Inelastic nuclear recoils

Simon Knapen

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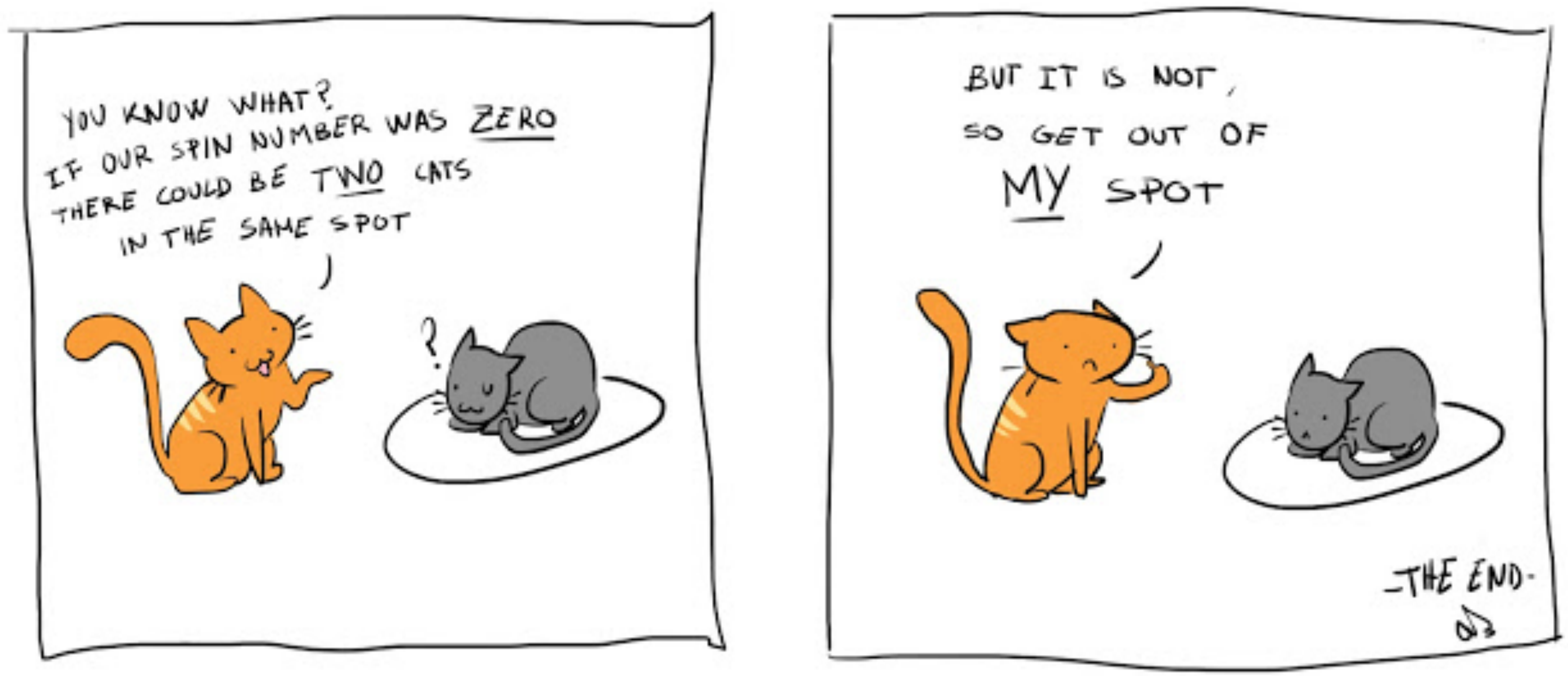

http://dingercatadventures.blogspot.com/2012/08/

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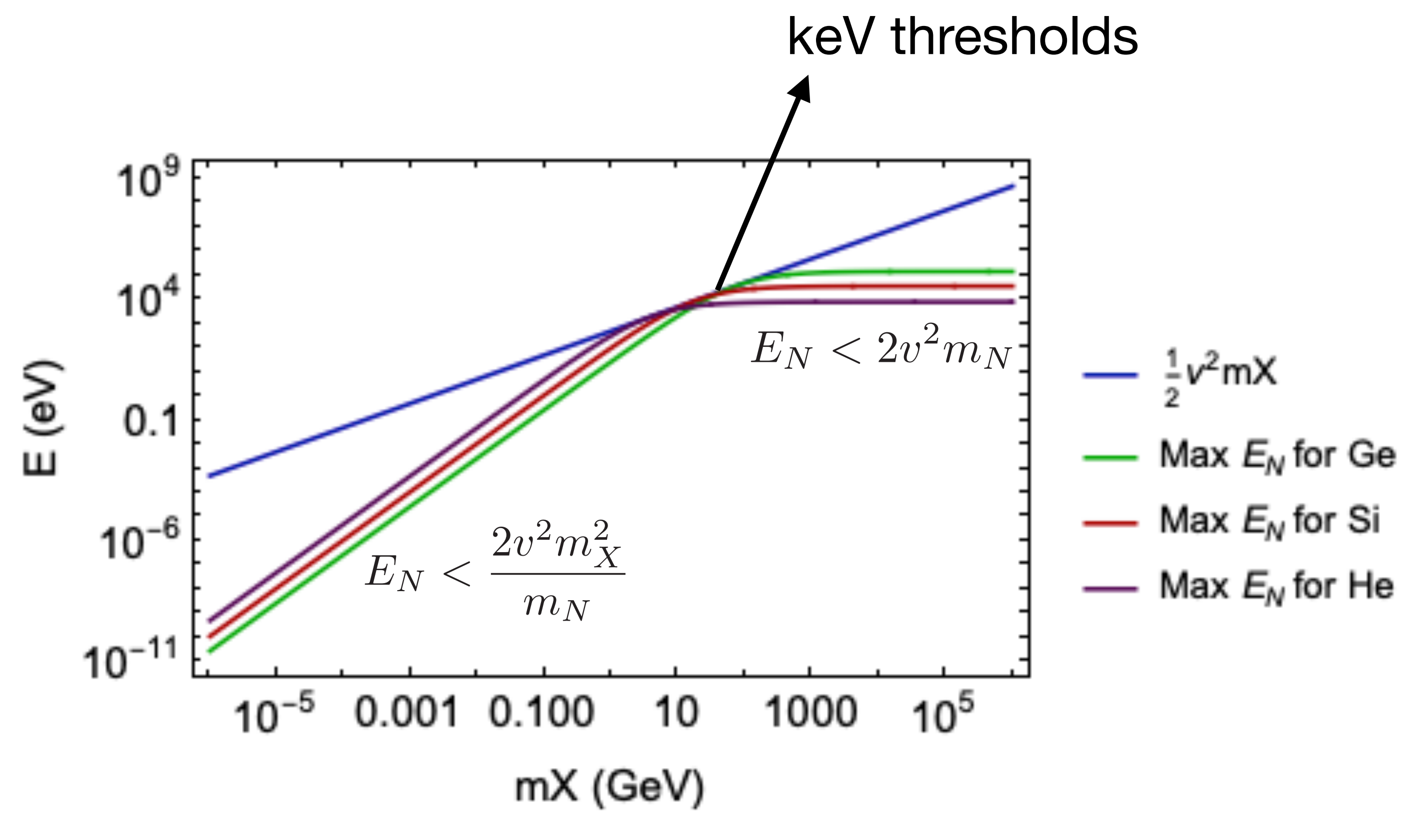
Elastic nuclear recoil kinematics

Momentum conservation implies that for elastic nuclear recoils we have

$$E_N < \frac{(2v\mu_{XN})^2}{2m_N}$$

For mX << mN, we are not accessing the vast majority of the kinetic energy of the dark matter



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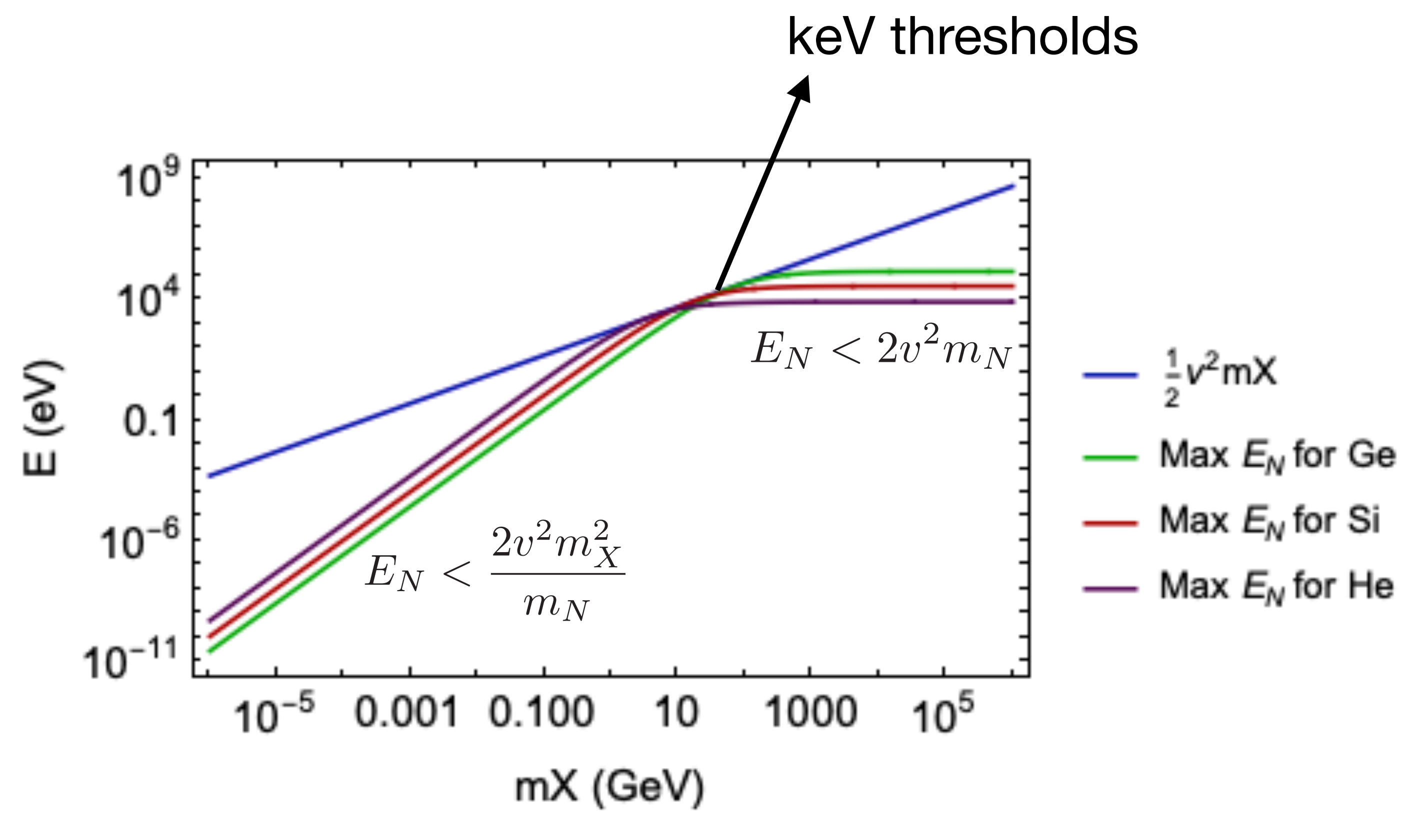

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1. When does the elastic billiard ball picture break?
(Tomorrow's lecture)

2. What about *inelastic* recoils?
(Today's lecture)

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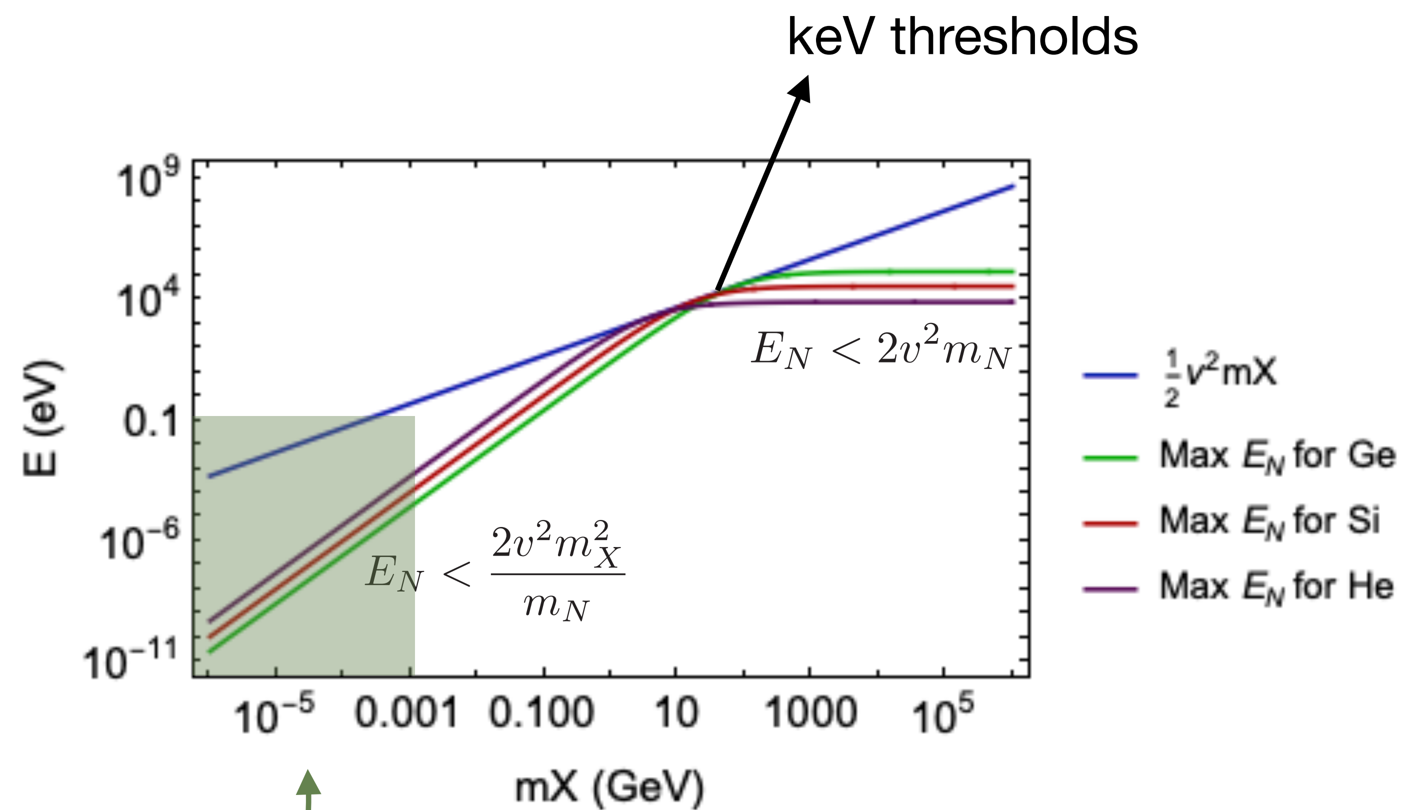

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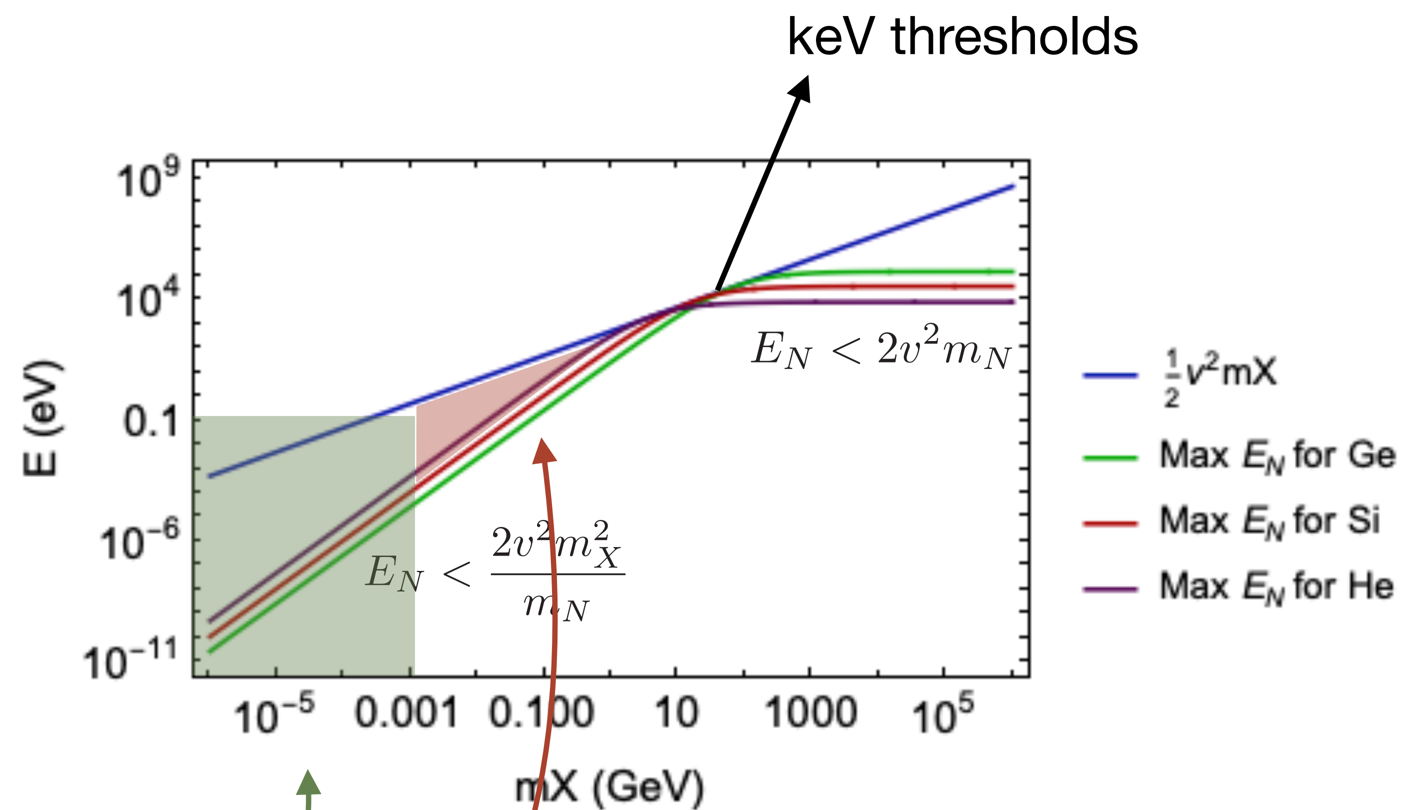

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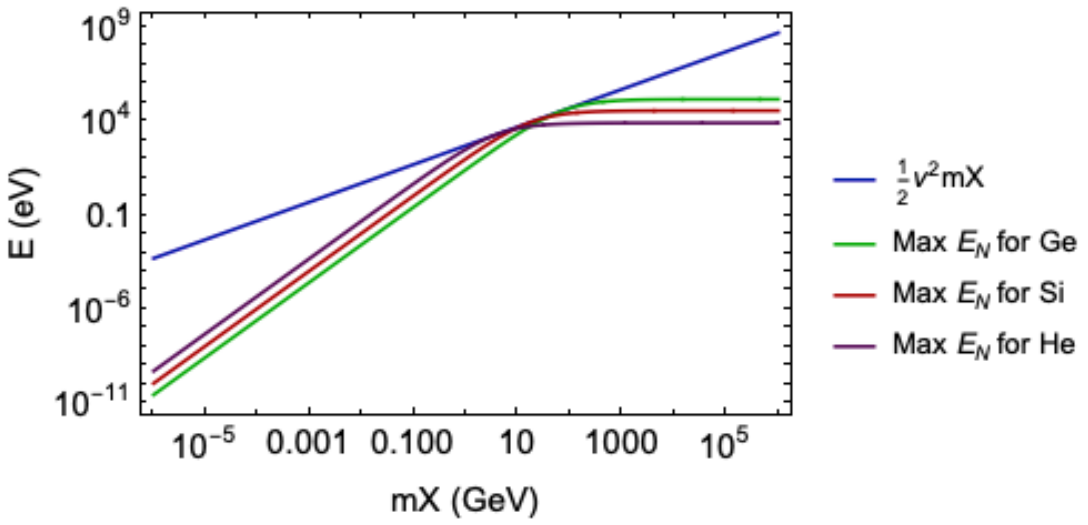
Inelastic nuclear recoils

