Summations of large logarithms by parton showers

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Perturbative cross sections

The main focus of this workshop is to calculate the pQCD cross sections as precise as possible, thus we have a pretty integral

$$\sigma[O_J] = \sum_m \frac{1}{m!} \sum_{\{a,b,f_1,\dots,f_m\}} \int_0^1 d\eta_{\rm a} \overbrace{\int_{\eta_{\rm a}}^1 \frac{dz}{z}\, \Gamma^{-1}_{aa'}(z,\mu^2)\, f_{a'/A}(\eta_{\rm a}/z,\mu^2)}^{\text{Bare PDF}}$$

$$\times \int_0^1 d\eta_{\rm b} \int_{\eta_{\rm b}}^1 \frac{d\bar z}{\bar z}\, \Gamma^{-1}_{bb'}(\bar z,\mu^2)\, f_{b'/A}(\eta_{\rm b}/\bar z,\mu^2)$$

$$\times \int d\phi(\eta_{\rm a}\eta_{\rm b} s, \{p,f\}_m)\, \langle M(\{p,f\}_m) | \underbrace{O_J(\{p,f\}_m)}_{\text{IR safe measurement operator}} \overbrace{| M(\{p,f\}_m)\rangle}^{\text{Partonic matrix element}}$$

$$+\, \mathcal{O}\left(\frac{\Lambda^2_{QCD}}{\mu_J^2}\right)$$

Error of the factorization

(Cannot be beaten by calculating higher and higher order.)

and here the MSbar parton in parton renormalised PDF is

$$\Gamma_{aa'}(z,\mu^2) = \delta(1-z)\delta_{aa'} - \frac{\alpha_{\rm s}(\mu^2)}{2\pi}\frac{1}{\epsilon}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} P_{aa'}(z) + \cdots \quad .$$

Statistical space

Introducing the statistical space we can represent the QCD density operator as a vector

$$\sigma[O_J] = \underbrace{(1|}_{\text{All the initial and final state sums and integrals}} \mathcal{O}_J \overbrace{\left[\mathcal{F}(\mu^2) \circ \mathcal{Z}_F(\mu^2)\right]}^{\text{Bare PDFs for both incoming hadrons}} \underbrace{|\rho(\mu^2))}_{|M\rangle\langle M|}$$

Bare PDFs for both incoming hadrons

All the initial and final state sums and integrals

QCD density operator
Describes the fully exclusive partonic final states.

The physical cross section is RG invariant as well as the QCD density operator and the bare PDF.

$$\mu^2 \frac{d}{d\mu^2} |\rho(\mu^2)) = \mu^2 \frac{d}{d\mu^2} \left[\mathcal{F}(\mu^2) \circ \mathcal{Z}_F(\mu^2)\right] = 0 + \mathcal{O}(\alpha_s^{k+1})$$

Perturbative expansion of the density operator

$$|\rho(\mu^2)) = \sum_{n=0}^{k} \left[\frac{\alpha_s(\mu^2)}{2\pi}\right]^n \sum_{\substack{n_R=0 \\ n_R+n_V=n}}^{n} \sum_{n_V=0}^{n} |\rho^{(n_R,n_V)}(\mu^2))$$

Number of real radiations

Number of loops

Statistical space

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A vector in the statistical space can be translated as

$$\big(\{p,f,c,s,c',s'\}_m\big|\rho\big) \Longleftrightarrow \big\langle\{c,s\}_m\big|M(\{p,f\}_m)\big\rangle\big\langle M(\{p,f\}_m)\big|\{c',s'\}_m\big\rangle$$

An operator in the statistical space corresponds to a direct products of the corresponding quantum operators:

