct capture could be rare: $0.02 \sim 5 \text{ yr}^{-1} \text{ Gpc}^{-3}$ (Hong & Lee ‘15)
  – How many binaries would be formed at a galaxy center with a SMBH?: O’Leary, Kocsis, Loeb (‘09)

  ➡ Eccentricity distributions:

  \[
  \Gamma_{\text{IGN}} = \int_{r_{\text{min}}}^{r_{\text{max}}} dr 4\pi r^2 \int_{M_{\text{min}}}^{M_{\text{max}}} dM \int_{M_{\text{min}}}^{M} dm \\
  \times \int_{x_m, x_M > 10, J > J_{\text{LC}}} d^3 v_m d^3 v_M f_m(r, v_m) f_M(r, v_M) \sigma_{\text{cs}} w,
  \]

  $\sim 10^{-8}$ and $10^{-10} \text{ yr}^{-1}$

  ➡ $1 \sim 1000/\text{yr at aLIGO}$!

N-body simulations (Hong & Lee ‘13): $\sim 0.02 \sim 20/\text{yr}$
NR simulations

- Gravitational Radiation Capture: “through GW emissions”

- The marginal capturing gives

\[
b_{\text{max}} = \frac{L_{\text{cr}}(E)}{\mu v_\infty} = \frac{L_{\text{cr}}(E)}{\sqrt{2\mu E}}
\]

\[
\sigma_{\text{cap}} = \pi b_{\text{max}}^2
\]

“Capturing Cross-section”
Two non-spinning equal mass black holes: w/ J. Hansen, P. Diener, F. Loeﬂer & H. Kim ('13)

Identifying this point is computationally very expensive!
• Features of orbits and waveforms:
• Eccentric orbits ⇒ Non-negligible multi mode contributions?

Energy budget (L=1.00):

l=2: ~98.9%
l=3: ~0.1%
l=4: ~1%
Rest: less than 0.01%
• Maximum impact parameter or capturing cross-section:

- Less capturing for large initial energies

- 2.5PN deviates from NR as $E$ increases: $\approx 40\%$ maximally

- But, 3.5PN is still in good agreement with error less than $\approx 4\%$

- For any given energy, the GR result gives the strongest capturing.
Un-equal masses without spin: w/ Y. Bae, H. Lee & J. Hansen ('17)

- EPO (Exact parabolic orbit)
- PNCO (PN corrected orbit)
- NR orbit
Effects of spin with equal masses: w/ Y. Bae & H. Lee ('20)

- How to design the parameter space?:
  - Energy flux emitted at 2.5PN order

\[ \mathcal{F} = \frac{G^3}{c^5} \left\{ f_{\text{NS}} + f_{\text{SO}} + f_{\text{SS}} + \mathcal{O}(\frac{1}{c^6}) \right\} \]

\[ \mathcal{F}_{\text{SO}} = AM \chi_{\text{eff}}^{(+)} + B \delta m \chi_{\text{eff}}^{(-)} + \ldots \]

\[ \delta m = m_1 - m_2, \]

Radiated energy through a single encounter

\[ \chi_{\text{eff}}^{(\pm)} = \chi_{\pm} \cdot \ell = \left( \frac{m_2}{M} \chi_2 \pm \frac{m_1}{M} \chi_1 \right) \cdot \ell, \]

\[ \chi_{\text{eff}}^{(+)} = \chi_{\text{eff}} = 0 \]

\[ \chi_{\text{eff}}(|\chi_1| = |\chi_2| = 0.5) \]
- Two BHs with anti-aligned spins will be captured or form a binary more easily.

- The EPO approximation underestimates the capture cross-section.

- Error for the EPO approximation goes about 5% at $v_\infty = 0.01 \, c \sim$ about 35% at $v_\infty = 0.2 \, c$ with anti-aligned spins.
III. Waveform modeling with general eccentricity

• IMR waveforms with eccentricity:
  - Cao & Han 2017; Hinderer & Babak 2017; Hinder et al. 2018; Huerta et al. 2018; Ireland et al. 2019
  - Klein+2018, Tiwari+2019 (PN), Chiaramello & Nagar 2020 (EOB)

• Ex) Cao & Han 2017: works up to $e \sim 0.2$ with overlap factor $\gtrsim 0.98$, compared to NR simulations. Near circular orbit though...