Kinetic energy is much to low to excite nuclear excitations

We are therefore of thinking of electromagnetic excitations



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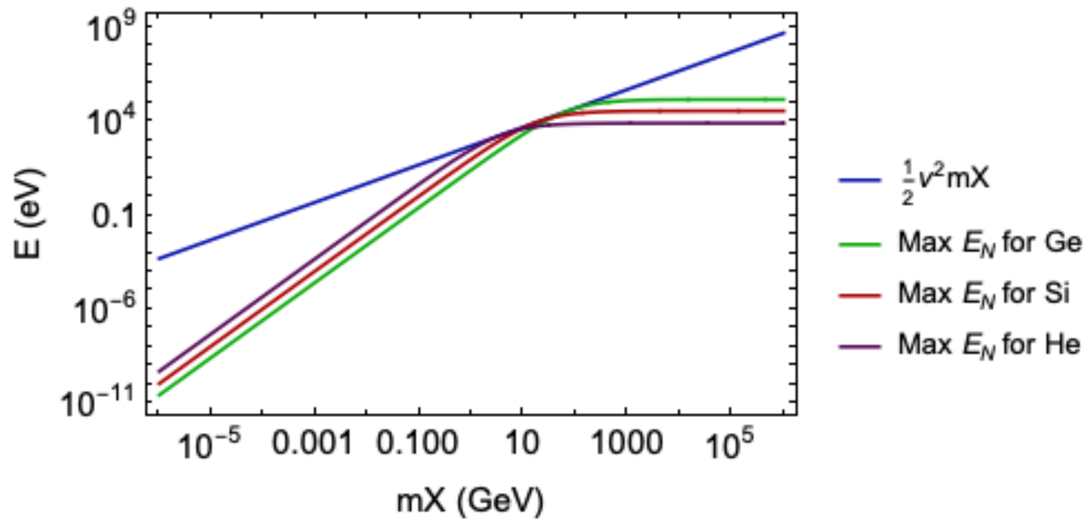
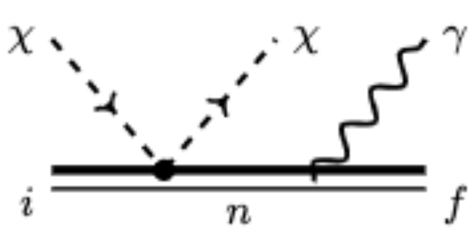
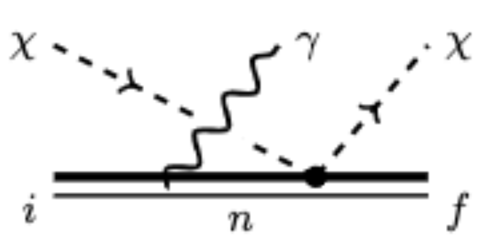


Photons:

Brehmstrallung
C. Kouvaris, J. Pradler: arXiv 1607.01789





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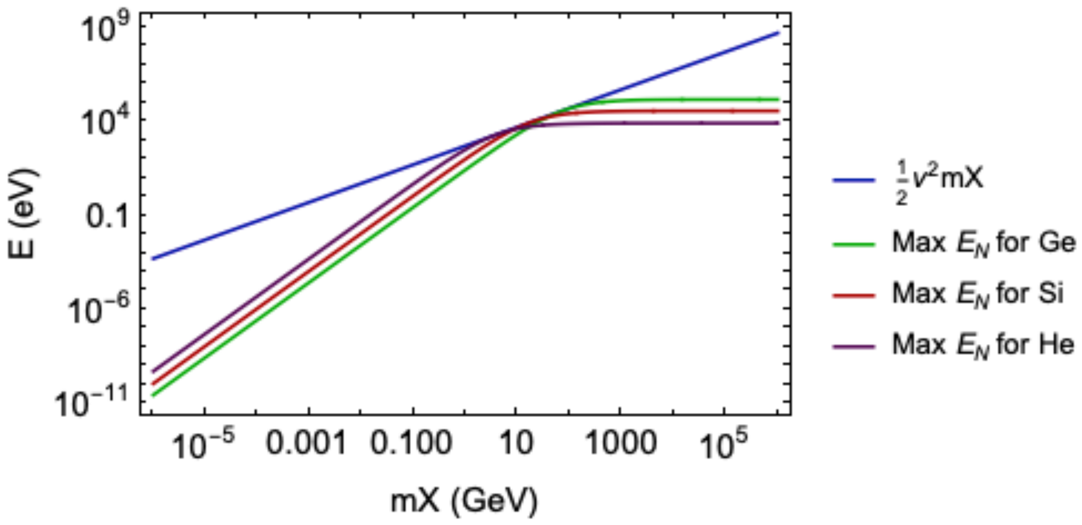

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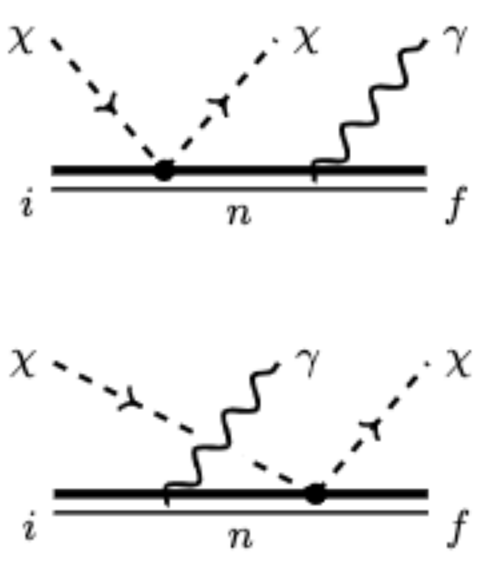


<u>Electrons (Migdal effect):</u>

In atoms (Xe, Ar, He, etc)
A. Migdal (1939)
R. Bernabei et. al.: arXiv 0706.1421
M. Ibe, W. Nakano, Y. Shoji and K. Suzuki: arXiv 1707.07258
…

In crystals (Si, Ge, GaAs, etc)
SK, J. Kozaczuk, T. Lin: arXiv 2011.09496
Z.-L. Liang, C. Mo, F. Zheng and P. Zhang: arXiv 2011.13352

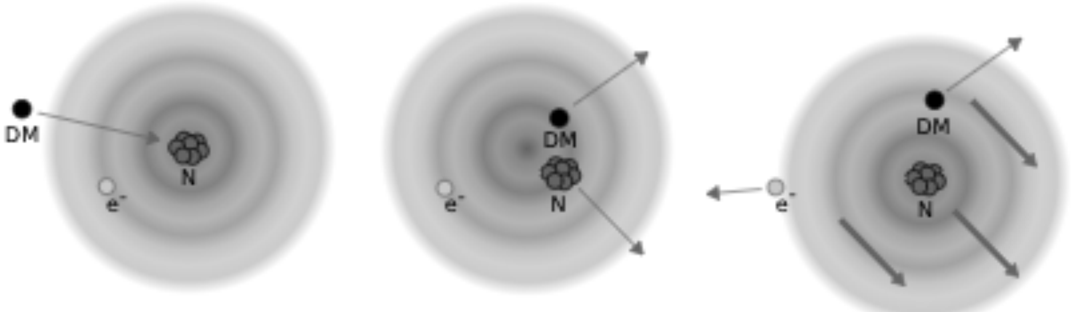
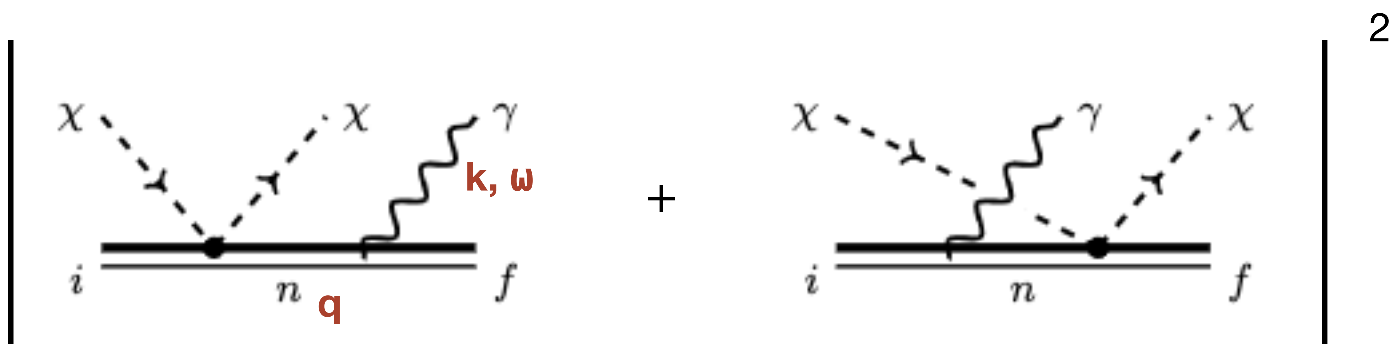


Fig from arXiv 1711.09906

Brehmstrallung

Main features:

- Leading order in α
- Destructive interference
- Three body phase space



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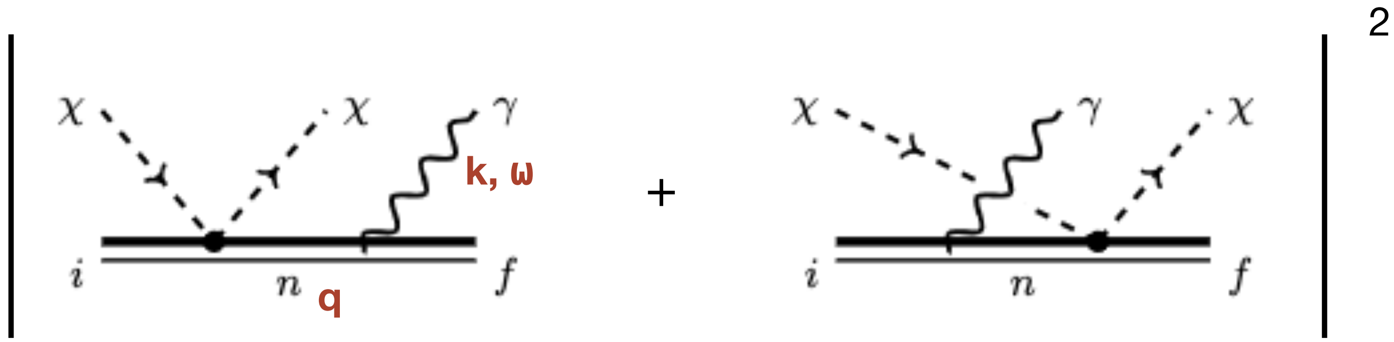


Soft limit:

If k << q, we can treat the photon emission as a small correction on top of an elastic nuclear recoil

Holds if

$$\omega \ll qv = \sqrt{m_N E_R} v \approx 10\ \mathrm{keV} \times \sqrt{\frac{A}{130}} \times \sqrt{\frac{E_R}{1\ \mathrm{keV}}}$$

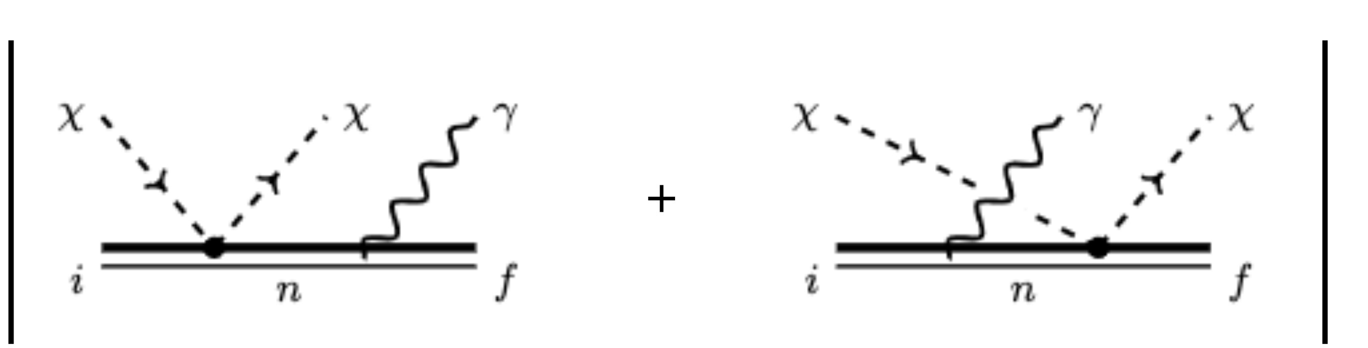
No problem, detectors like XENON, CMDS etc can easily see ~ keV photons, as they leave a strong *ionization* signal

C. Kouvaris, J. Pradler: arXiv 1607.01789

Brehmstrallung

Free ion approximation:

$$\left.\frac{d^2\sigma}{dE_R d\omega}\right|_{\text{naive}} = \frac{4Z^2\alpha}{3\pi}\frac{1}{\omega}\frac{E_R}{m_N} \times \frac{d\sigma}{dE_R}\Theta(\omega_{\max}-\omega).$$



Not very good, because electrons provide screening

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C. Kouvaris, J. Pradler: arXiv 1607.01789

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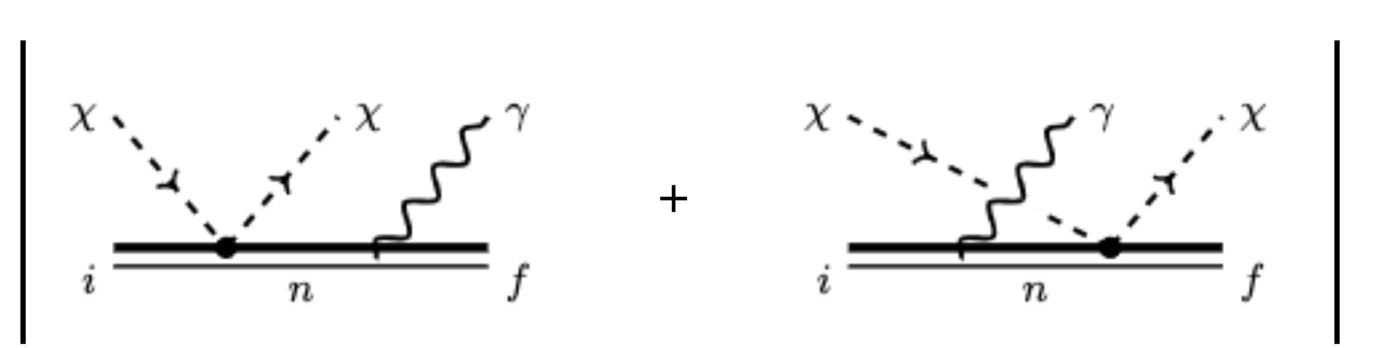


Not very good, because electrons provide screening

Instead

Frequency-dependent effective charge

$$\frac{d^2\sigma}{d\omega dE_R} = \frac{4\alpha}{3\pi\omega}\frac{E_R}{m_N}|Z(\omega)|^2 \times \frac{d\sigma}{dE_R}\theta\left(\frac{m_X v^2}{2} - \omega\right)$$

The effective charge is related to the polarizability of the atom

$$Z(\omega) = -\frac{\alpha(\omega)}{\alpha} m_e \omega^2$$

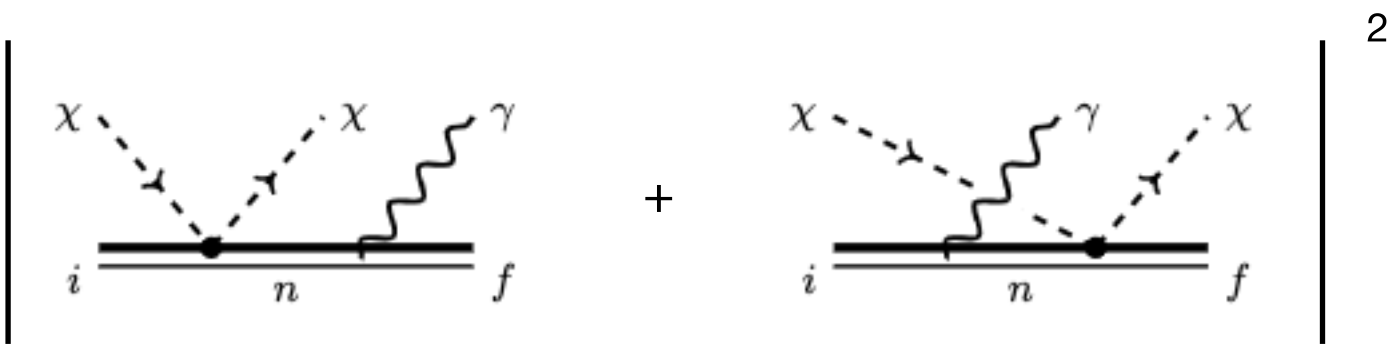
In the high energy limit this reduces to the free-ion ion result $Z(\omega) \to Z \quad \text{for} \quad \omega \to \infty$

C. Kouvaris, J. Pradler: arXiv 1607.01789

Brehmstrallung

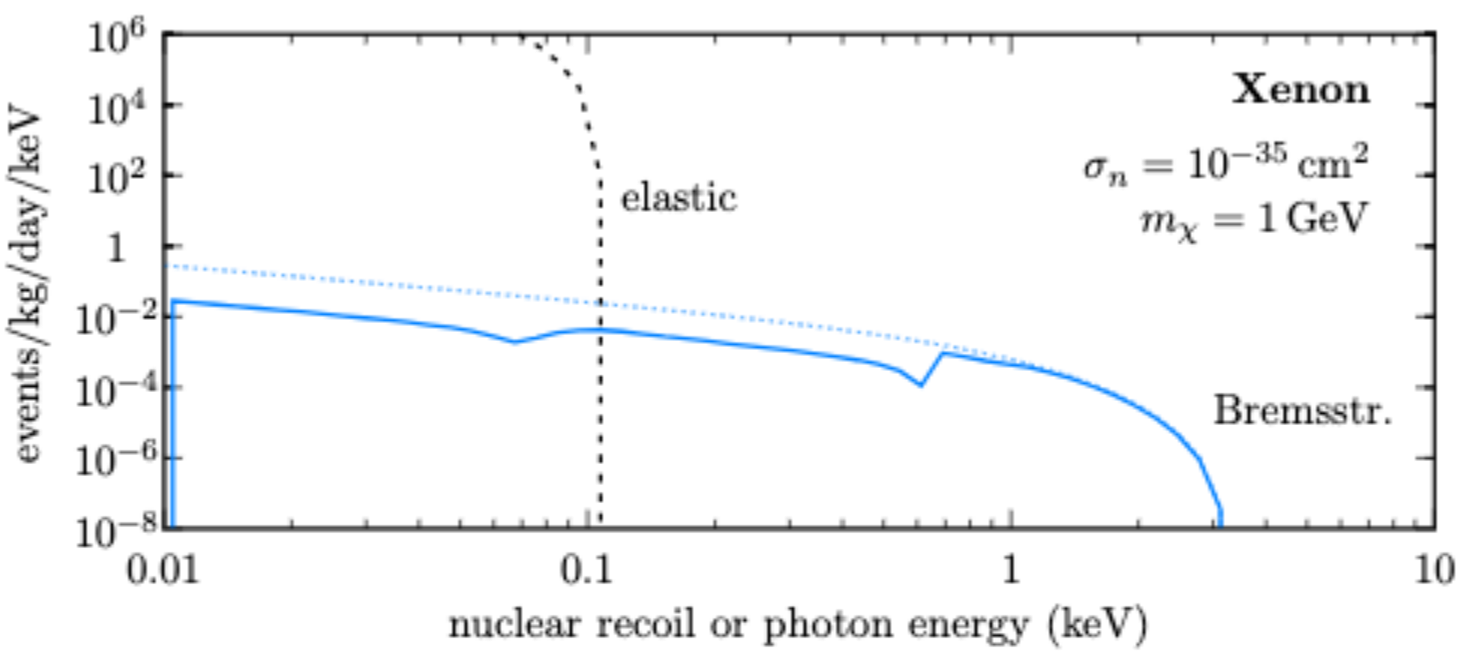
Full result

$$\frac{d^2\sigma}{d\omega dE_R} = \frac{4\alpha}{3\pi\omega}\frac{E_R}{m_N}|Z(\omega)|^2 \times \frac{d\sigma}{dE_R}\theta\left(\frac{m_X v^2}{2} - \omega\right)$$



Extract Z(ω) from measured data

Result



C. Kouvaris, J. Pradler: arXiv 1607.01789

The Migdal effect

<u>Usual explanation:</u>

Nucleus is suddenly kicked and rushes away. Not all the electron wave functions have time to respond and one or more electron is left behind