$$\mathcal{A}(\mu^2) \Longleftrightarrow A^L(\mu^2)\otimes A^R(\mu^2)^\dagger$$

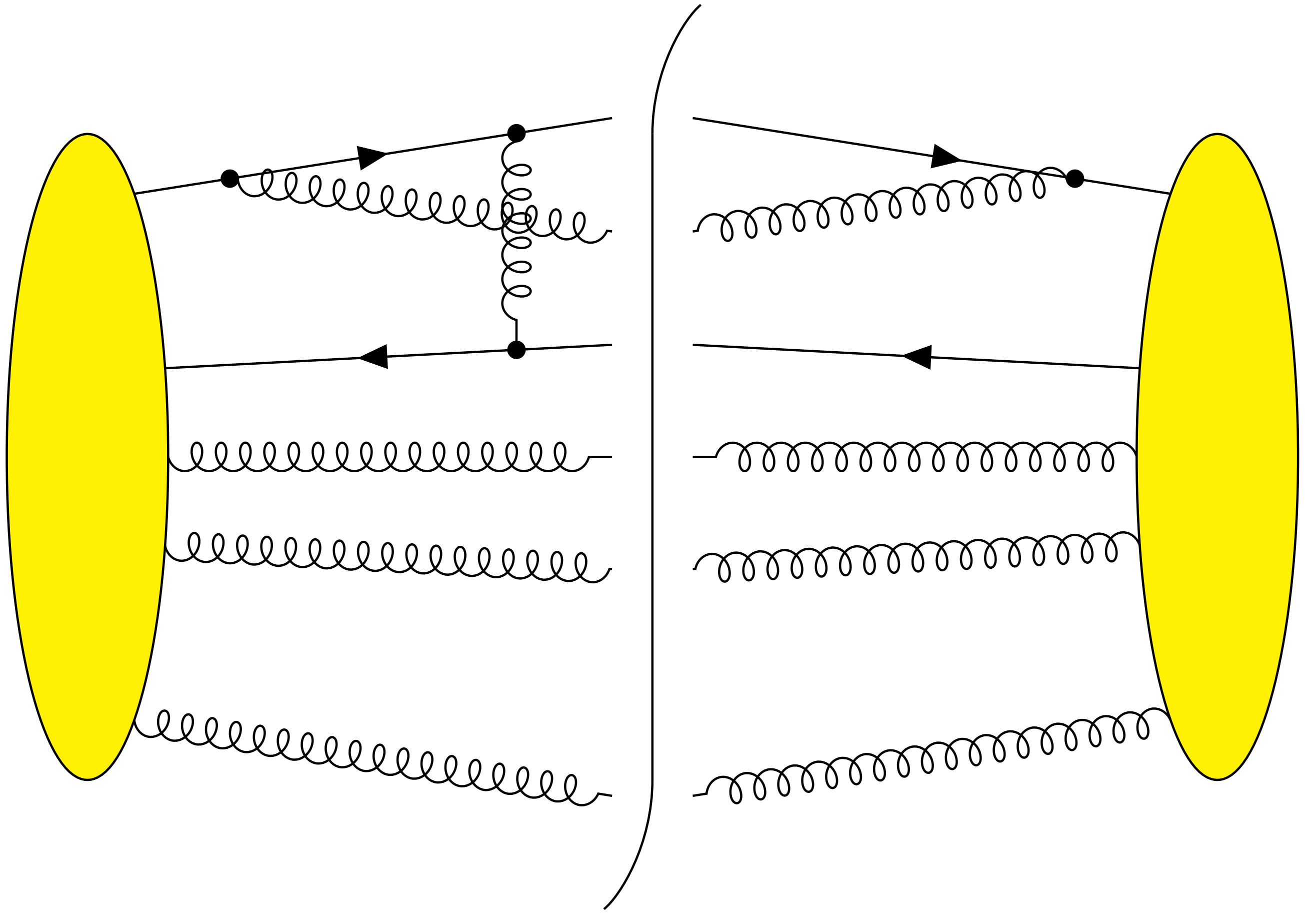
When operators act on a state we have

$$\cdots\mathcal{A}_3(\mu_3^2)\mathcal{A}_2(\mu_2^2)\mathcal{A}_1(\mu_1^2)\big|\rho\big) \Longleftrightarrow \cdots A_3^L(\mu_3^2)A_2^L(\mu_2^2)A_1^L(\mu_1^2)\big|M\big\rangle\big\langle M\big|A_1^R(\mu_1^2)^\dagger A_2^R(\mu_2^2)^\dagger A_3^R(\mu_3^2)^\dagger\cdots$$

Fixed order cross sections

Infrared sensitive operator

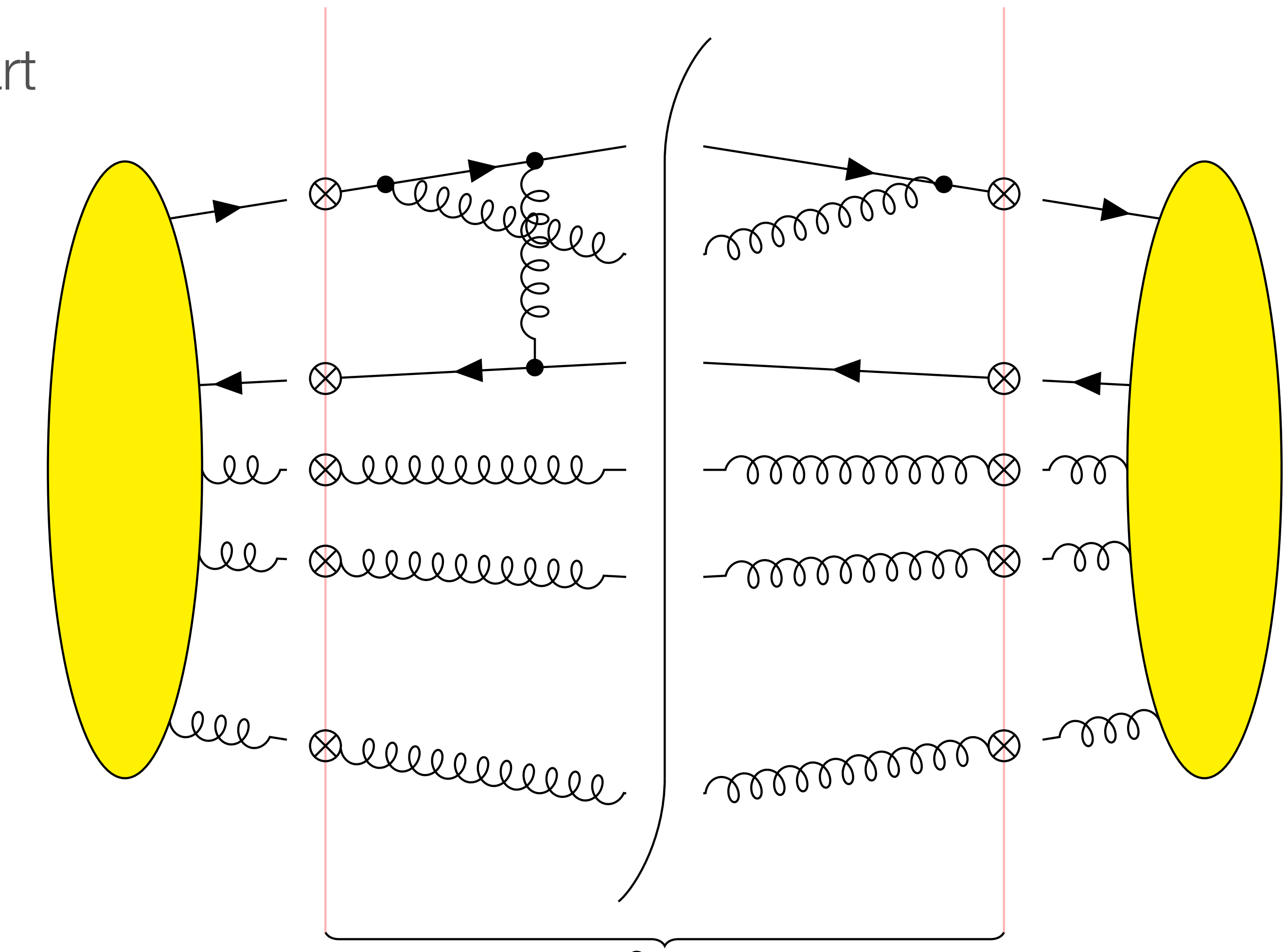
Amplitudes have soft or collinear singularities and they have divergences $1/\varepsilon$ from the loops