\[
\begin{align*}
\dot{r} &= \frac{\partial H}{\partial p}, \\
\dot{p} &= -\frac{\partial H}{\partial r} + \vec{F}.
\end{align*}
\]

\[H = M \sqrt{1 + 2\eta \left(\frac{H_{\text{eff}}}{M\eta} - 1\right)},\]

\[\vec{F} = \frac{1}{M\eta\omega_\Phi|\vec{r} \times \vec{p}|} \frac{dE}{dt} \vec{p}, \quad -\frac{dE}{dt} = \frac{1}{16\pi} \sum_{\ell} \sum_{m=-\ell}^{\ell} |\hat{h}_{\ell m}|^2.\]

\[l=2, 3, \ldots, 8\]

- Conservative part
- Same as SEOBNR’s

\[
h_{22}^{\text{insp-plan}} = h_{22}^{(C)} + h_{22}^{(PNE)}, \quad h_{22}^{(PNE)} = h_{22} - h_{22}|_{\dot{r}=0},
\]

\[
h_{\ell m}^{\text{merger-RD}} = \sum_{n=0}^{N-1} A_{\ell mn} e^{-i\sigma_{\ell mn}(t-t_{\text{match}})}.
\]

SEOBNRE (Spinning EOBNR Eccentric)
3PM Hamiltonian:

- Recently, Bern et al (PRL, ’19) have obtained the Hamiltonian at the third post-Minkowskian (3PM) order describing the scattering amplitude for two massive spinless particles in the context of effective field theory.

\[
H(p, r) = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + V(p, r), \quad V(p, r) = \sum_{i=1}^{\infty} c_i(p^2) \left( \frac{G}{|r|} \right)^i,
\]

\[
c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi} (1 - 2\sigma^2), \quad c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi} \left[ \frac{3}{4} (1 - 5\sigma^2) - \frac{4\nu \sigma (1 - 2\sigma^2)}{\gamma \xi} - \frac{\nu^2 (1 - \xi)(1 - 2\sigma^2)^2}{2\gamma^3 \xi^2} \right],
\]

\[
c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi} \left[ \frac{1}{12} (3 - 6\nu + 206\nu \sigma - 54\sigma^2 + 108\nu \sigma^2 + 4\nu \sigma^3) \right] - \frac{4\nu (3 + 12\sigma^2 - 4\sigma^4) \arcsinh \sqrt{\frac{\sigma - 1}{2}}}{\sqrt{\sigma^2 - 1}}
\]

\[
- \frac{3\nu \gamma (1 - 2\sigma^2)(1 - 5\sigma^2)}{2(1 + \gamma)(1 + \sigma)} - \frac{3\nu \sigma (7 - 20\sigma^2)}{2\gamma \xi} - \frac{\nu^2 (3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma \sigma^2 + 15\xi \sigma^2)(1 - 2\sigma^2)}{4\gamma^3 \xi^2}
\]

\[
+ \frac{2\nu^3 (3 - 4\xi) \sigma (1 - 2\sigma^2)^2}{\gamma^4 \xi^3} + \frac{\nu^4 (1 - 2\xi)(1 - 2\sigma^2)^3}{2\gamma^6 \xi^4}
\]

\[
E_{1,2} = \sqrt{p^2 + m_{1,2}^2}, \quad m = m_1 + m_2, \quad \nu = m_1 m_2 / m^2, \quad E = E_1 + E_2, \quad \xi = E_1 E_2 / E^2,
\]

\[
\gamma = E / m, \quad \sigma = p_1 \cdot p_2 / m_1 m_2.
\]
An EOB Hamiltonian for a system of two BHs in general

\[ H^{\text{EOB}} = M \sqrt{1 + 2\nu \left( \frac{H^{\text{eff}}}{\mu} - 1 \right)} \]

For \( \nu = \frac{m_1m_2}{(m_1+m_2)^2} \to 0 \), i.e., \( m_1 \gg m_2 \), we know the answer

\[ H_S^2 = \left( 1 - \frac{2GM}{r} \right) \left[ \mu^2 + \frac{L^2}{r^2} + \left( 1 - \frac{2GM}{r} \right) p_r^2 \right] \]

✓ Antonelli, Buonanno+ 2019:

Then, the post-Schwazschild EOB Hamiltonian at 3PM would be

\[ (\hat{H}^{\text{eff,PS}})^2 = H_S^2 + (1 - 2u)[u^2 q_{2\text{PM}} + u^3 q_{3\text{PM}} + O(G^4)] \]

\[ \hat{H}^{\text{eff}} = \frac{H^{\text{eff}}}{\mu}, \ldots \]

\[ q_{2\text{PM}} = \frac{3}{2} \left( 5\hat{H}_S^2 - 1 \right) \left( 1 - \frac{1}{\sqrt{1 + 2\nu(\hat{H}_S - 1)}} \right), \quad q_{3\text{PM}} = -\frac{2\hat{H}_S - 1}{\hat{H}_S - 1} q_{2\text{PM}} + \frac{4}{3} \nu \hat{H}_S + \frac{14\hat{H}_S^2 + 25}{1 + 2\nu(\hat{H}_S - 1)} \]

Angular momentum:

\[ l \equiv |L|/(GM\mu) \]
- **PN/PM corrected:**

\[
H^\text{eff} = G^0 \left( \ldots + \left( \frac{\nu}{c} \right)^{2n} + \ldots \right) + G \left( \ldots + \left( \frac{\nu}{c} \right)^{2n} + \ldots \right) + G^2 \left( \ldots + \left( \frac{\nu}{c} \right)^{2n} + \ldots \right) + G^3 \left( \ldots + \left( \frac{\nu}{c} \right)^{2n} + \ldots \right) + G^4 \left( \ldots + \left( \frac{\nu}{c} \right)^{2n} + \ldots \right) + \ldots
\]

\[
H^\text{EOB} = M \sqrt{1 + 2\nu \left( \frac{H^\text{eff}}{\mu} - 1 \right)}, \quad H_S^2 = \left( 1 - \frac{2GM}{r} \right) \left[ \mu^2 + \frac{L^2}{r^2} + \left( 1 - \frac{2GM}{r} \right) p_r^2 \right]
\]

\[
\hat{H}^\text{eff} = \frac{H^\text{eff}}{\mu}, \quad \hat{H}_S = \frac{H_S}{\mu}, \quad u = \frac{GM}{r}, \quad \hat{p}_r = \frac{p_r}{\mu}, \quad l \equiv \hat{p}_\phi = \frac{L}{GM\mu},
\]

\[
[\hat{H}^\text{eff,PS}(u, \hat{p}_r, l)]^2 = \hat{H}_S^2 + (1 - 2u) \hat{Q}^\text{PS}(u, \hat{H}_S, \nu)
\]

\[
\hat{H}_S^2 = (1 - 2u)[1 + l^2 u^2 + (1 - 2u) \hat{p}_r^2]
\]

\[
\hat{Q}^\text{PS} = u^2 q^\text{2PM}(\hat{H}_S, \nu) + u^3 q^\text{3PM}(\hat{H}_S, \nu) + \Delta_3^\text{PN}(u, \hat{H}_S, \nu) + \Delta_4^\text{PN}(u, \hat{H}_S, \nu) + \mathcal{O}(5\text{PN})
\]

- **QPN terms:**

\[
q^\text{2PM} = \frac{3}{2} \frac{5\hat{H}_S^2 - 1}{\hat{H}_S^2 - 1} \left( 1 - \frac{1}{\sqrt{1 + 2\nu(\hat{H}_S - 1)}} \right)
\]

\[
q^\text{3PM} = -\frac{2\hat{H}_S^2 - 1}{\hat{H}_S^2 - 1} q^\text{2PM} + \frac{4}{3} \nu \hat{H}_S \left( \frac{14\hat{H}_S^2 + 25}{1 + 2\nu(\hat{H}_S - 1)} \right)
\]

\[
+ \frac{8\nu}{\sqrt{\hat{H}_S^2 - 1}} \left( \frac{4\hat{H}_S^4 - 12\hat{H}_S^2 - 3}{1 + 2\nu(\hat{H}_S - 1)} \right) \text{ sinh}^{-1} \sqrt{\frac{\hat{H}_S - 1}{2}}
\]

\[
\Delta_3^\text{PN} = \left( \frac{175}{3} \nu - \frac{41\pi^2}{32} \nu - \frac{7}{2} \nu^2 \right) u^4
\]

\[
\Delta_4^\text{PN} = \sum_{n=2}^{5} \alpha_{4n} u^n (\hat{H}_S^2 - 1)^{5-n}
\]