↓

Ionization

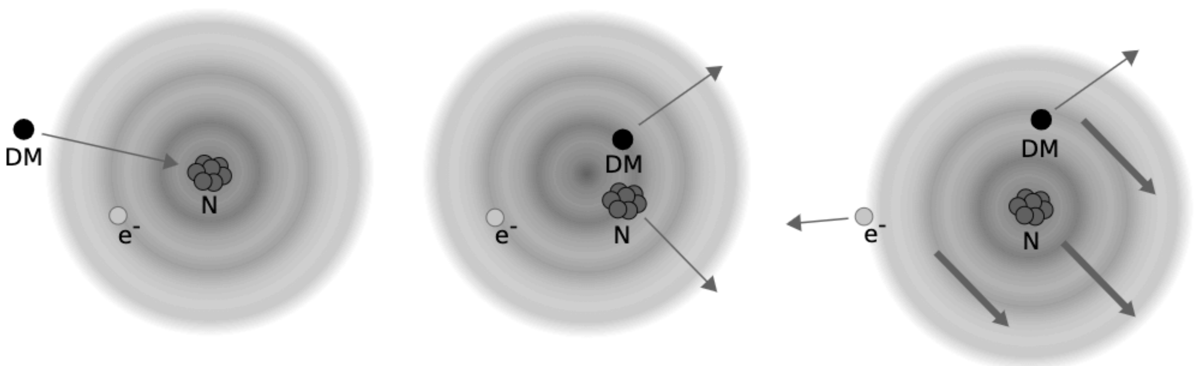


Figure from arXiv 1711.09906

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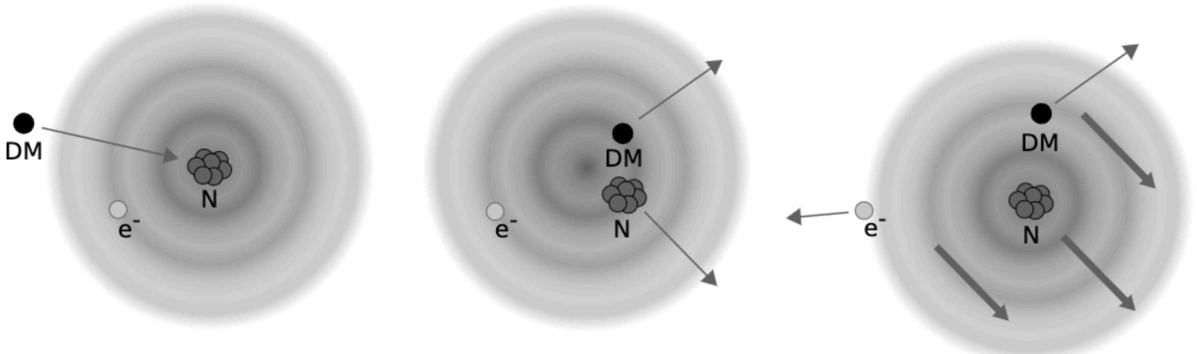
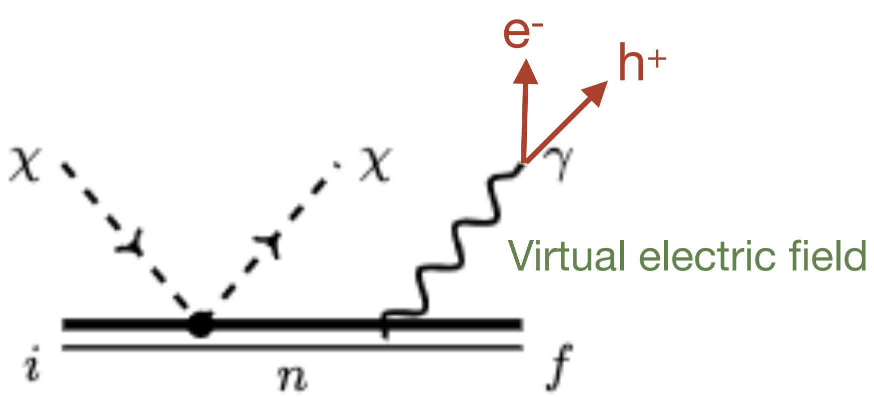


Figure from arXiv 1711.09906

More microscopic explanation:

The change in the Coulomb field felt by the electrons causes energy transfer from the DM to the electrons, and causes the ionization

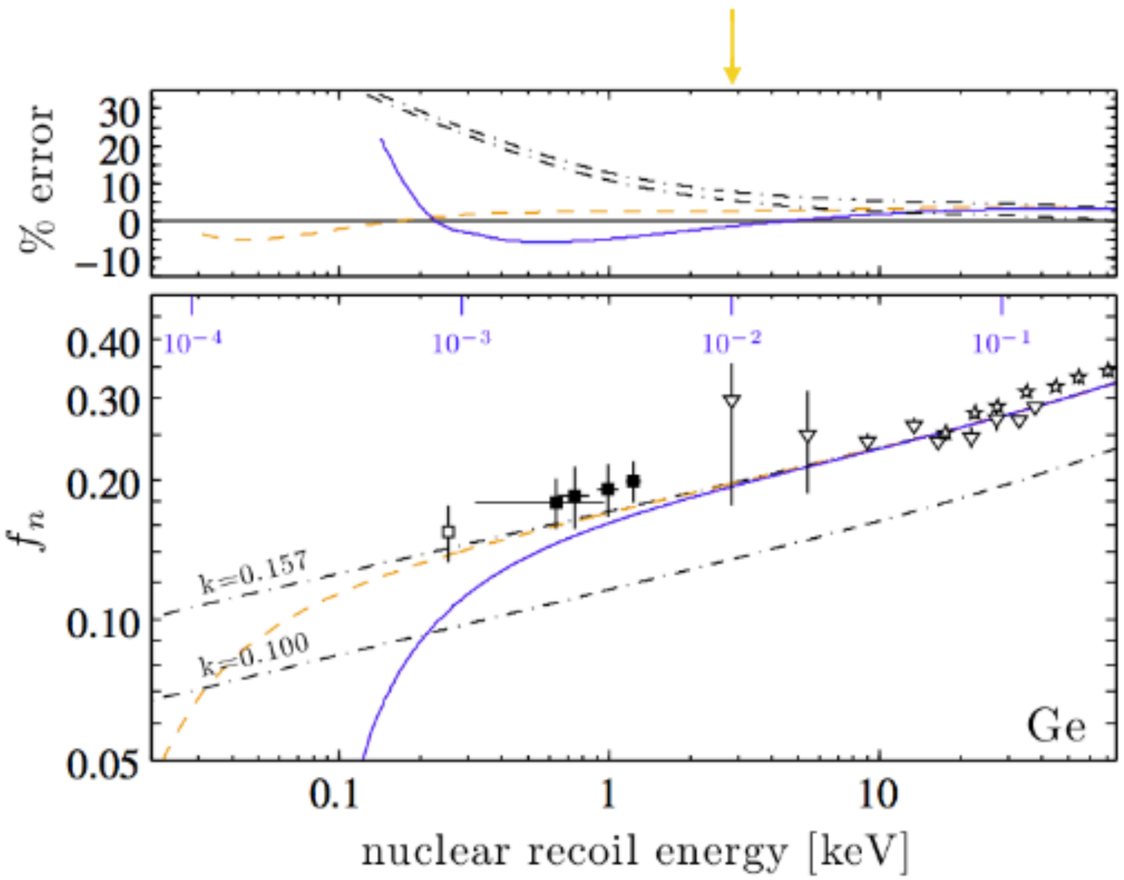
The Migdal effect is very analogous to the brehmstrallung process, but now energy is dissipated into e- h+ pairs instead of a photon

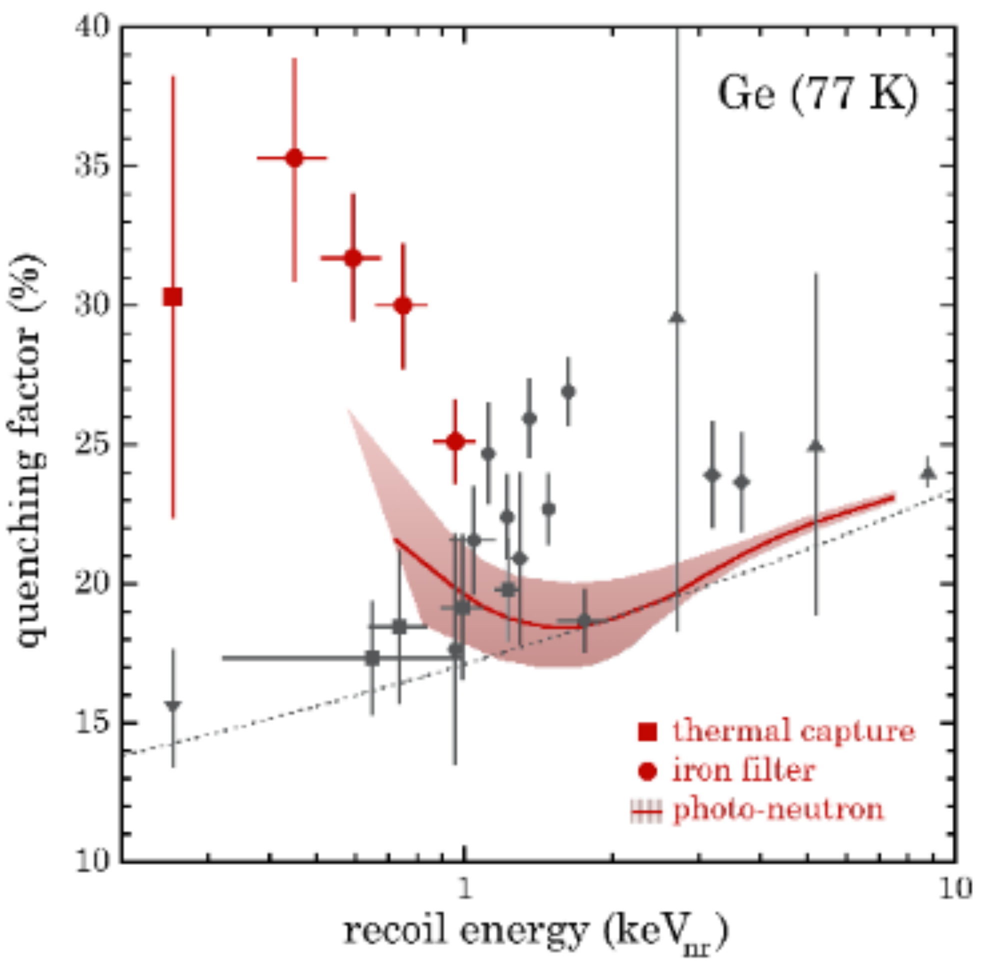


What the Migdal effect is not

The Migdal effect describes ionization/electronic excitations during the initial hard DM-nucleus collision

The recoiling nucleus produces secondary ionization e- when encountering *other* nuclei in the crystal.
This is described by the *quenching factor*:





This is important, but unrelated to the Migdal effect

P. Sorensen: arXiv 1412.3028

J. Collar et. al.: arXiv 2102.10089

Notation

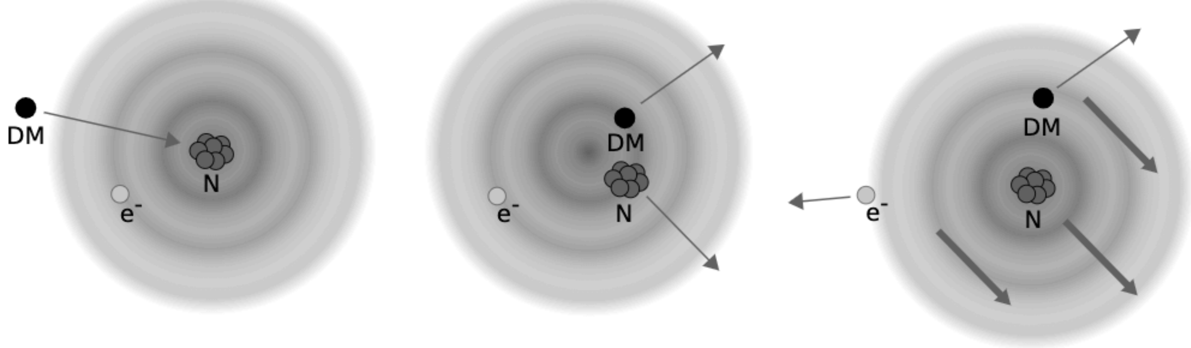
$\vert i\rangle, \vert f\rangle$	Initial and final state of the atom or crystal
E_i, E_f	Energy of the initial and final state
E_N, v_N	Energy and velocity of the recoiling nucleus
$\mathbf{r}_N, \mathbf{r}_\alpha$	Position operator corresponding to nucleus and electron labeled with α
ω, k	Energy and momentum deposited to the electrons

Calculation with Migdal's trick

This is how the calculation goes using the original method by Migdal:

If $E_N >> \omega$, the electron cloud cannot adjust itself to on the time scale of the DM-nucleus impact

This means that the excited electron wave functions in the rest frame of the recoiling nucleus, are simply the ground state wave function, boosted to the frame of the recoiling nucleus.



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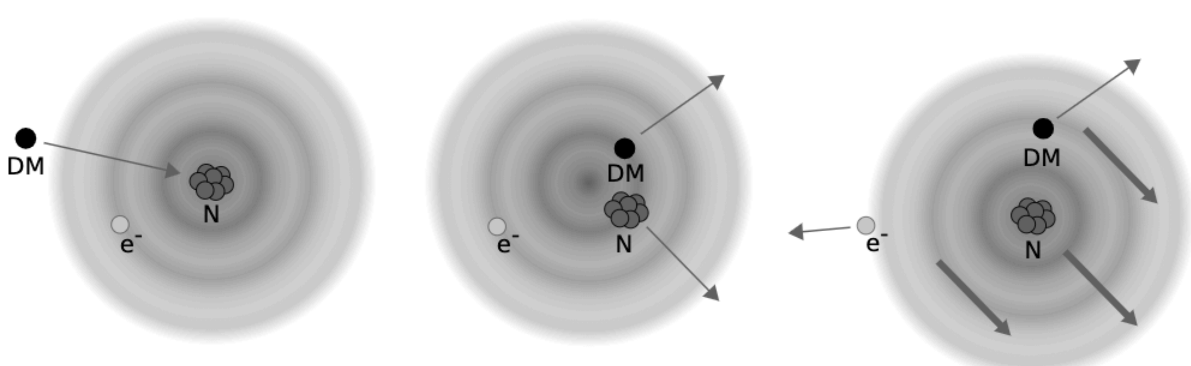
In equations, boosting the wave function with a velocity v_N is equivalent to multiplying it with a phase factor:

$$|i\rangle \to e^{im_e \mathbf{v}_N \cdot \sum_\beta \mathbf{r}_\beta} |i\rangle$$

The transition matrix element to a particular final state f is therefore just

$$\mathcal{M}_{if} = \langle f| e^{im_e \mathbf{v}_N \cdot \sum_\beta \mathbf{r}_\beta} |i\rangle \approx im_e \mathbf{v}_N \cdot \langle f| \sum_\beta \mathbf{r}_\beta |i\rangle$$

Transition dipole moment



Migdal's trick

Numerical evaluation

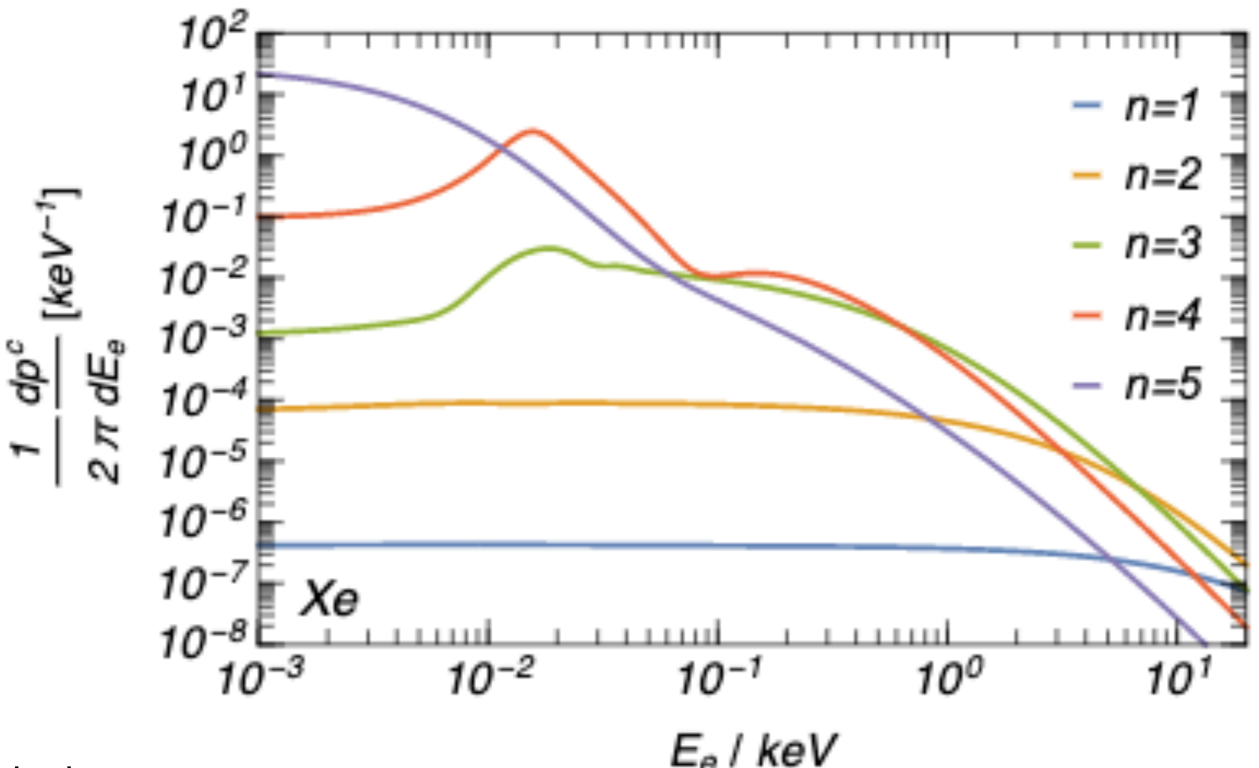
Ibe, Nakano, Shoji & Suzuki (1707.07258) used the numerical package "Flexible Atomic Code" (FAC) to compute the wave functions for a large collection of atoms

This is fairly painful, but once you have them, you can compute the transition probabilities*

$$\frac{dP_{i\to f}}{d\omega} = m_e^2 \left| \mathbf{v}_N \cdot \langle f | \sum_\beta \mathbf{r}_\beta |i\rangle \right|^2 \delta(E_i - E_f + \omega)$$

Thankfully they provide tabulated ionization probabilities, so one can easily reconstruct their results

<u>Example:</u>



* My normalization here is a bit different from theirs, but the physical object is the same

Alternative calculation

Migdal's calculation is cute, but has a few drawbacks:

- The "brehmstrallung" analogy is not so clear. E.g. Where is the dependence on the ion charge?
- The boosting business feels awkward. Is it really legal in all cases?

We should be able to do a straight-up calculation in the lab frame, with old fashioned time-dependent perturbation theory!

Appendix D of SK, J. Kozaczuk, T. Lin: arXiv 2011.09496

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$$H(t) = H_0 + H_1(t)$$

$$H_0 = -\sum_\beta \frac{Z_N \alpha}{|\mathbf{r}_\beta|}$$

$$H_1(t) = -\sum_\beta \frac{Z_N \alpha}{|\mathbf{r}_\beta - \mathbf{R}_N(t)|} + \sum_\beta \frac{Z_N \alpha}{|\mathbf{r}_\beta|}$$

With $\mathbf{R}_N(t) = \theta(t)\mathbf{v}_N t$

$$\approx -Z_N \alpha \sum_\beta \frac{\hat{\mathbf{r}}_\beta \cdot \mathbf{v}_N}{\mathbf{r}_\beta^2} t\theta(t)$$

→ Dipole potential

Z_N is the charge of the ion; for the moment we tread this as fixed, lumping the inner-shell electrons together with the nucleus.

Appendix D of SK, J. Kozaczuk, T. Lin: arXiv 2011.09496

Alternative calculation

The transition probability is

$$P_{i\to f} = \left|\frac{1}{\omega}\int_0^\infty dt\, e^{i(\omega+i\eta)t}\langle f|\frac{dH_1(t)}{dt}|i\rangle\right|^2 = \left|\langle f|\frac{1}{\omega^2}\sum_\beta \frac{Z_N\alpha\hat{\mathbf{r}}_\beta\cdot\mathbf{v}_N}{\mathbf{r}_\beta^2}|i\rangle\right|^2$$

Let’s compare the results at the level of the matrix element:

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Let's compare the results at the level of the matrix element:

<u>Migdal's trick</u>

$$\mathcal{M}_{if} = i m_e \mathbf{v}_N \cdot \langle f| \sum_\beta \mathbf{r}_\beta \,|i\rangle$$

<u>Perturbation theory</u>

$$\mathcal{M}_{if} = i \langle f| \frac{1}{\omega^2} \sum_\beta \frac{Z_N \alpha \hat{\mathbf{r}}_\beta \cdot \mathbf{v}_N}{\mathbf{r}_\beta^2} |i\rangle$$

Which one is right???