- ➠ We want to describe the singularity structure in a **process independent way**.
- ➠ Everything in the yellow blobs is considered hard.

Infrared sensitive operator

Consider the momenta coming from the hard part as fixed and on shell.



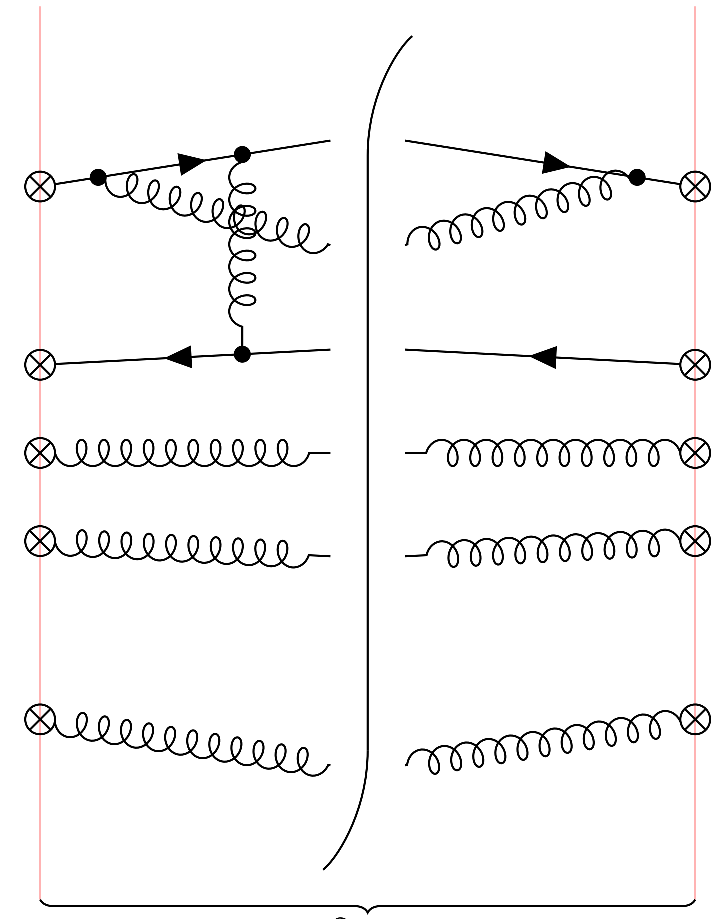
$$\left[\frac{\alpha_s(\mu^2)}{2\pi}\right]^2 \mathcal{D}^{(1,1)}(\mu^2)$$

This gives us an operator as

$$\begin{aligned}
&\big(\{\hat{p},\hat{f},\hat{s},\hat{s}',\hat{c},\hat{c}'\}_{m+n_\mathrm{R}}\big|\rho(\mu^2)\big) \\
&\quad\sim \frac{1}{m!}\int [d\{p\}_m] \sum_{\{f\}_m} \sum_{\{s,s',c,c'\}_m} \\
&\qquad\times \big(\{\hat{p},\hat{f},\hat{s},\hat{s}',\hat{c},\hat{c}'\}_{m+n_\mathrm{R}}\big|{\color{red}\mathcal{D}(\mu^2)}\big|\{p,f,s,s',c,c'\}_m\big) \\
&\qquad\times \big(\{p,f,s,s',c,c'\}_m\big|\rho_\mathrm{hard}(\mu^2)\big)
\end{aligned}$$

Infrared sensitive operator

We can consider a more constructive approach to build the full infrared sensitive operator. This operator basically represents the QCD density operator of a $m \to X$ *(anything)* process.



$$\left[\frac{\alpha_s(\mu^2)}{2\pi}\right]^2 \mathcal{D}^{(1,1)}(\mu^2)$$

$$\mathcal{D}(\mu^2) = 1 + \sum_{n=1}^{k}\left[\frac{\alpha_s(\mu^2)}{2\pi}\right]^n \sum_{\substack{n_R=0 \\ n_R+n_V=n}}^{n}\sum_{n_V=0}^{n} \mathcal{D}^{(n_R,n_V)}(\mu^2)$$