\[
+ (\alpha_{44,\text{ln}} u^4 (\hat{H}_S^2 - 1) + \alpha_{45,\text{ln}} u^5) \ln u
\]
\[ \alpha_{42} = \left( -\frac{1027}{12} - \frac{147432}{5} \ln 2 + \frac{1399437}{160} \ln 3 + \frac{1953125}{288} \ln 5 \right) \nu, \]

\[ \alpha_{43} = \left( -\frac{78917}{300} - \frac{14099512}{225} \ln 2 + \frac{14336271}{800} \ln 3 + \frac{4296875}{288} \ln 5 \right) \nu. \]

\[ \alpha_{44} = \left( -\frac{43807}{225} + \frac{296 \gamma_E}{15} - \frac{33601 \pi^2}{6144} - \frac{9771016}{225} \ln 2 + \frac{1182681}{100} \ln 3 + \frac{390625}{36} \ln 5 \right) \nu + \left( -\frac{405}{4} + \frac{123 \pi^2}{54} \right) \nu^2 + \frac{13}{2} \nu^3, \]

\[ \alpha_{45} = \left( -\frac{34499}{1800} + \frac{136 \gamma_E}{3} - \frac{29917 \pi^2}{6144} - \frac{254936}{25} \ln 2 + \frac{1061181}{400} \ln 3 + \frac{390625}{144} \ln 5 \right) \nu + \left( -\frac{2387}{24} + \frac{205 \pi^2}{64} \right) \nu^2 + \frac{9}{4} \nu^3, \]

and

\[ \alpha_{44,\ln} = \frac{148}{15} \nu, \quad \alpha_{45,\ln} = \frac{68}{3} \nu. \]

\[ \gamma_E = 0.57721... \text{ is the Euler-Mascheroni constant.} \]
- Scattering angles: NR vs Newtonian vs EOB vs 3PM

\[ H(r, p_r, L) = E \]

\[ p_r(r, E, L) \]

\[
\left( \frac{\partial H}{\partial L} \right)(r, p_r, \frac{\partial p_r}{\partial L}, L) = 0
\]

\[
\Delta \phi = \pi + \chi(E, L) = -2 \int_{r_{\text{min}}}^{\infty} dr \frac{\partial}{\partial L} p_r(r, E, L)
\]
- Results:

<table>
<thead>
<tr>
<th>Case: $\vec{p} = (P_x, P_y)$</th>
<th>NR</th>
<th>Newtonian</th>
<th>3PM</th>
<th>EOB</th>
<th>$\Delta\chi/\chi$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.090, 0.099)</td>
<td>90.13</td>
<td>87.97</td>
<td>90.34</td>
<td>89.78</td>
<td>-0.388</td>
</tr>
<tr>
<td>(0.214, 0.058)</td>
<td>165.11°</td>
<td>155.00°</td>
<td>165.73°</td>
<td>165.30</td>
<td>0.115</td>
</tr>
<tr>
<td>(0.0326, 0.015)</td>
<td>300.11°</td>
<td>258.51</td>
<td>299.98</td>
<td>300.52</td>
<td>0.137</td>
</tr>
<tr>
<td>(0.0331, 0.015)</td>
<td>299.08</td>
<td>257.63</td>
<td>299.12</td>
<td>299.67</td>
<td>0.197</td>
</tr>
<tr>
<td>(0.0376, 0.012)</td>
<td>356.05</td>
<td>273.65</td>
<td>370.20</td>
<td>359.91</td>
<td>1.084</td>
</tr>
<tr>
<td>(0.043, 0.0185)</td>
<td>376.41</td>
<td>262.64</td>
<td>359.44</td>
<td>384.72</td>
<td>2.208</td>
</tr>
<tr>
<td>(0.00957, 0.004)</td>
<td>187</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Design the parameter space in which we can check the validity of our EOB Hamiltonian:
- High velocity and strong interaction

- We have finished NR simulations for these parameters.

- The analysis and comparisons with the predictions of the known waveform models, which are valid in the limited regimes, are currently work in progress.
IV. Conclusion

- Gravitational radiation capture processes for two BHs have been analyzed numerically.

- Effects of unequal masses (upto $m_1:m_2 = 1:16$) and spin configurations ($\chi^{(+)}_{eff}$) are shown.

- Scattering angles for BH encounters with arbitrary eccentricities are calculated in NR.

- A 3PM/4PN EOB Hamiltonian for arbitrary eccentricities has been constructed and various tests with NR simulations are work in progress. (w/ Y.-H. Hyun & Y.-B. Bae)
THANKS!