Making sense of this

For the Coulomb Hamiltonian $$H_0 = \sum_\beta \frac{|\mathbf{p}_\beta|^2}{2m_e} + V(\mathbf{r}_\beta, \mathbf{r}_N)$$

We have a number of operator identities:

$$[\mathbf{r}_\beta, H_0] = i\frac{1}{m_e}\mathbf{p}_e$$

And

$$[p_\beta, H_0] = -i\frac{dV}{d\mathbf{r}_\beta}$$

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Total force exerted on the electron

$$\begin{aligned}
\mathcal{M}_{if}^{(Migdal)} &= im_e\mathbf{v}_N \cdot \langle f| \sum_\beta \mathbf{r}_\beta |i\rangle \\
&= -i\frac{m_e}{\omega}\mathbf{v}_N \cdot \langle f| \sum_\beta [\mathbf{r}_\beta, H_0] |i\rangle \qquad \text{used} \qquad \omega = E_f - E_i \\
&= \frac{1}{\omega}\mathbf{v}_N \cdot \langle f| \sum_\beta \mathbf{p}_\beta |i\rangle \\
&= -\frac{1}{\omega^2}\mathbf{v}_N \cdot \langle f| \sum_\beta [\mathbf{p}_\beta, H_0] |i\rangle \\
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\end{aligned}
$$

Proportional to total force exerted in the electron

Making sense of this

$$\mathcal{M}_{if}^{(Migdal)} = i m_e \mathbf{v}_N \cdot \langle f | \sum_\beta \mathbf{r}_\beta | i \rangle$$

$$= -i \frac{m_e}{\omega} \mathbf{v}_N \cdot \langle f | \sum_\beta [\mathbf{r}_\beta, H_0] | i \rangle \qquad \text{used} \qquad \omega = E_f - E_i$$

$$= \frac{1}{\omega} \mathbf{v}_N \cdot \langle f | \sum_\beta \mathbf{p}_\beta | i \rangle$$

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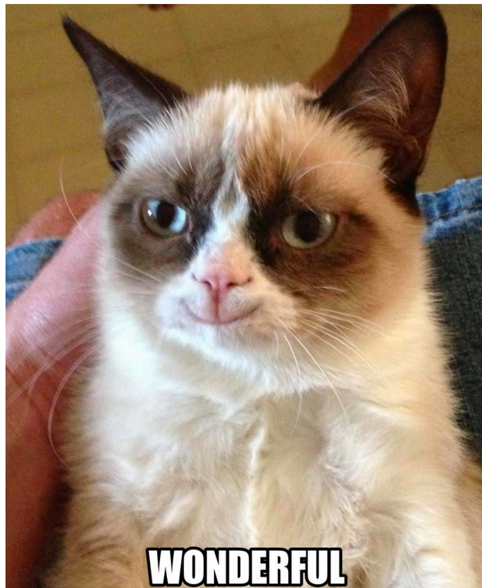
Proportional to total force exerted in the electron

Electron-electron interactions cancel out in the same, only the force from the nucleus remains

$$= i \frac{Z_N \alpha}{\omega^2} \mathbf{v}_N \cdot \langle f | \sum_\beta \frac{\hat{\mathbf{r}}_\beta}{|\mathbf{r}_\beta - \mathbf{r}_N|^2} | i \rangle$$

$$= i \frac{Z_N \alpha}{\omega^2} \mathbf{v}_N \cdot \langle f | \sum_\beta \frac{\hat{\mathbf{r}}_\beta}{|\mathbf{r}_\beta|^2} | i \rangle \qquad \text{taking} \qquad \mathbf{r}_\beta \gg \mathbf{r}_N$$

$$= \mathcal{M}_{if}^{(pert)}$$



A closer look…

Just removing some intermediate steps here, same derivation…

$$\mathcal{M}_{if}^{(Migdal)} = i\frac{1}{\omega^2}\mathbf{v}_N \cdot \langle f| \sum_\beta \frac{dV}{d\mathbf{r}_\beta} |i\rangle$$

$$= i\frac{Z_N \alpha}{\omega^2}\mathbf{v}_N \cdot \langle f| \sum_\beta \frac{\hat{\mathbf{r}}_\beta}{|\mathbf{r}_\beta - \mathbf{r}_N|^2} |i\rangle$$

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This step assumes that only the recoiling nucleus exerts a force on the electrons!

The Migdal trick and the perturbative calculation are equivalent, *but only for atomic targets!*

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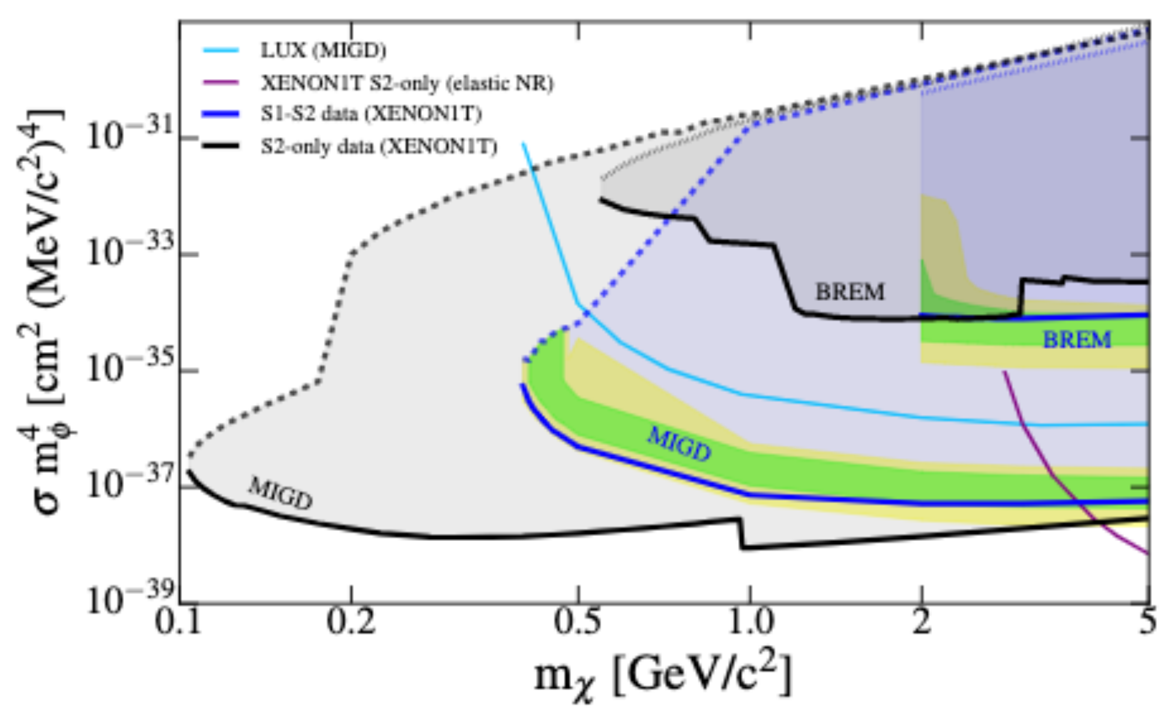
In hindsight, the reason is rather obvious: The boosting trick doesn't work for a crystal, because we'd be boosting all the spectator ions as well! Those contribution would need to be subtracted off in Migdal's calculation, which are exactly the terms that are missing above.

Intermediate conclusions

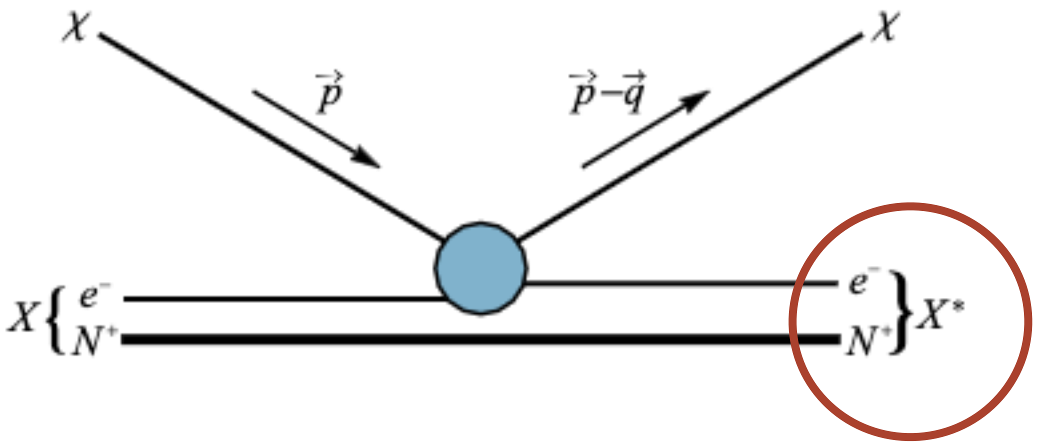
- The Migdal effect is very analogous to brehmstrallung, with the difference that the photon is virtual and excites one or more electrons
- We've shown this by showing that the Migdal trick is equivalent to a perturbative computation…
- … but only for isolated atoms!
- The numerical Ibe et al results are super useful but can only relied for atomic targets, such as Xe, Ar, He etc

Now let us look at crystals…



Xenon1T arXiv: 1907.12771

Crystals are complicated



e^- are not free

e^- are not at rest

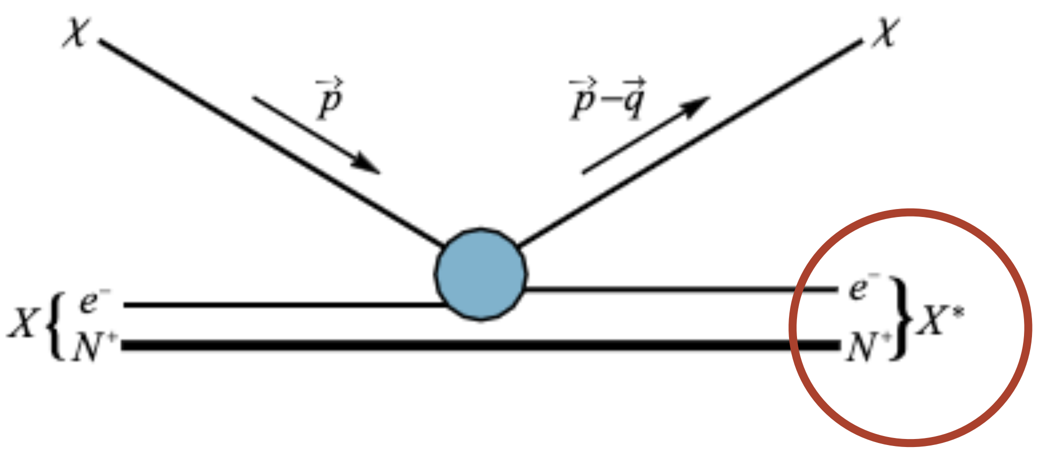
e^- are not localized

e^- are not alone

→ screening

See Tien-Tien's lectures

Crystals are complicated



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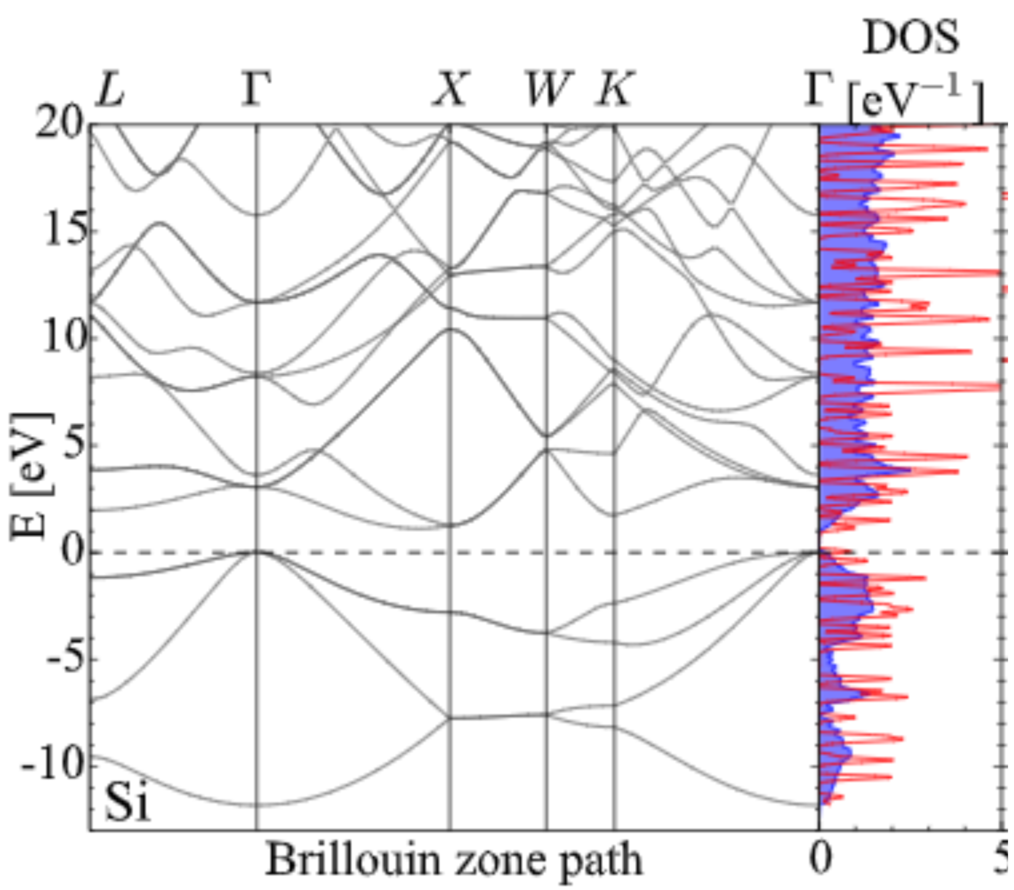
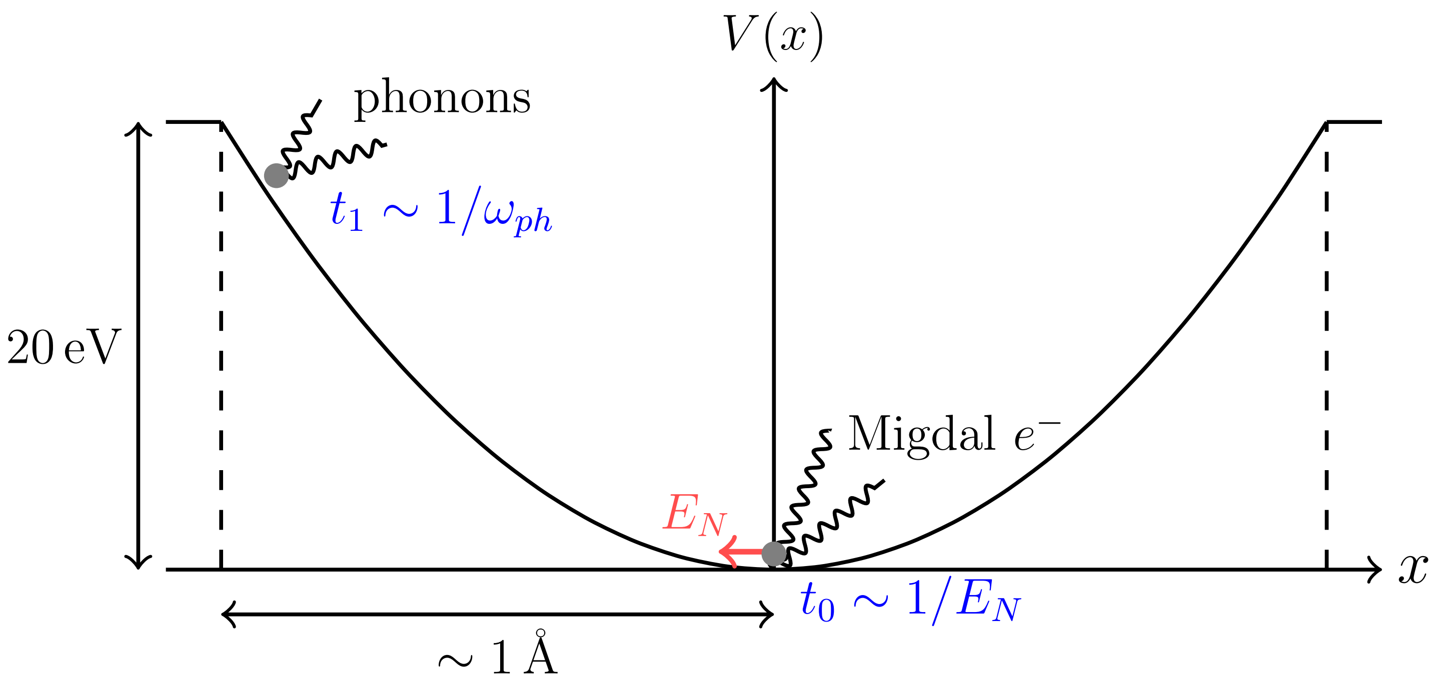


→ Bloch wave functions

Obtain with density functional theory (DFT)

See Tien-Tien's lectures

The impulse approximation in a crystal

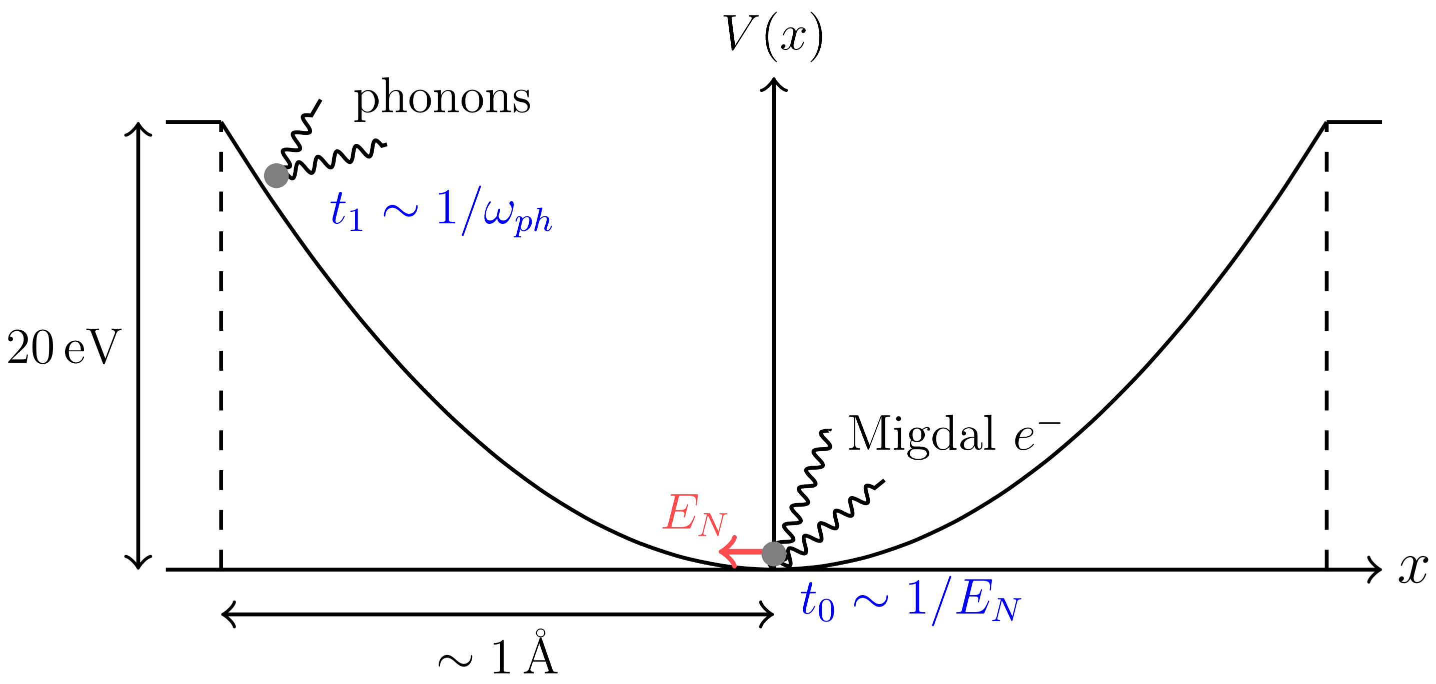


If the DM is heavy enough, most collisions take place at an energy well above the typical phonon energy (~ 30 meV)

If this is the case, the nucleus doesn't feel the crystal potential during the initial hard recoil