The structure is rather straightforward:

$$\begin{aligned}&\big(\{\hat{p},\hat{f},\hat{s}',\hat{c}',\hat{s},\hat{c}\}_{m+n_R}\big|\mathcal{D}^{(n_R,n_V)}(\mu^2,\mu_S^2)\big|\{p,f,s',c',s,c\}_m\big)\\ &= \sum_{G\in\mathrm{Graphs}}\int d^d\{\ell\}_{n_V}\ {}_D\big\langle\{\hat{s},\hat{c}\}_{m+n_R}\big|\boldsymbol{V}_L(G;\{\hat{p},\hat{f}\}_{m+n_R},\{\ell\}_{n_V},\mu^2)\big|\{s,c\}_m\big\rangle\\ &\quad\times\big\langle\{s,c\}_m\big|\boldsymbol{V}_R^\dagger(G;\{\hat{p},\hat{f}\}_{m+n_R},\{\ell\}_{n_V},\mu^2)\big|\{\hat{s},\hat{c}\}_{m+n_R}\big\rangle_D\\ &\quad\times\sum_{I\in\mathrm{Regions(G)}}\big(\{\hat{p},\hat{f}\}_{m+n_R}\big|\mathcal{P}_G(I)\big|\{p,f\}_m\big)\underbrace{\Theta_G(I;\{\hat{p},\hat{f}\}_{m+n_R},\{\ell\}_{n_V};\mu_S^2)}\end{aligned}$$

Constrains the off-shellness of the hard partons

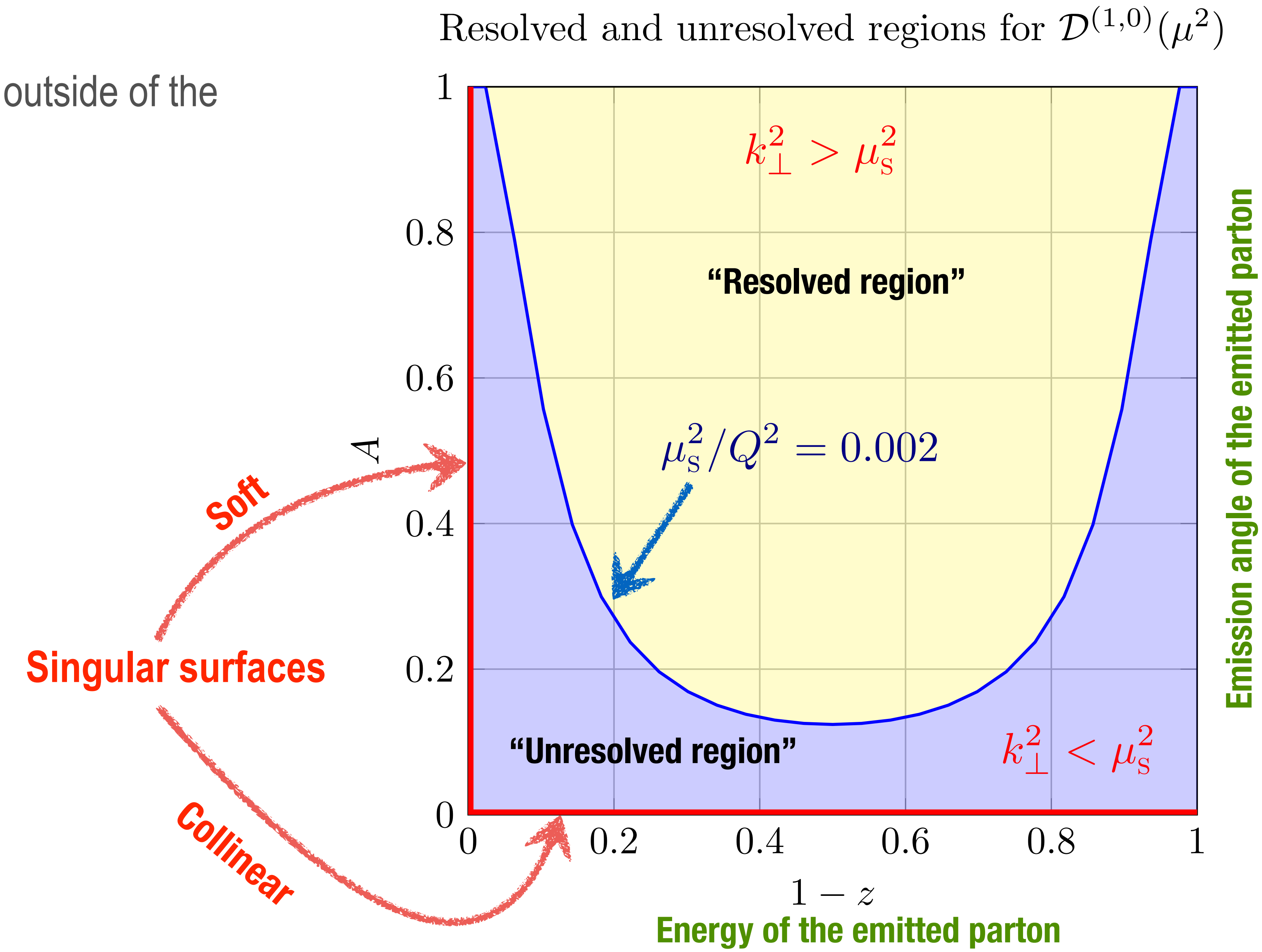
Infrared sensitive operator

We have to introduce an **ultraviolet cutoff to capture only the IR part** of the amplitudes. At first order level in the real graphs it is just a cut on an infrared sensitive variable of the splitting:

$$\Theta_G(I;\{\hat{p},\hat{f}\}_{m+n_{\mathrm{R}}},\{\ell\}_{n_{\mathrm{V}}};\mu_{\mathrm{S}}^2) \sim \theta(k_\perp^2 < \mu_{\mathrm{S}}^2)$$