SK, J. Kozaczuk, T. Lin: arXiv 2011.09496

The impulse approximation in a crystal



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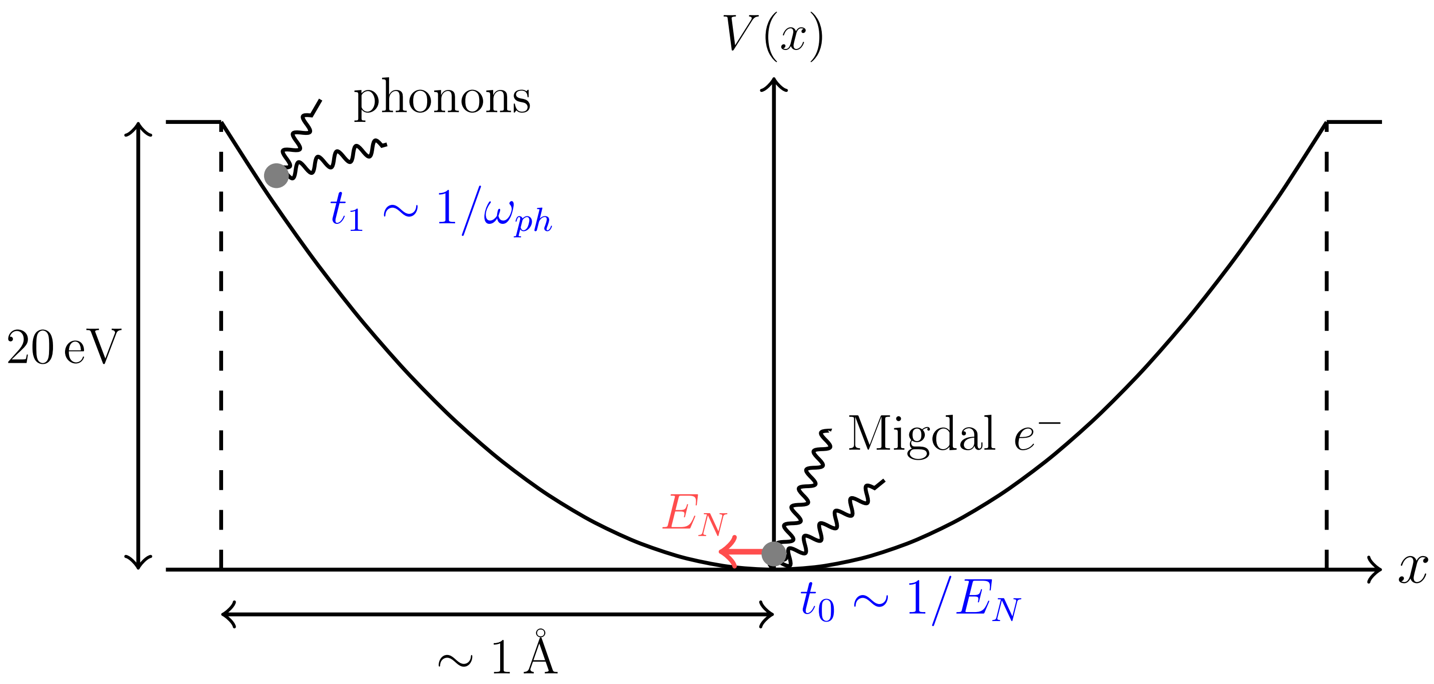

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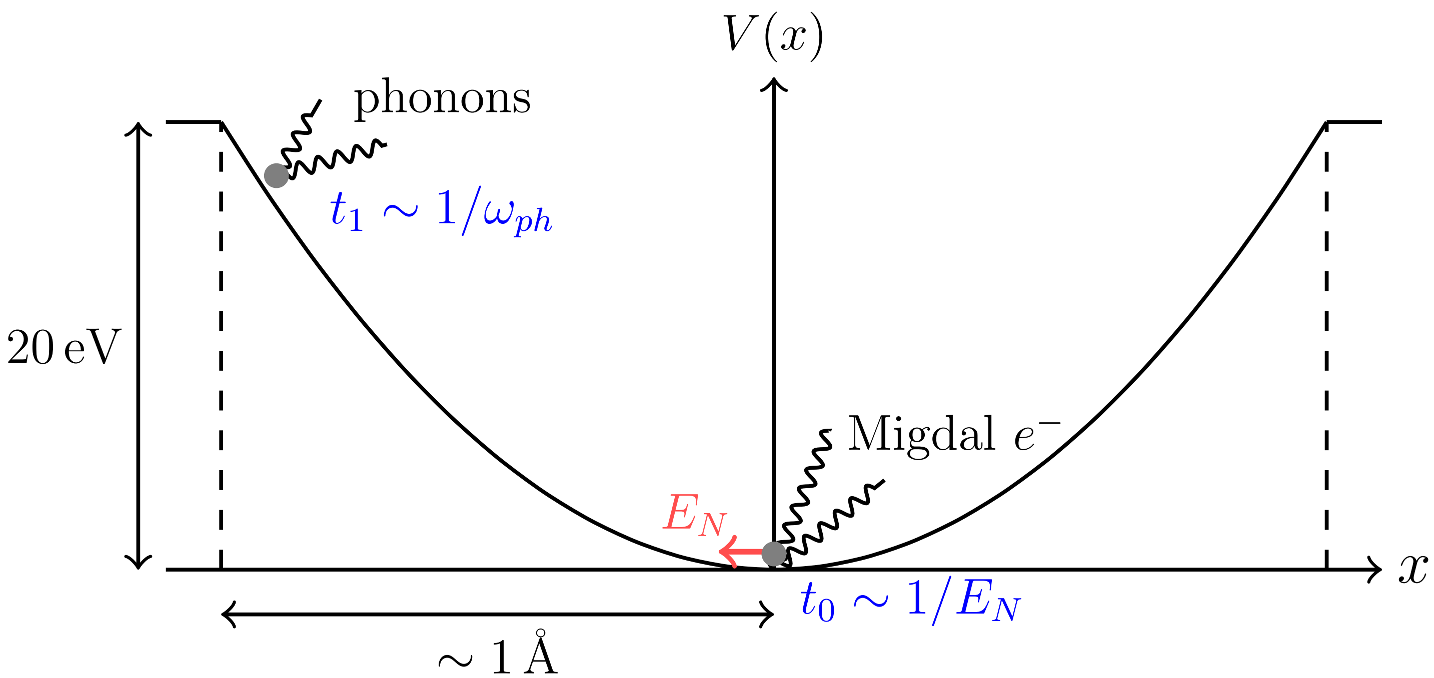

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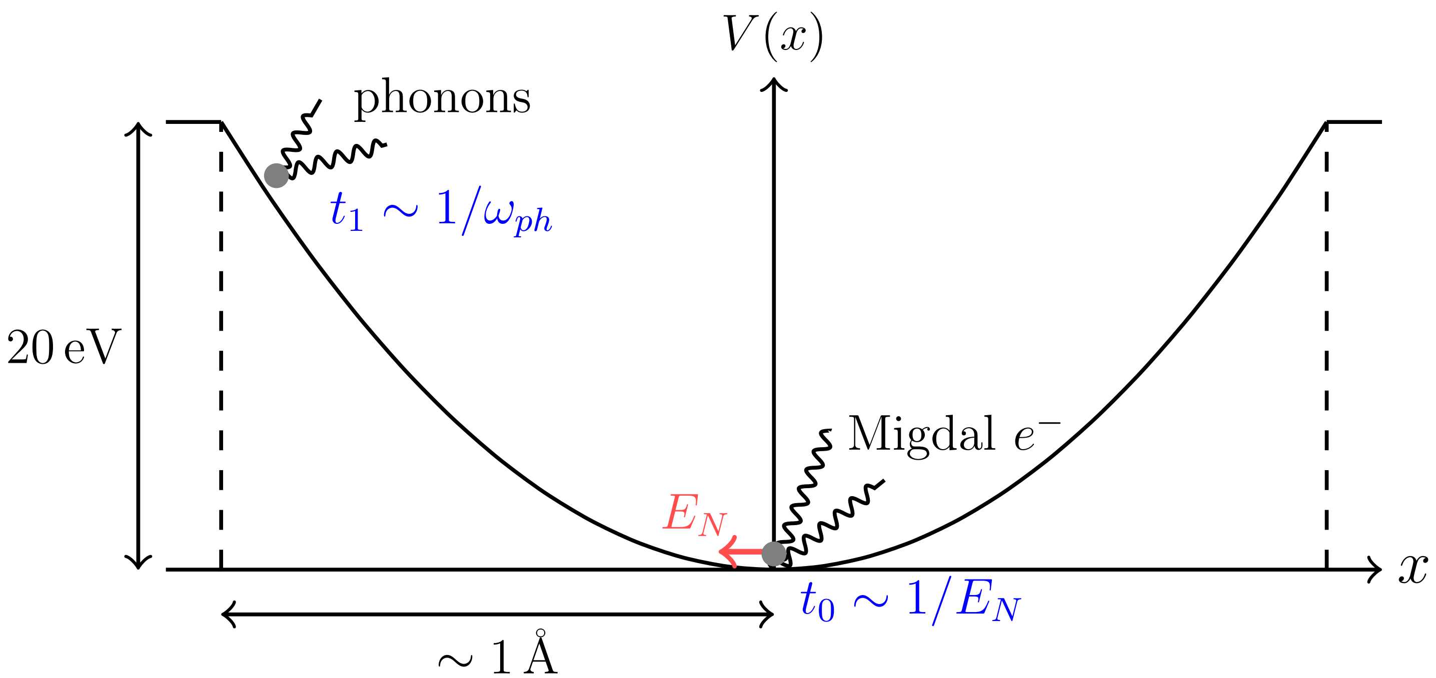
When it is valid we can factorize the long distance physics (phonons) from the short distance physics (Migdal effect).

SK, J. Kozaczuk, T. Lin: arXiv 2011.09496

Soft nuclear recoils



Nuclei are not free

Nuclei are roughly at rest

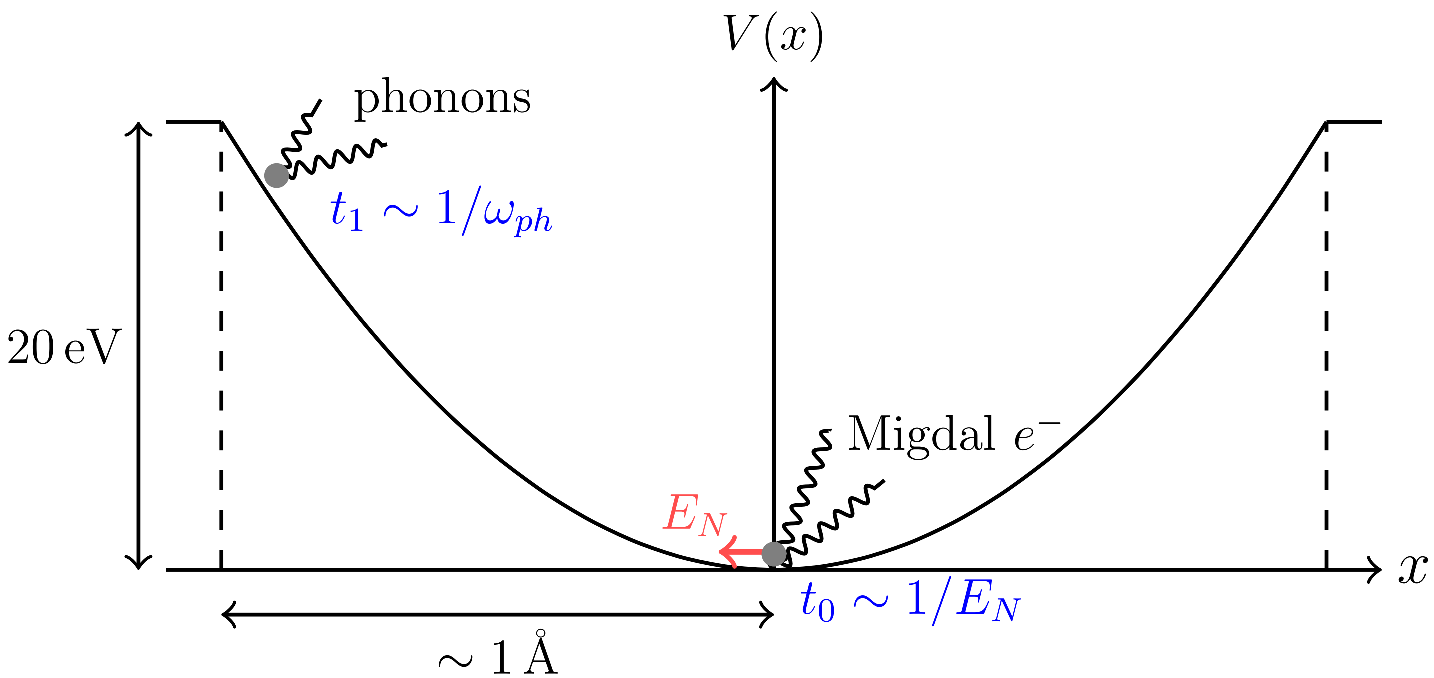
Nuclei are pretty localized

Lets first look at soft nuclear recoils without Migdal effect (bit of a preview for tomorrow)

A short-ranged interaction is described by a delta-function potential:

$$\mathcal{V}(\mathbf{r}) = V_0 \delta(\mathbf{r}_N - \mathbf{r}) \to \tilde{V}(\mathbf{q}) = \tilde{V}_0 e^{i\mathbf{q}\cdot\mathbf{r}_N}$$

Soft nuclear recoils



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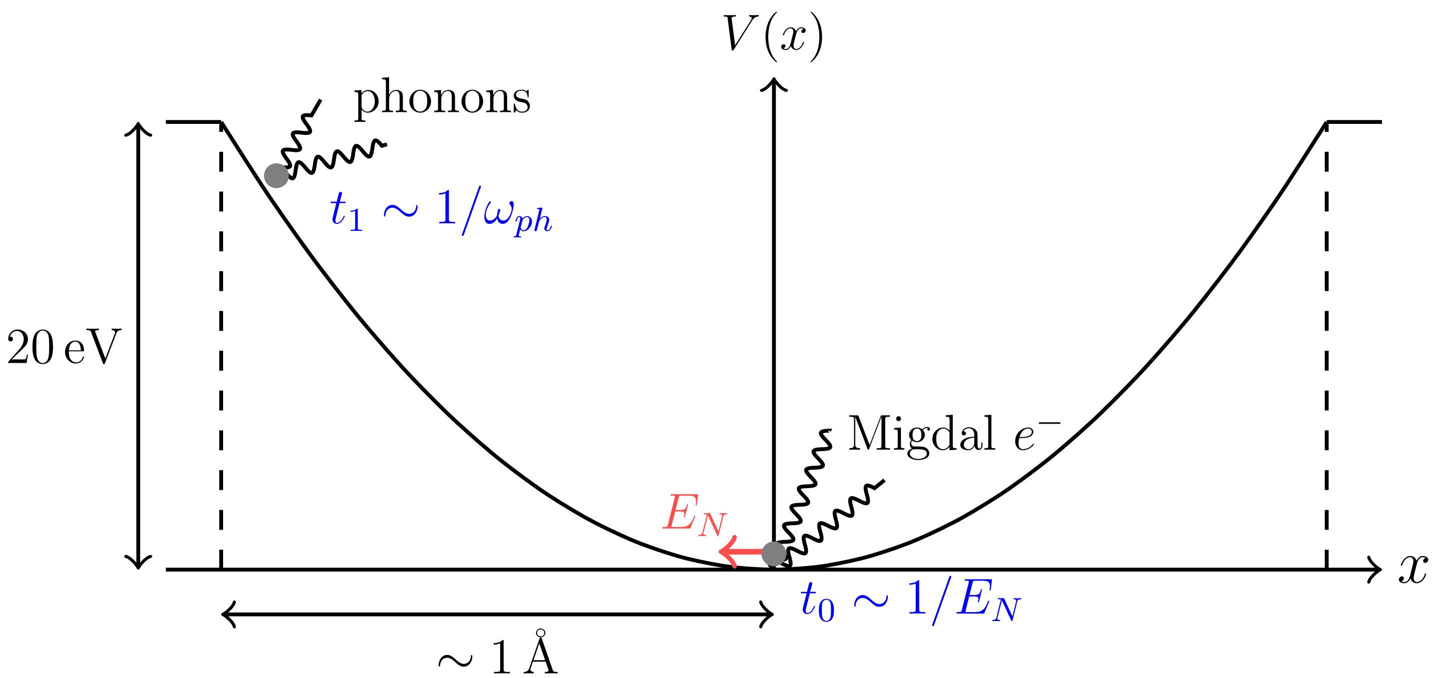
The scattering process is described by the "dynamical structure factor" or "response function"

$$S(\mathbf{q},\omega) \equiv \sum_{\lambda_f} \left| \langle \lambda_f | \, e^{-i\mathbf{q}\cdot\mathbf{r}_N} \, | \lambda_i \rangle \right|^2 \delta(E_{\lambda_f} - E_{\lambda_i} - \omega)$$

Initial and final states of the nucleus, sitting in its potential well

Soft nuclear recoils



$$
\begin{aligned}
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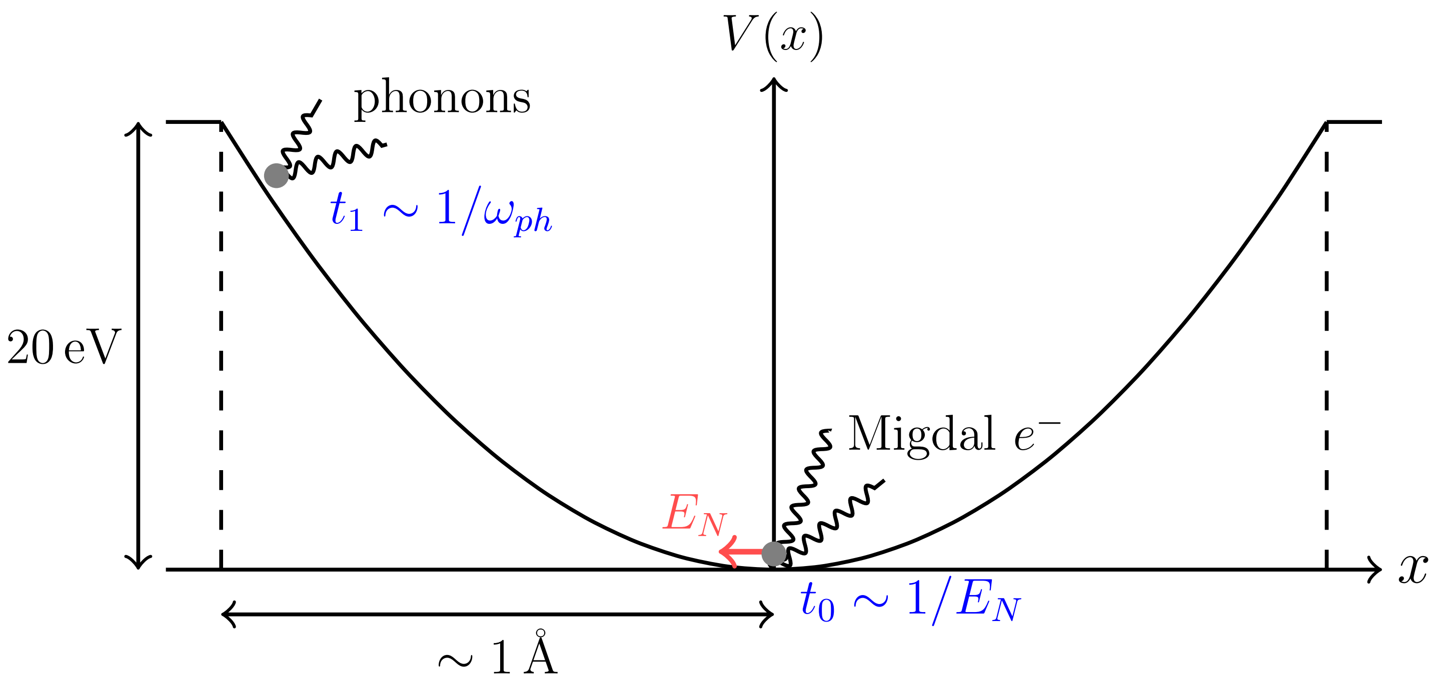
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \sum_{\lambda_f} \langle \lambda_i | \, e^{-i\mathbf{q}\cdot\mathbf{r}_N} \, | \lambda_f \rangle \langle \lambda_f | \, e^{iE_{\lambda_f} t} e^{i\mathbf{q}\cdot\mathbf{r}_N} e^{-iE_{\lambda_i} t} \, | \lambda_i \rangle \, e^{-i\omega t} \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \, \langle e^{-i\mathbf{q}\cdot\mathbf{r}_N(0)} e^{i\mathbf{q}\cdot\mathbf{r}_N(t)} \rangle e^{-i\omega t} \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \, e^{-2W(\mathbf{q})} e^{\langle \mathbf{q}\cdot\mathbf{r}_N(0) \, \mathbf{q}\cdot\mathbf{r}_N(t) \rangle} e^{-i\omega t}
\end{aligned}
$$