The singular surfaces may not extend outside of the unresolved region.

There can be **no naked singularity**!



N^kLO calculations

Subtractions

Singularities cancel each other here

$$\sigma[O_J] = \overbrace{\left(1\middle|\mathcal{O}_J\left[\mathcal{F}(\mu^2)\circ\mathcal{Z}_F(\mu^2)\right]\mathcal{D}(\mu^2)\right.}\underbrace{\mathcal{D}^{-1}(\mu^2)\big|\rho(\mu^2)\big)}_{=|\rho_{\rm H}(\mu^2))}$$

Hard part, finite in d=4 dimension

$$+\,\mathcal{O}(\alpha_s^{k+1}L^{2k+2})$$
$$+\,\mathcal{O}(\Lambda_{QCD}^2/\mu_J^2)$$

Usually $\mathcal{D}^{-1}(\mu^2)$ is constructed by hand and $\mathcal{D}(\mu^2)$ is its inverse.

- This is a good approximation as long as
$$\mu^2 < \mu_J^2$$
- the D operator doesn't create resolvable partons, thus
$$\mathcal{D}(\mu^2)\mathcal{O}_J \approx \mathcal{O}_J\mathcal{D}(\mu^2)$$
- otherwise we have to deal with large logarithms,
$$L = \log\frac{\mu^2}{\mu_J^2}$$

$$\begin{aligned}\mathcal{D}^{-1}(\mu_{\rm R}^2)\big|\rho(\mu_{\rm R}^2)\big) = &\overbrace{\big|\rho^{(0)}(\mu_{\rm R}^2)\big)}^{\text{Born term}} + \frac{\alpha_s(\mu_{\rm R}^2)}{2\pi}\overbrace{\left[\big|\rho^{(1)}(\mu_{\rm R}^2)\big) - \mathcal{D}^{(1)}(\mu_{\rm R}^2)\big|\rho^{(0)}(\mu_{\rm R}^2)\big)\right]}^{\text{NLO contributions}}\\ &+\left[\frac{\alpha_s(\mu_{\rm R}^2)}{2\pi}\right]^2\underbrace{\left\{\big|\rho^{(2)}(\mu_{\rm R}^2)\big) - \mathcal{D}^{(1)}(\mu_{\rm R}^2)\big|\rho^{(1)}(\mu_{\rm R}^2)\big) - \left[\mathcal{D}^{(2)}(\mu_{\rm R}^2) - \mathcal{D}^{(1)}(\mu_{\rm R}^2)\mathcal{D}^{(1)}(\mu_{\rm R}^2)\right]\big|\rho^{(0)}(\mu_{\rm R}^2)\big)\right\}}_{\text{NNLO contributions}}\\ &+\mathcal{O}(\alpha_s^3)\end{aligned}$$

N^{k}LO calculations

We define an operator that is **finite** and **doesn't** change the number of patrons and their momenta and flavours in such way that

$$\underbrace{(1|\mathcal{V}(\mu^2)}_{} = \underbrace{(1|\left[\mathcal{F}(\mu^2)\circ\mathcal{Z}_F(\mu^2)\right]\mathcal{D}(\mu^2)\mathcal{F}^{-1}(\mu^2)}_{}$$

- IR **finite** operator
- **doesn't** create new patrons
- **doesn't** change momenta or flavours
- its definition is **ambiguous**

- IR **singular** operator
- **does** create new patrons
- **does** change momenta and flavours

With the help of this we can define a normalised IR singular operator as

$$\mathcal{X}_1(\mu^2) = \left[\mathcal{F}(\mu^2)\circ\mathcal{Z}_F(\mu^2)\right]\mathcal{D}(\mu^2)\mathcal{F}^{-1}(\mu^2)\mathcal{V}^{-1}(\mu^2)$$

$\Rightarrow$ from definition $(1|\mathcal{X}_1(\mu^2) = (1|$

The cross section can be written as

$$\sigma[O_J] = (1|\underbrace{\mathcal{O}_J\,\mathcal{X}_1(\mu^2)}_{}\,\mathcal{V}(\mu^2)\mathcal{F}(\mu^2)\,|\rho_{\mathrm{H}}(\mu^2))$$

when we don't have to worry about large logs, these operators **commute**

Useful notations

It is proven to be useful to generalise the procedure of defining operator $\mathcal{V}(\mu^2)$ from $\mathcal{D}(\mu^2)$.