(This step is a page of algebra)

Appendix B of SK, J. Kozaczuk, T. Lin: arXiv 2011.09496

Soft nuclear recoils



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Debeye-Waller factor, measures how delocalized the nucleus is

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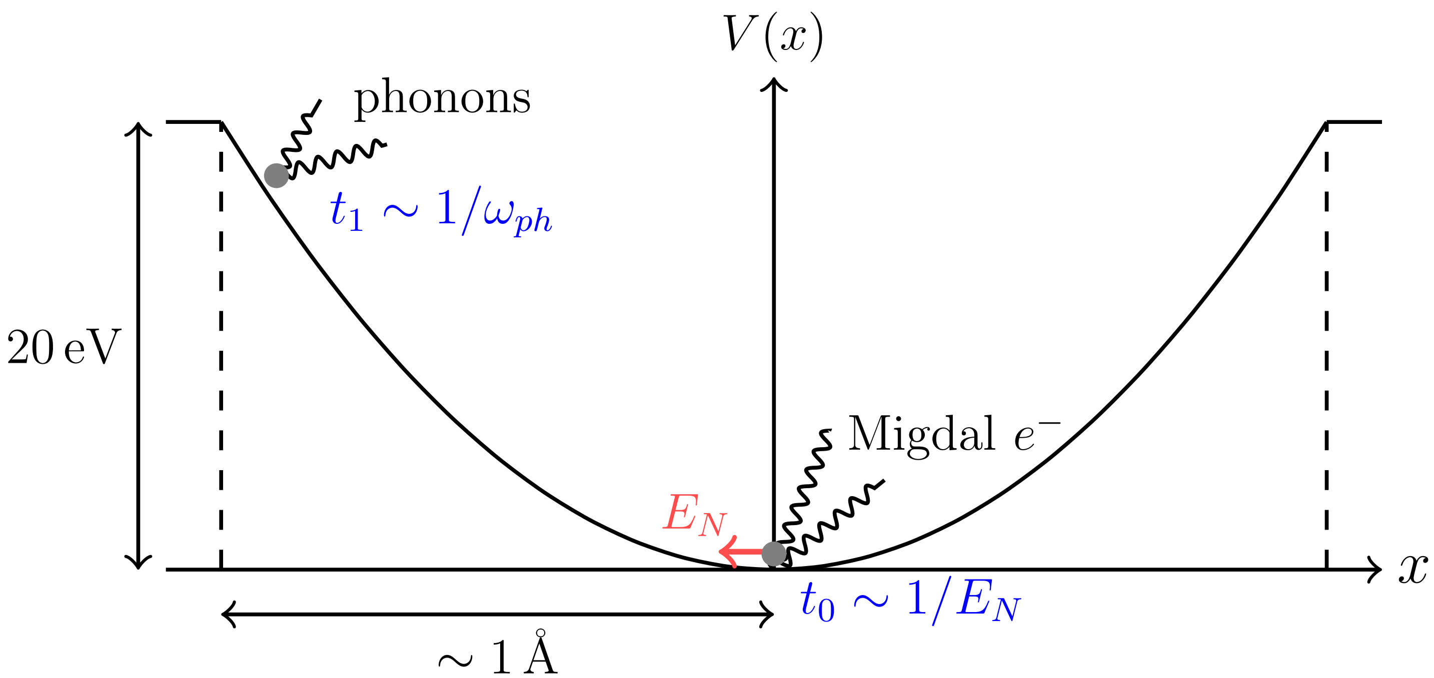
Soft nuclear recoils



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(This step is a page of algebra)

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In a harmonic potential

$$
\langle \mathbf{q}\cdot\mathbf{r}_N(0)\, \mathbf{q}\cdot\mathbf{r}_N(t)\rangle = \frac{q^2}{2m_N}\int d\omega' \frac{D(\omega')}{\omega'}\left[\cos(\omega' t)\coth\left(\frac{\omega'}{2T}\right) + i\sin(\omega' t)\right]
$$

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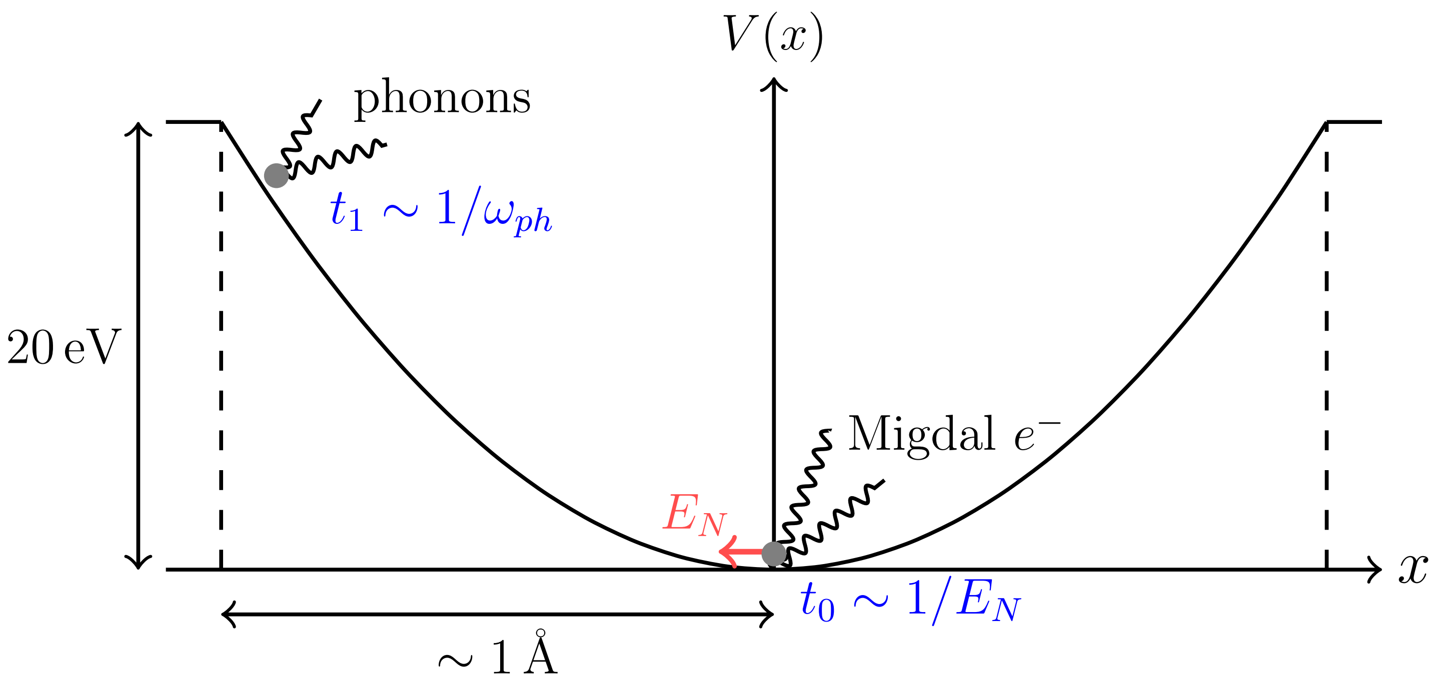
Soft nuclear recoils



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Density of states

In a harmonic potential

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Soft nuclear recoils

In the impulse approximation, the response function is gaussian

$$S^{IA}(\mathbf{q},\omega) = \frac{1}{\sqrt{2\pi\Delta^2}} e^{-\frac{\left(\omega - \frac{q^2}{2m_N}\right)^2}{2\Delta^2}} \qquad \text{with} \qquad \Delta^2 \equiv \frac{q^2\bar{\omega}}{2m_N}$$

Typical phonon frequency

Asymptotes to a δ-function for $q^2/2m_N \gg \bar{\omega}$ (Free limit)

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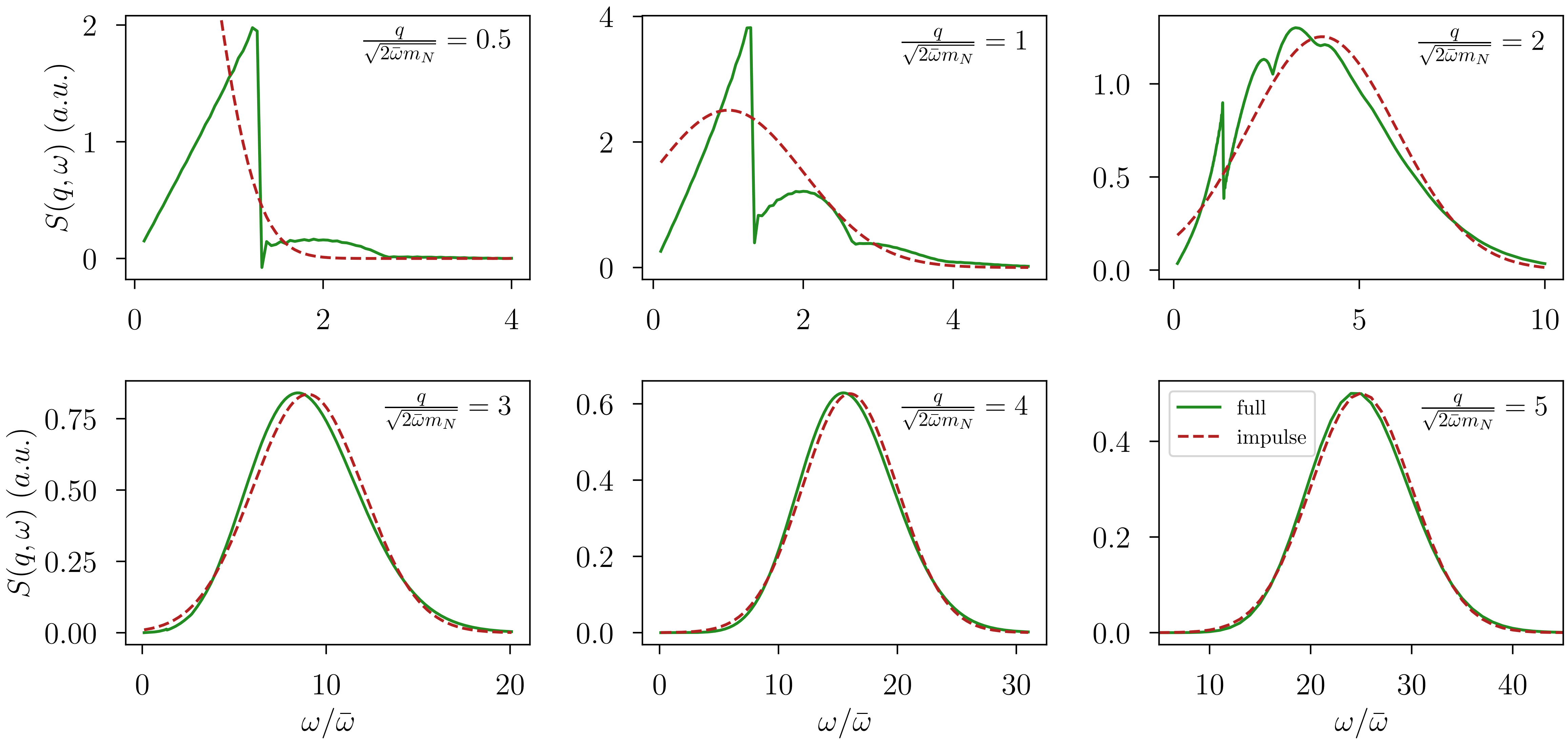

We can use the impulse approximation as long as $\frac{q^2}{2m_N\bar{\omega}} \gtrsim 4$

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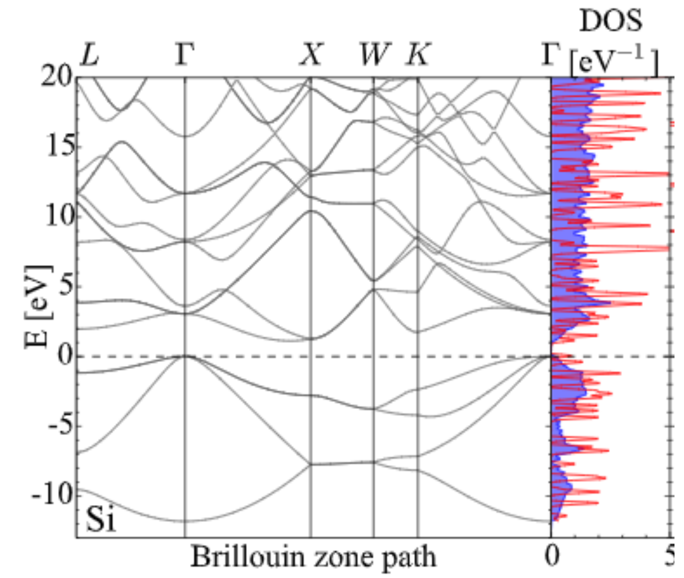
Back to electrons

A hard nuclear recoil can cause valence -> conduction band transition

Confusing to calculate in semi-conductors, since the electrons don't belong to any particular atom



From 1509.01598 (Essig et al.)

Remember we cannot boost because the crystal frame is a preferred frame

SK, J. Kozaczuk, T. Lin: arXiv 2011.09496
Liang et.al. : arXiv 2011.13352

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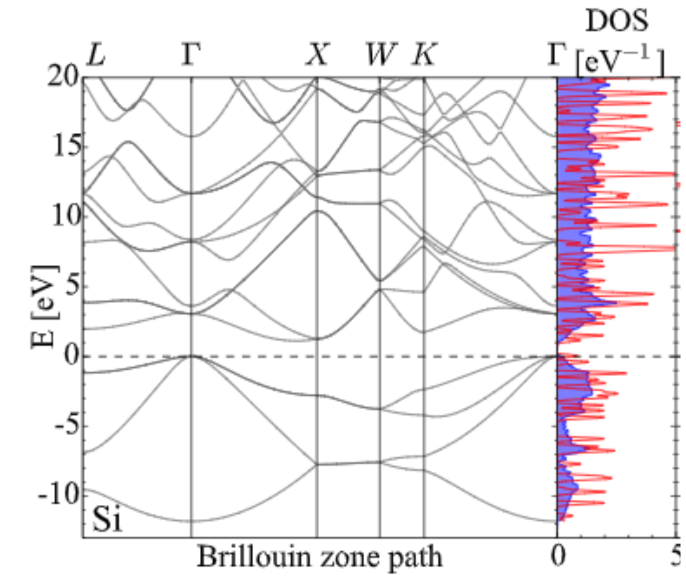

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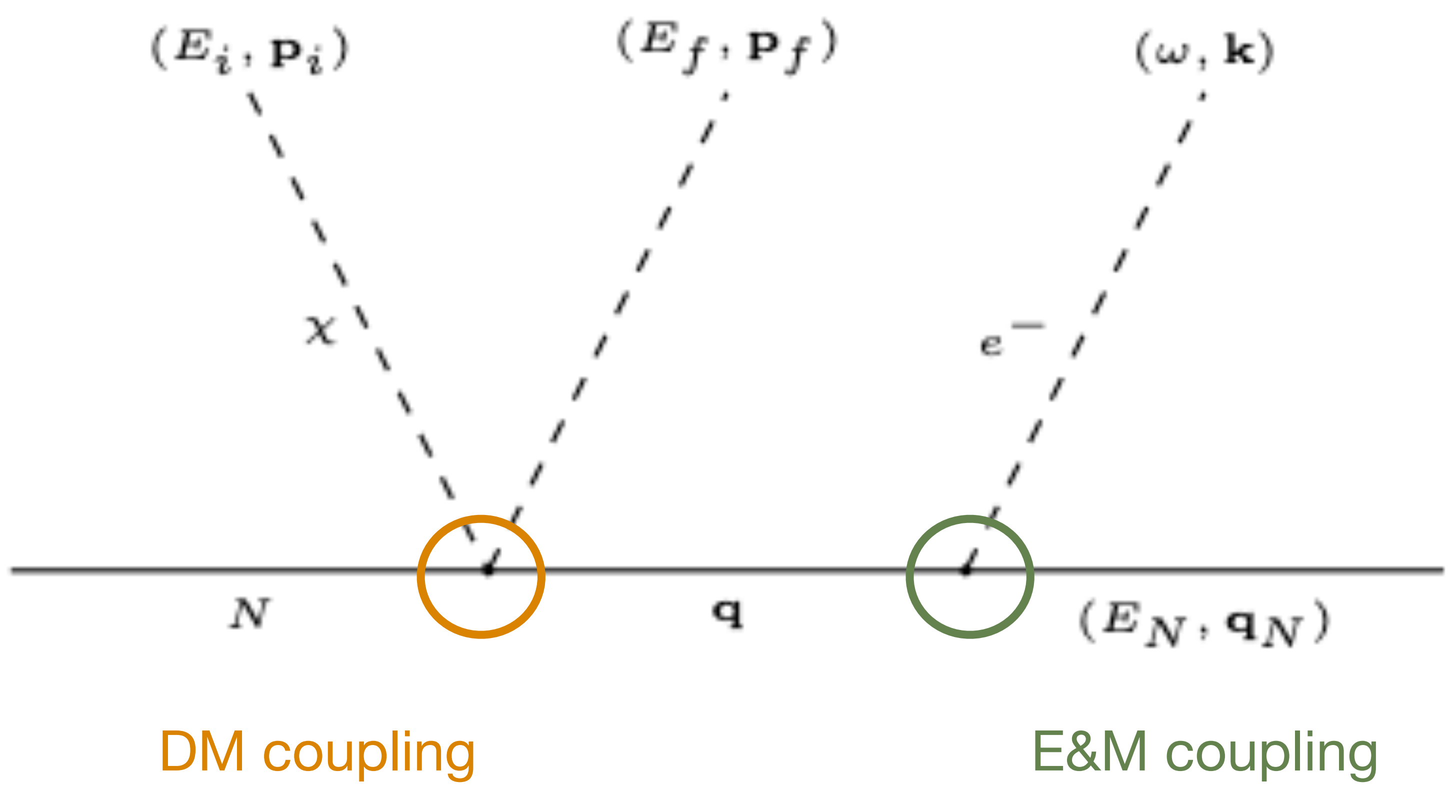
Leading order calculation in E&M force



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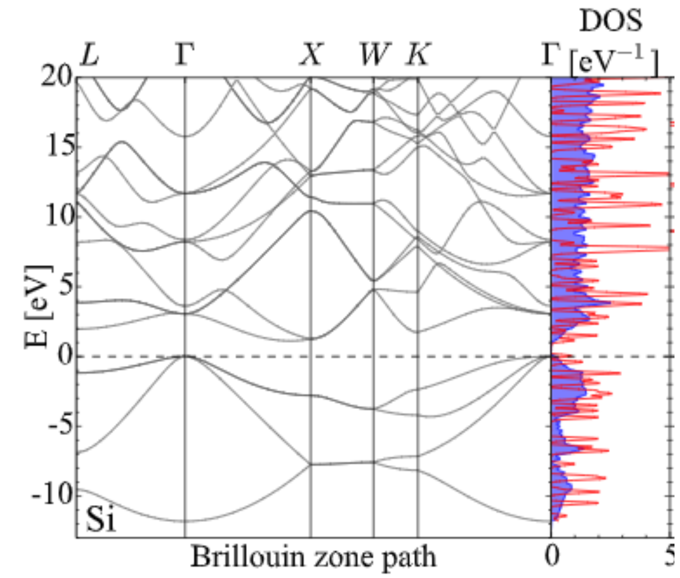