Let $\mathscr{A}$ be a linear operator in the statistical space (may or mayn't change the number of partons):

$$\mathcal{A}\big|\{p,f,c,c',s,s'\}_m\big) = \int d\{\hat{p},\hat{f},\hat{c},\hat{c}',\hat{s},\hat{s}'\}_{\hat{m}}\,\big|\{\hat{p},\hat{f},\hat{c},\hat{c}',\hat{s},\hat{s}'\}_{\hat{m}}\big)\,\big(\{\hat{p},\hat{f},\hat{c},\hat{c}',\hat{s},\hat{s}'\}_{\hat{m}}\big|\mathcal{A}\big|\{p,f,c,c',s,s'\}_m\big)$$

We define a mapping, $[\cdot]_{\mathbb{P}} : \mathcal{A} \longrightarrow \mathcal{B} = [\mathcal{A}]_{\mathbb{P}}$, in such a way that

$$\mathcal{B}\big|\{p,f,c,c',s,s'\}_m\big) = \int d\{\hat{c},\hat{c}',\hat{s},\hat{s}'\}_m\,\big|\{p,f,\hat{c},\hat{c}',\hat{s},\hat{s}'\}_m\big)\,\big(\{p,f,\hat{c},\hat{c}',\hat{s},\hat{s}'\}_m\big|\mathcal{B}\big|\{p,f,c,c',s,s'\}_m\big)$$

and

$$(1|\,[\mathcal{A}]_{\mathbb{P}} = (1|\mathcal{A}$$

The combination $\mathcal{A} - [\mathcal{A}]_{\mathbb{P}}$ appears frequently, thus it is useful to define: $[\mathcal{A}]_{1-\mathbb{P}} = \mathcal{A} - [\mathcal{A}]_{\mathbb{P}}$.

$$\mathcal{V}(\mu_R^2) = \Big[\big[\mathcal{F}(\mu_R^2)\circ\mathcal{Z}_F(\mu_R^2)\big]\,\mathcal{D}(\mu_R^2)\Big]_{\mathbb{P}}\mathcal{F}^{-1}(\mu_R^2)$$

Fixed order cross sections

Parton showers

(in only two slides)

Shower Cross Section

The fixed order cross section is fine as long as we can calculate at "**all order level**". But life is not that easy…

- **truncated** at NLO, NNLO level
- prefers **large scale,** $\mu^2 \approx Q^2$

$$\sigma[O_J] = \big(1\big|\underbrace{\mathcal{O}_J\,\mathcal{X}_1(\mu^2)}\,\mathcal{V}(\mu^2)\,\overbrace{\mathcal{F}(\mu^2)\,\big|\rho_{\mathrm{H}}(\mu^2)\big)}$$

- prefers **small scale**, $\mu^2 \ll \mu_J^2$
- that is **in conflict** with the hard part

- Choose a **hard scale**, $\mu_{\mathrm{H}}^2 \approx Q^2$
- Choose a **cutoff scale**, $\mu_J^2 \gg \mu_{\mathrm{f}}^2 \approx 1\mathrm{GeV}^2$
- Insert a unit operator before the measurement operator as,

$$1 = \mathcal{X}_1(\mu_{\mathrm{f}}^2)\mathcal{X}_1^{-1}(\mu_{\mathrm{f}}^2)$$

$$\sigma[O_J] = \underbrace{\big(1\big|\mathcal{O}_J\,\mathcal{X}_1(\mu_{\mathrm{f}}^2)}_{=(1|\mathcal{O}_J}\,\overbrace{\mathcal{X}_1^{-1}(\mu_{\mathrm{f}}^2)\mathcal{X}_1(\mu_{\mathrm{H}}^2)}^{\mathcal{U}(\mu_{\mathrm{f}}^2,\mu_{\mathrm{H}}^2)}\,\mathcal{V}(\mu_{\mathrm{H}}^2)\mathcal{F}(\mu_{\mathrm{H}}^2)\,\big|\rho_{\mathrm{H}}(\mu_{\mathrm{H}}^2)\big)$$

No resolvable radiation come from $\mathcal{X}_1(\mu_{\mathrm{f}}^2)$ operator, thus these operators commute, $\mathcal{O}_J\mathcal{X}_1(\mu_{\mathrm{f}}^2) \approx \mathcal{X}_1(\mu_{\mathrm{f}}^2)\mathcal{O}_J$.

$$\mathcal{U}(\mu_{\mathrm{f}}^2, \mu_{\mathrm{H}}^2) = \mathbb{T}\exp\left\{\int_{\mu_{\mathrm{f}}^2}^{\mu_{\mathrm{H}}^2}\frac{d\mu^2}{\mu^2}\,\mathcal{S}(\mu^2)\right\}$$

$$\frac{1}{\mu^2}\mathcal{S}(\mu^2) = \lim_{\epsilon\to 0}\mathcal{X}_1^{-1}(\mu^2)\frac{d\mathcal{X}_1(\mu^2)}{d\mu^2}$$

First order shower

The generators of the unitary shower can be expanded in the coupling:

$$S(\mu^2) = \frac{\alpha_s(\mu^2)}{2\pi} S^{(1)}(\mu^2) + \left[\frac{\alpha_s(\mu^2)}{2\pi}\right]^2 S^{(2)}(\mu^2) + \cdots$$

and the first order term is rather simple

$$\frac{1}{\mu^2} S^{(1)}(\mu^2) = \Bigg[\underbrace{\mathcal{F}(\mu_R^2)\frac{\partial\mathcal{D}^{(1,0)}(\mu^2,\mu_S^2)}{\partial\mu_S^2}\mathcal{F}^{-1}(\mu^2)}_{} - \underbrace{\frac{\partial\left[\mathcal{F}(\mu_R^2)\mathcal{D}^{(1,0)}(\mu^2,\mu_S^2)\right]_{\mathbb{P}}}{\partial\mu_S^2}\mathcal{F}^{-1}(\mu^2)}_{} + \underbrace{\mathrm{Im}\frac{\partial\mathcal{D}^{(0,1)}(\mu^2,\mu_S^2)}{\partial\mu_S^2}}_{}\Bigg]_{\mu_S^2=\mu^2}$$