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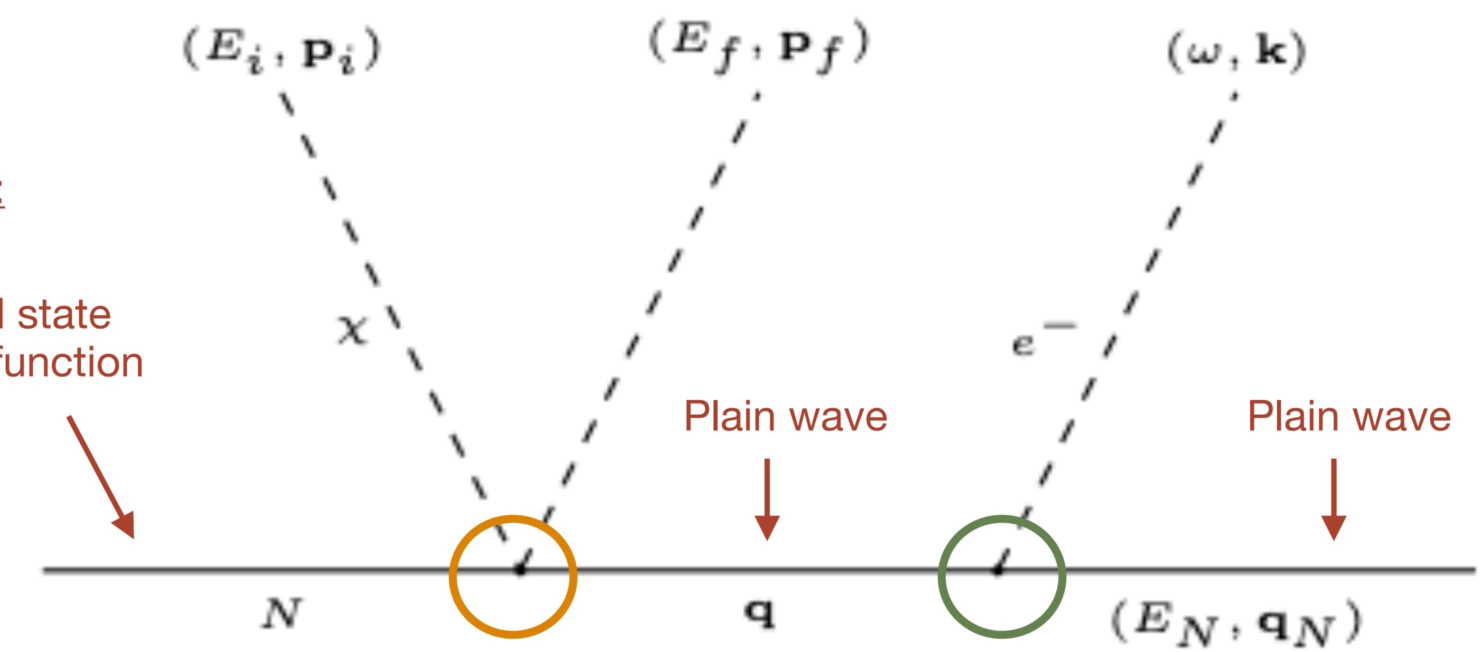
With impulse approximation:

Bound state
wave function



Plain wave

Plain wave

DM coupling

E&M coupling

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Back to electrons

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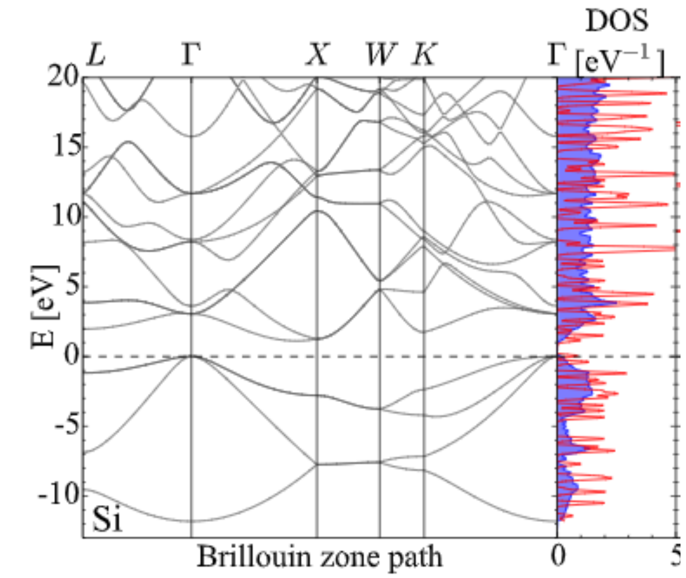

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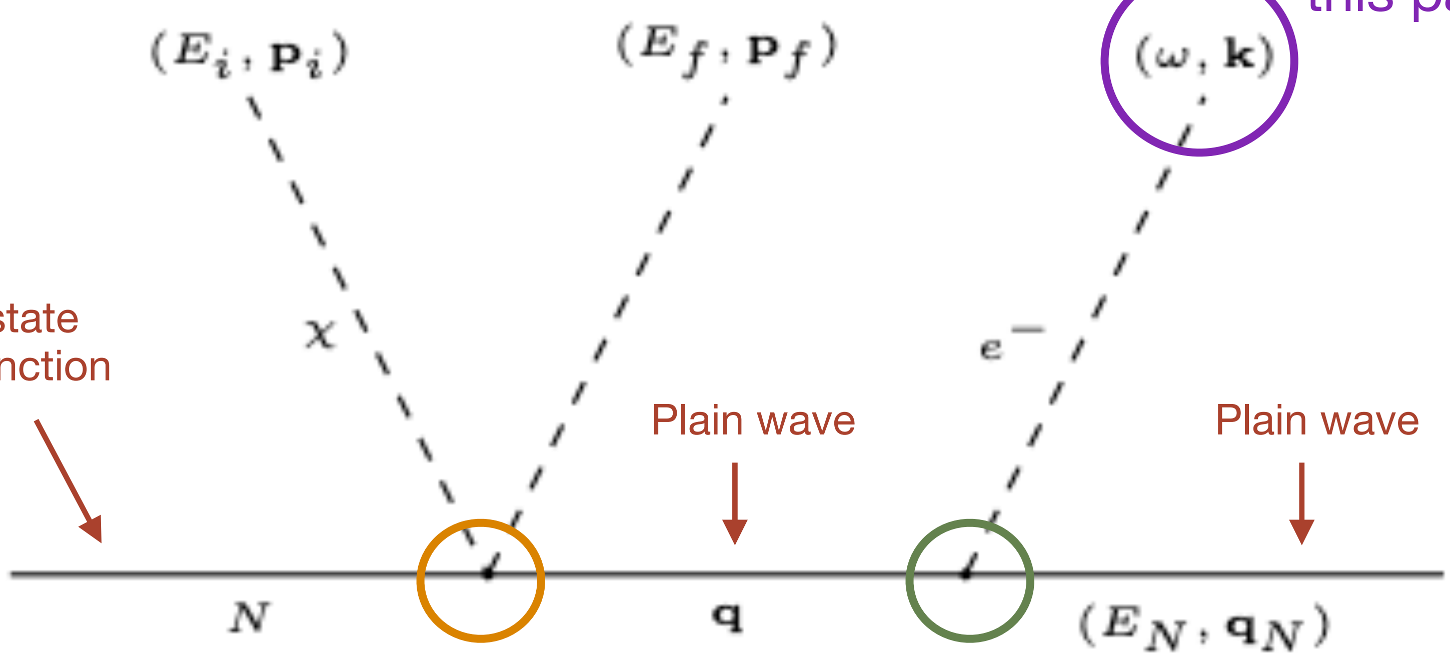
How to deal with this part?

With impulse approximation:

Bound state wave function

Plain wave

Plain wave



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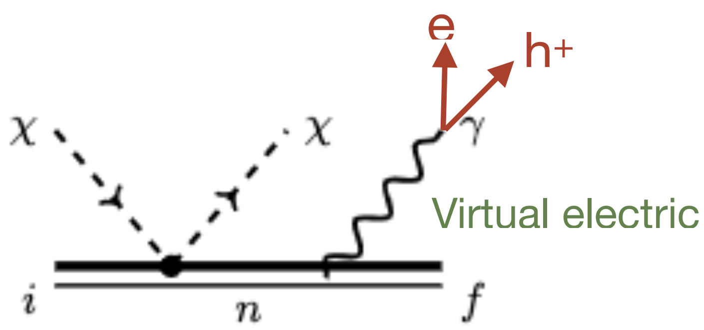
The energy loss function (ELF)

Recall that we are looking for a virtual photon splitting into an electron-hole pair



Coulomb potential in a dielectric:

$$H = eQ_\chi \int \frac{d^3\mathbf{k}}{(2\pi)^2} \frac{1}{\epsilon(\mathbf{k},\omega)} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2}$$

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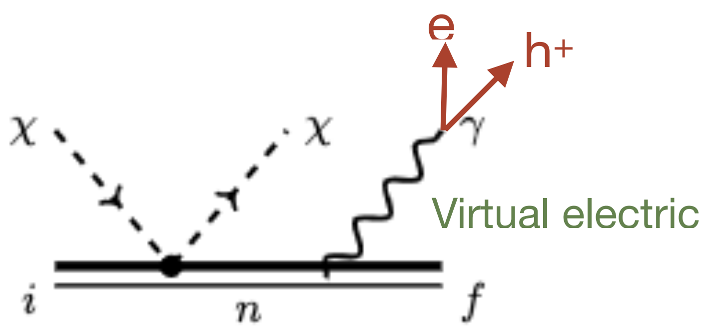

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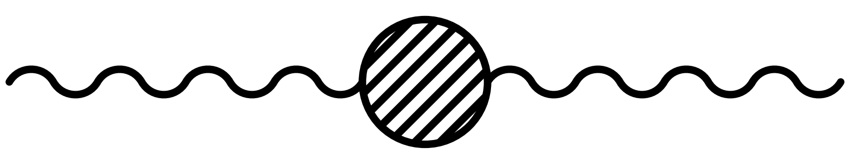
In QFT language:



$$\sim \frac{1}{\epsilon(\mathbf{k},\omega)} \frac{1}{k^2}$$

(Non-relativistic limit)

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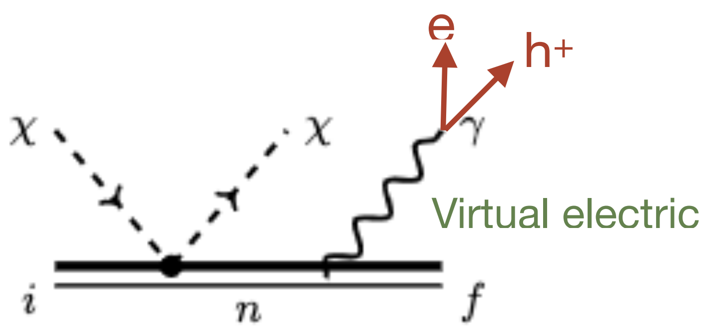

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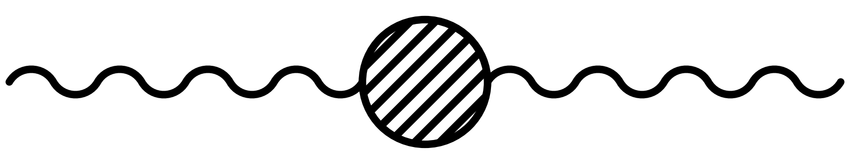

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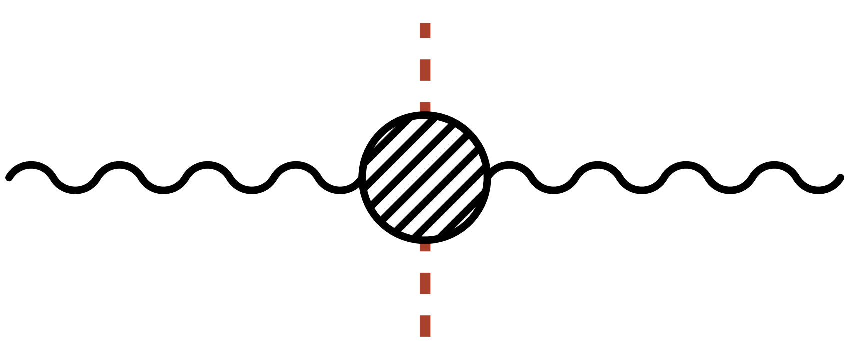

$$\sim \mathrm{Im}\left[\frac{-1}{\epsilon(\mathbf{k},\omega)}\right] = \frac{\mathrm{Im}\,\epsilon(\mathbf{k},\omega)}{|\epsilon(\mathbf{k},\omega)|^2}$$

"Energy Loss Function" (ELF)

The energy loss function (ELF)

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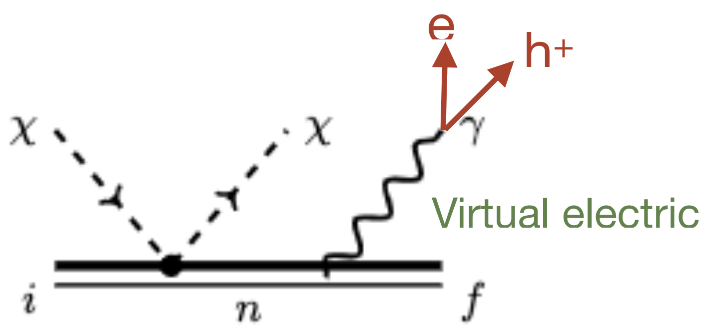

Coulomb potential in a dielectric:

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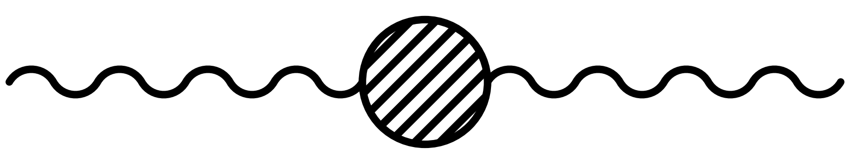

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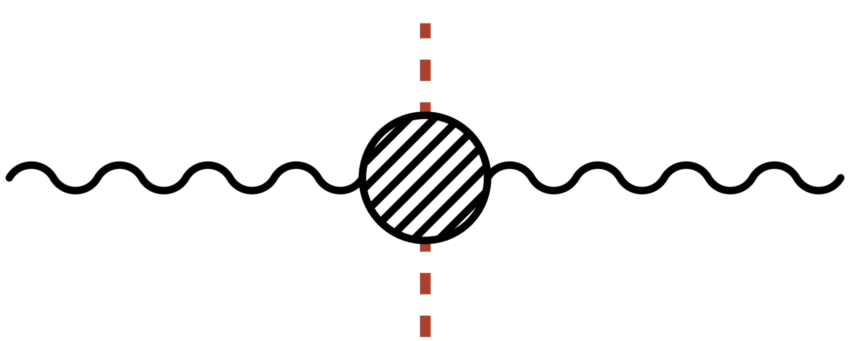


Tien-Tien's matrix element

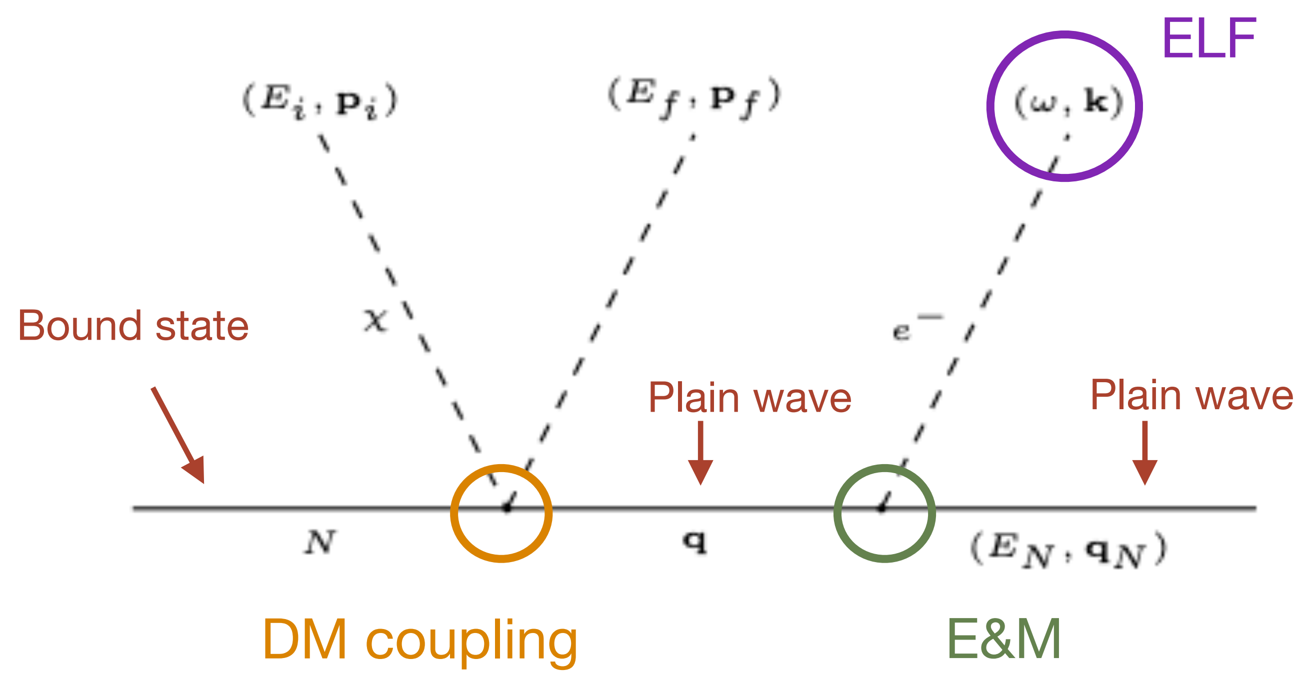
$$\sim \mathrm{Im}\left[\frac{-1}{\epsilon(\mathbf{k},\omega)}\right] = \frac{\mathrm{Im}\,\epsilon(\mathbf{k},\omega)}{|\epsilon(\mathbf{k},\omega)|^2}$$

Screening

"Energy Loss Function" (ELF)

Result

With impulse approximation:



Explicit calculation is a little tedious since we need Bloch functions etc, like you learned from Tien-Tien. The derivation is straightforward, but the formulas tend to be fairly long etc

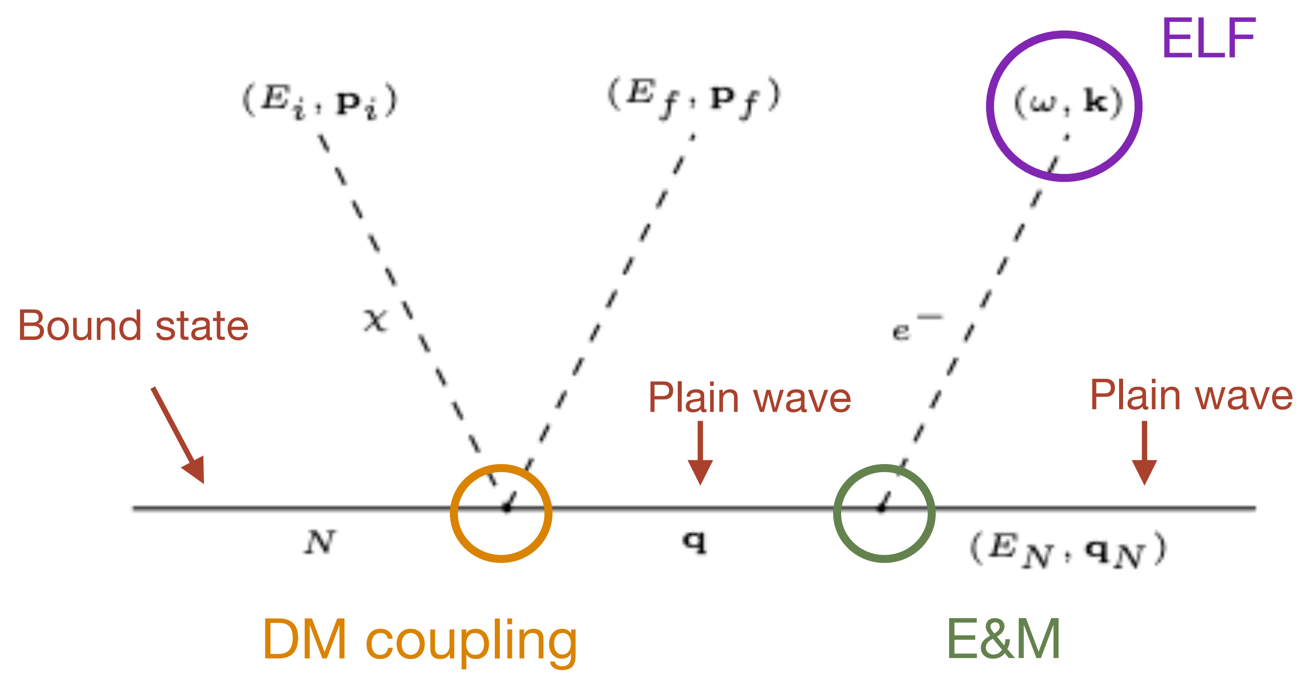
Result:

$$R = \frac{8\pi^2 Z_{\text{ion}}^2 \alpha A^2 \rho_\chi \bar{\sigma}_n}{m_N m_\chi \mu_{\chi n}^2} \int d^3v f_\chi(v) \int d\omega \int \frac{d^3\mathbf{q}_N}{(2\pi)^3} \int \frac{d^3\mathbf{p}_f}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{k^2} \mathrm{Im}\left[\frac{-1}{\epsilon(\mathbf{k},\omega)}\right] \left[\frac{1}{\omega - \frac{\mathbf{q}_N \cdot \mathbf{k}}{m_N}} - \frac{1}{\omega}\right]^2$$
$$\times |F_{DM}(\mathbf{p}_i - \mathbf{p}_f)|^2 |F(\mathbf{p}_i - \mathbf{p}_f - \mathbf{q}_N - \mathbf{k})|^2 \, \delta\left(E_i - E_f - E_N - \omega\right).$$