Real operator
all the quantum numbers of the emitted parton is **resolved**

Integrated real operator
- all the quantum numbers of the emitted parton is **integrated out**
- it is **not** the contribution of the virtual graphs

Glauber gluon
imaginary part of the virtual graphs

$$\sim \mathrm{i}\pi$$

Note, the first order kernel is independent of the real part of the virtual graphs.

Leading Color Approx. (LC)

Leading Color Approximation is widely used in parton shower implementations.

- **No colour interferences** are considered. The colour space is diagonal in every step of the shower

$$\big|\{p,f,c\}_m\big)_{\rm LC} \equiv \big|\{p,f,c,c\}_m\big)$$

- Colour group is **reduced to U(3)**, the colour overlaps at the end of the shower is trivial

$$C_F = \frac{C_A}{2} = \frac{N_c}{2} \qquad \text{and} \qquad \big(1\big|\{c,c\}_m\big) = \big\langle\{c\}_m\big|\{c\}_m\big\rangle = 1 + \mathcal{O}(1/N_c^2)$$

- In general the error terms are suppressed by $1/N_c^2$ but they are **LL contributions**.
- One **can tweak** the $C_A/2,\ C_F$ factor to obtain LL and NLL for some observables. *(See thrust result, later!)*
- We don't see how it can be improved systematically. No clear way to treat the error terms perturbatively.

But DEDUCTOR doesn't use it at all!

LC+ Approximation

Despite of the name it is **not an approximation of the colour space**, it is an **approximation of the shower evolution operator**.

LC+ part

- Diagonal operator in the color space
- **Exact** in the **collinear** limit
- Some soft interferences are included but not all
- **Easy to exponentiate**

$$\mathcal{S}^{(1)}(\mu^2) = \overbrace{\mathcal{S}^{(1)}_{\mathrm{LC+}}(\mu^2)} + \underbrace{\Delta\mathcal{S}^{(1)}(\mu^2)}$$

Wide angle soft part

- Only **wide angle soft** singularities
- Only single log contribution
- Leads to only $1/N_c^2$ suppressed terms
- Can be treated **perturbatively**

This decomposition **preserves unitary**,

$$(1|\mathcal{S}^{(1)}(\mu^2) = (1|\mathcal{S}^{(1)}_{\mathrm{LC+}}(\mu^2) = (1|\Delta\mathcal{S}^{(1)}(\mu^2) = 0$$

and it allows us to treat the wide angle soft part perturbatively in a very efficient and flexible way.

- No approximation of the colour group, it is the **full SU(3)** algebra
- Can handle **any colour interferences**

$$\{c\}_m \neq \{c'\}_m$$

- At the end of the shower we calculate the **full SU(3)** colour overlap without approximation,

$$\langle \{c'\}_m | \{c\}_m \rangle$$

We have a **very fast algorithm** to do this, and can deal with hundreds of partons.

- **No** need of **tweaking** the $C_A/2,\ C_F$ colour factors.

Fixed order cross sections

Parton showers

Summing large logarithms with parton showers

Summing logarithms

I don't trust in eye measure to clam LL or NLL accuracy of any parton shower. One way to check the summation property of the shower is to **gain analytical control** on the shower cross section. Is it *possible* to do it? Is it *simple*?

$$\sigma[O_J] = \left(1\middle|\mathcal{O}_J\,\mathbb{T}\exp\left\{\int_{\mu_\mathrm{f}^2}^{\mu_\mathrm{H}^2}\frac{d\mu^2}{\mu^2}\,\mathcal{S}(\mu^2)\right\}\mathcal{V}(\mu_\mathrm{H}^2)\mathcal{F}(\mu_\mathrm{H}^2)\,\middle|\rho_\mathrm{H}(\mu_\mathrm{H}^2)\right)$$

- **"infinite"** number of partons
- make measurement on these multi-parton states
- **impossible task** to study the log structure analytically

We should **reformulate** the shower cross section, in such a way that:

- ➠ more suitable for analytical studies
- ➠ **without** extra approximation (all the approximations have been done in the shower operator $\mathcal{S}(\mu^2)$)
- ➠ the effect of the measurement operator should be **exponentiated**

We want to test the log summation property of the parton shower cross algorithms

- ➠ study observables that exponentiates (**thrust**, Drell-Yan pT-distributions,…)
- ➠ analytical results are available

Preparing observables

Consider the Dell-Yan kT distribution:

$$\hat{\mathcal{O}}(\boldsymbol{k}_\perp)\big|\{p,f,...\}_m\big) = (2\pi)^2\delta^{(2)}(\boldsymbol{k}_\perp - \boldsymbol{k}_Z(\{p\}_m))\big|\{p,f,...\}_m\big)$$