Full derivation in appendix A of SK, J. Kozaczuk, T. Lin: arXiv 2011.09496

Result

<u>With impulse approximation:</u>



Explicit calculation is a little tedious since we need Bloch functions etc, like you learned from Tien-Tien. The derivation is straightforward, but the formulas tend to be fairly long etc

<u>Result:</u>

ELF   Nucleus propagator

$$R = \frac{8\pi^2 Z_{\text{ion}}^2 \alpha A^2 \rho_\chi \bar{\sigma}_n}{m_N m_\chi \mu_{\chi n}^2} \int d^3v f_\chi(v) \int d\omega \int \frac{d^3\mathbf{q}_N}{(2\pi)^3} \int \frac{d^3\mathbf{p}_f}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{k^2} \boxed{\text{Im}\left[\frac{-1}{\epsilon(\mathbf{k},\omega)}\right]} \boxed{\left[\frac{1}{\omega - \frac{\mathbf{q}_N \cdot \mathbf{k}}{m_N}} - \frac{1}{\omega}\right]^2}$$
$$\times \boxed{|F_{DM}(\mathbf{p}_i - \mathbf{p}_f)|^2} \boxed{|F(\mathbf{p}_i - \mathbf{p}_f - \mathbf{q}_N - \mathbf{k})|^2} \delta\left(E_i - E_f - E_N - \omega\right).$$

DM form factor   Crystal form factor

Full derivation in appendix A of SK, J. Kozaczuk, T. Lin: arXiv 2011.09496

In the soft limit

Analogous to the brehmstrallung case. Valid for k << v m_X:

$$\frac{d\sigma_{\rm ion}}{dE_N d\omega} \approx \frac{d\sigma_{\rm el}}{dE_N}\frac{dP}{d\omega}$$

$$\frac{dP}{d\omega} = 4\alpha \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{Z_{\rm ion}^2(k)}{k^2}\frac{|\mathbf{v}_N \cdot \mathbf{k}|^2}{\omega^4}\mathrm{Im}\left[\frac{-1}{\epsilon(\mathbf{k},\omega)}\right].$$

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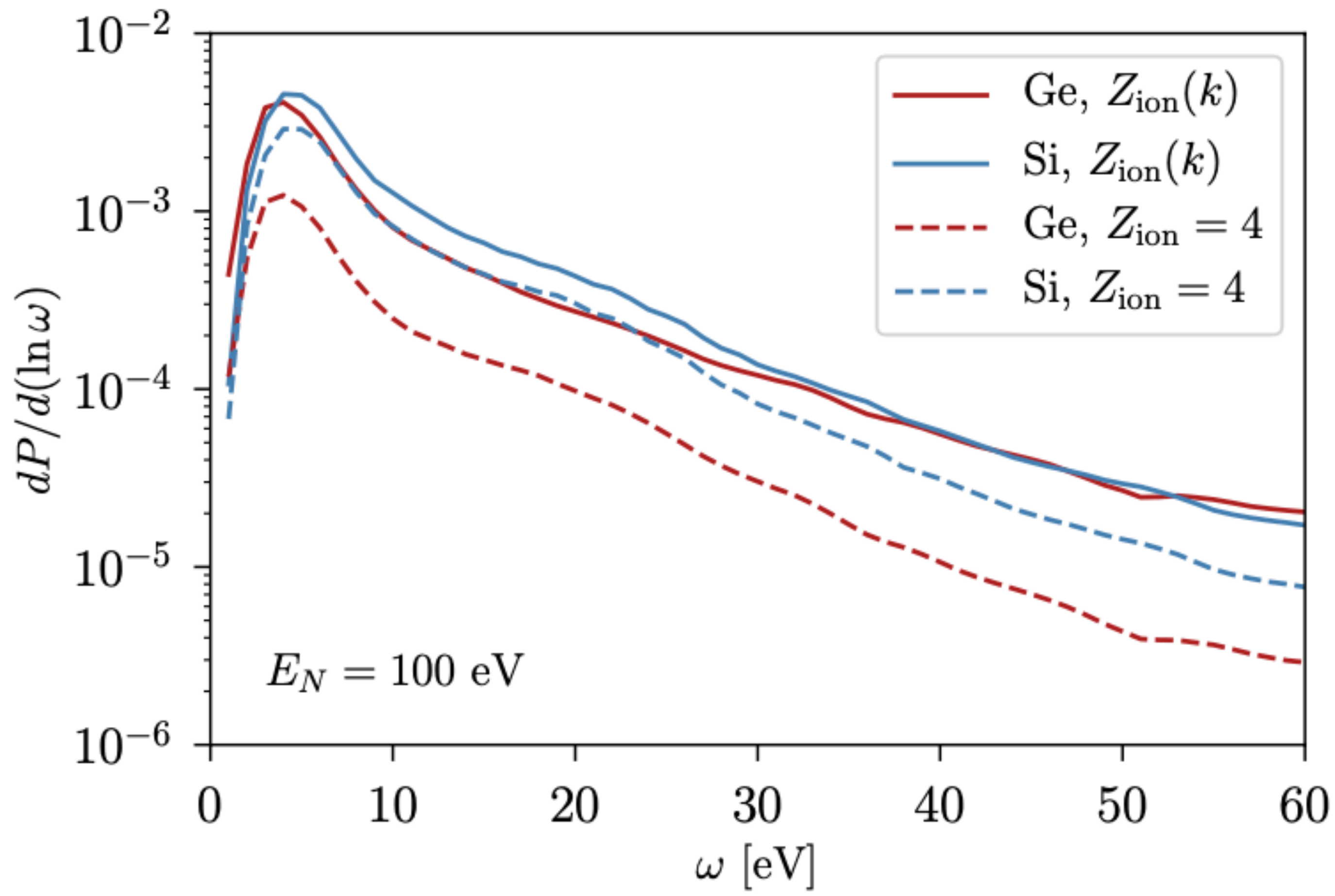
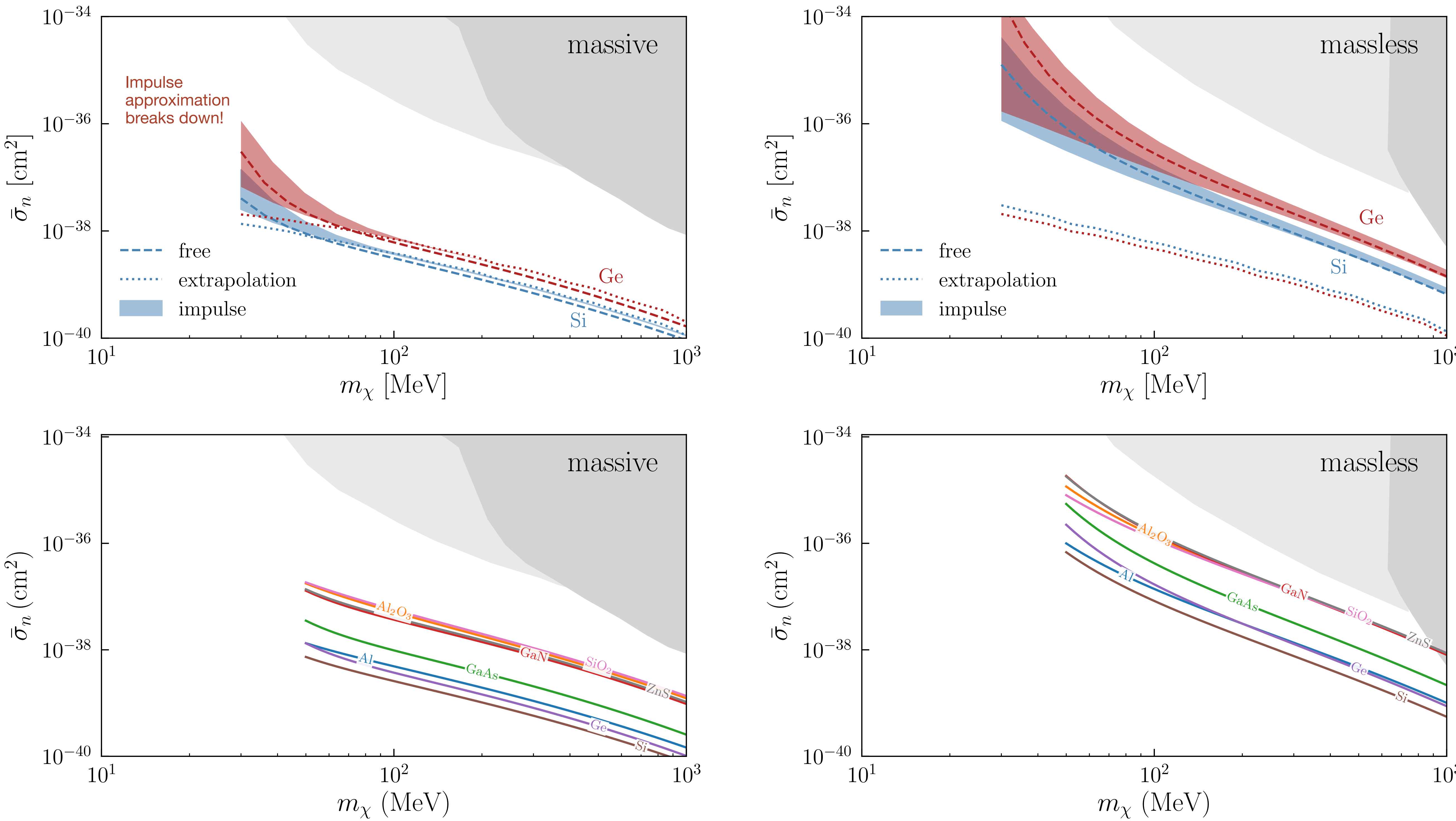


As for brehmstrallung the momentum dependence of the effective charge is quite important.

Because the probability is weighted towards fairly high k, screening isn't as effective

Migdal effect results



We believe the electronic response is on solid ground

Nuclear recoil (impulse approximation) is main source of uncertainty main 1 branch 0 tags Go to file Add file Code

Simon Knapen Merge branch 'main' of github.com:tongylin/DarkELF into main 18a4517 17 days ago 15 commits

darkelf	fixed loading error in Migdal module	17 days ago
data	initial commit	last month
examples	removed checkpoint files	last month
README.md	Update README.md	23 days ago

README.md

DarkELF

DarkELF is a python package capable of calculating interaction rates of light dark matter in dielectric materials, including screening effects. The full response of the material is parametrized in the terms of the energy loss function (ELF) of material, which DarkELF converts into differential scattering rates for both direct dark matter electron scattering and through the Migdal effect. In addition, DarkELF can calculate the rate to produce phonons from sub-MeV dark matter scattering via the dark photon mediator, as well as the absorption rate for dark matter comprised of dark photons. The package currently includes precomputed ELFs for Al,Al2O3, GaAs, GaN, Ge, Si, SiO2, and ZnS, and allows the user to easily add their own ELF extractions for arbitrary materials.

See arXiv 2104.12786 for a description of the implementation

Authors

Simon Knapen, Jonathan Kozaczuk and Tongyan Lin

Physics

ELF

Currently DarkELF contains ELF look-up tables obtained with the GPAW density functional theory code for Si and Ge, as well as data-driven Mermin model for the remaining materials. The Lindhard ELF is also included. DarkELF also comes with a number of measured ELFs in the optical limit for energy depositions below the electronic band gap, which is relevant for phonon processes. Additional materials and ELF computations may be added at a later date. When using a particular ELF computation, please refer to the relevant experimental papers and/or GPAW package. These references can be found in arXiv 2104.12786.

About

Calculating dark matter scattering and absorption rates with the energy loss functions (ELF)

Readme

Releases

No releases published
Create a new release

Packages

No packages published
Publish your first package

Contributors 2

simonknapen86
tongylin Tongyan Lin

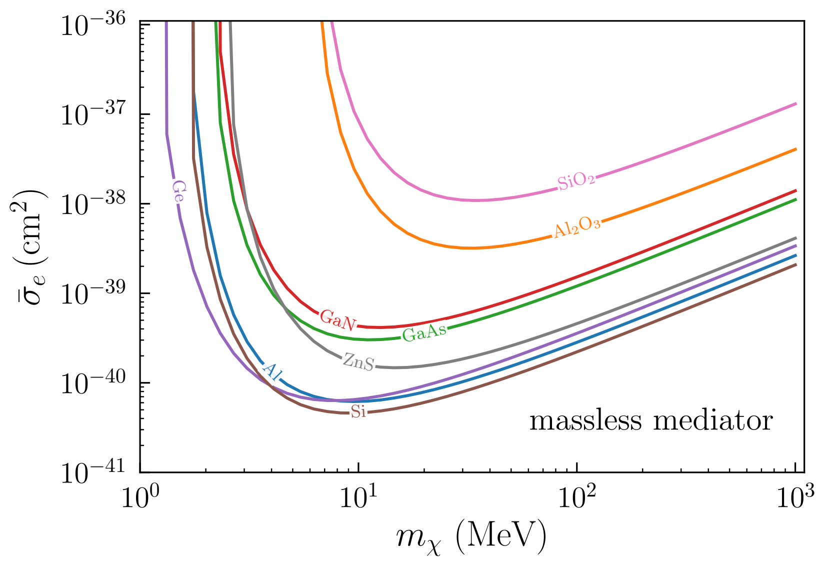
Languages

Python 100.0%

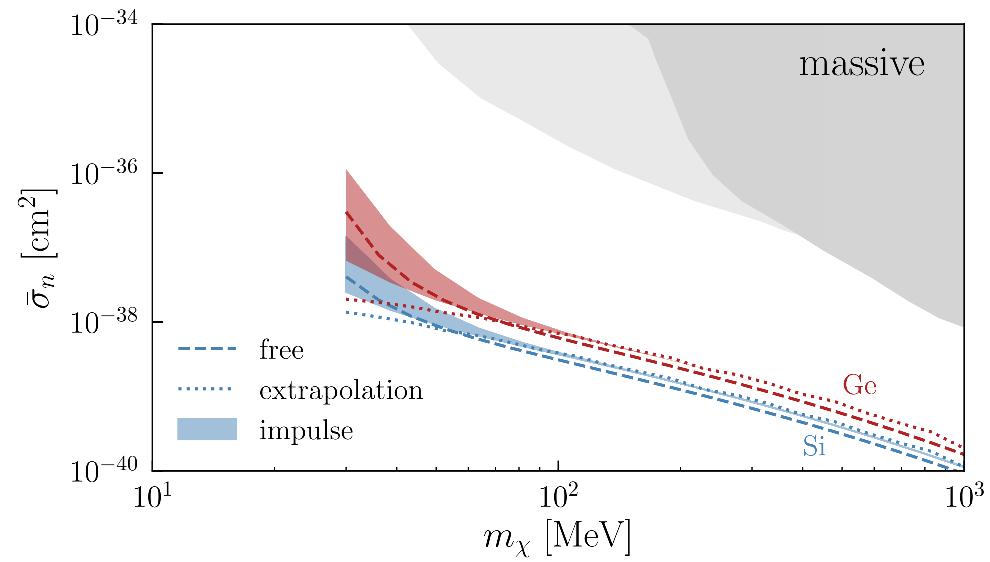
https://github.com/tongylin/DarkELF

DarkELF functions

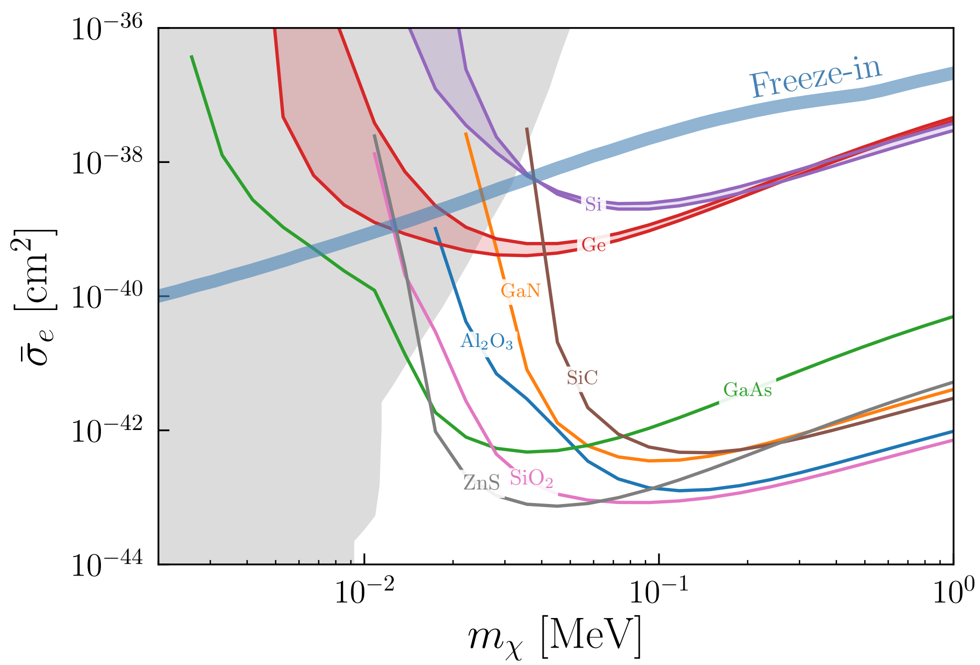
DM - electron scattering



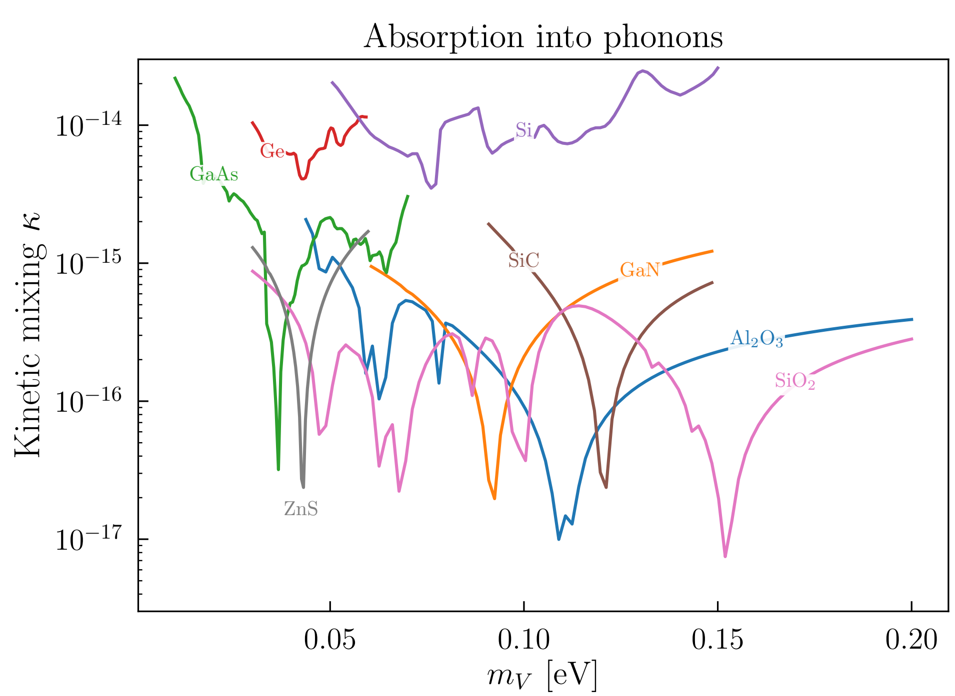
Migdal effect



DM - phonon scattering



Dark photon absorption



For tomorrow’s discussion session

If you don’t have them already, please install:

- python3 with numpy & scipy
- Jupyter

In my experience this is most straightforward by installing a full scientific python environment such as Anaconda

https://www.anaconda.com/

Once you have this, download and unpack the darkELF package.

https://github.com/tongylin/DarkELF

No compilation is needed. Try to run one of the example notebooks.

Summary

- Inelastic processes such as brehmstrallung and the Migdal effect give experiments access to DM candidates whose elastic recoils are below threshold
- This comes a price in scattering rate…
- …and a bit of pain/fun for the theorists
- The Migdal trick works for atomic targets, for crystals a direct calculation is needed
- For low DM mass, the impulse approximation breaks for both for nobel liquids and crystals. In the regime the correct answer is not yet known.

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Questions?