This operator is not invertible, but its **Fourier transform** is,

$$\mathcal{O}(\boldsymbol{b})\big|\{p,f,...\}_m\big) = e^{\mathrm{i}\boldsymbol{b}\cdot\boldsymbol{k}_Z(\{p\}_m)}\big|\{p,f,...\}_m\big) \ , \qquad \mathcal{O}^{-1}(\boldsymbol{b})\big|\{p,f,...\}_m\big) = e^{-\mathrm{i}\boldsymbol{b}\cdot\boldsymbol{k}_Z(\{p\}_m)}\big|\{p,f,...\}_m\big) \ .$$

Similarly for **thrust**, we use **Laplace transformation** to make the measurement operator invertible,

$$\mathcal{O}(\nu)\big|\{p,f,...\}_m\big) = e^{-\nu\tau(\{p\}_m)}\big|\{p,f,...\}_m\big) \ , \qquad \mathcal{O}^{-1}(\nu)\big|\{p,f,...\}_m\big) = e^{\nu\tau(\{p\}_m)}\big|\{p,f,...\}_m\big)$$

The formalism can deal with measurement operator that has an inverse, thus we almost **always need** some kind of **proxy** to do the analytical studies of the parton showers. Sometimes it is just a simple integral transformation, sometimes a generating functional. It is a good guideline to follow the footsteps of the analytic calculation.

$\hat{\mathcal{O}}(\boldsymbol{v}) \Longrightarrow \mathcal{O}(\boldsymbol{r})$ and $\mathcal{O}(\boldsymbol{r})$ always **has an inverse** over the **whole** statistical space

Observable dependent shower

We define an operator that is **finite** and **doesn't** change the number of patrons and their momenta and flavours but this time **with observable dependence**

$$\mathcal{Y}(\mu^2;\boldsymbol{r}) = \left[\mathcal{O}(\boldsymbol{r})\left[\mathcal{F}(\mu^2)\circ\mathcal{Z}_F(\mu^2)\right]\mathcal{D}(\mu^2)\mathcal{O}^{-1}(\boldsymbol{r})\right]_{\mathbb{P}} \times \left(\left[\left[\mathcal{F}(\mu^2)\circ\mathcal{Z}_F(\mu^2)\right]\mathcal{D}(\mu^2)\right]_{\mathbb{P}}\right)^{-1}$$

- IR **finite** operator
- **doesn't** create new patrons
- **doesn't** change momenta or flavours
- its definition obviously is **ambiguous**
- **normalised**

$$\mathcal{O}(\boldsymbol{r}) = 1 \implies \mathcal{Y}(\mu^2;\boldsymbol{r}) = 1$$

From the definition, it is easy to show that

$$\big(1\big|\mathcal{Y}(\mu^2;\boldsymbol{r})\mathcal{O}(\boldsymbol{r}) = \big(1\big|\mathcal{O}(\boldsymbol{r})\,\mathcal{U}(\mu_{\mathrm{f}}^2,\mu^2)$$

and the shower cross section becomes

$$\sigma(\boldsymbol{r}) = \big(1\big|\mathcal{Y}(\mu_{\mathrm{H}}^2,\boldsymbol{r})\,\mathcal{O}(\boldsymbol{r})\,\mathcal{V}(\mu_{\mathrm{H}}^2)\,\mathcal{F}(\mu_{\mathrm{H}}^2)\,\big|\rho_{\mathrm{H}}(\mu_{\mathrm{H}}^2)\big) = \big(1\big|\mathcal{O}(\boldsymbol{r})\,\mathcal{U}(\mu_{\mathrm{f}}^2,\mu_{\mathrm{H}}^2)\,\mathcal{V}(\mu_{\mathrm{H}}^2)\,\mathcal{F}(\mu_{\mathrm{H}}^2)\,\big|\rho_{\mathrm{H}}(\mu_{\mathrm{H}}^2)\big)$$

measurement **after the shower** (many patrons)

measurement on the **hard state** (only few patrons)

It is **really** an **equal sign**!

Observable dependent shower

The $\mathcal{Y}(\mu^2;\boldsymbol{r})$ operator can be **exponentiated** in the usual way,

$$\mathcal{Y}(\mu_{\rm H}^2;\boldsymbol{r}) = \mathbb{T}\exp\left\{\int_{\mu_{\rm f}^2}^{\mu_{\rm H}^2} \frac{d\mu^2}{\mu^2} \mathcal{S}_{\mathcal{Y}}(\mu^2;\boldsymbol{r})\right\} \quad , \quad \text{with} \quad \mathcal{Y}(\mu_{\rm f}^2;\boldsymbol{r}) = 1$$

where

$$\frac{1}{\mu^2}\mathcal{S}_{\mathcal{Y}}(\mu^2;\boldsymbol{r}) = \mathcal{Y}^{-1}(\mu^2;\boldsymbol{r})\frac{d\mathcal{Y}(\mu^2;\boldsymbol{r})}{d\mu^2} \quad .$$

- Here the exponent **has to be an all order** expression to maintain the equality with the shower cross section.
- The operator $\mathcal{S}_{\mathcal{Y}}(\mu^2;\boldsymbol{r})$ contains large logarithms of $L(\boldsymbol{r})$.
- We can relate the $\mathcal{S}_{\mathcal{Y}}(\mu^2;\boldsymbol{r})$ operator to the generator of the parton shower $\mathcal{S}(\mu^2)$ via

$$\big(1\big|\mathcal{Y}(\mu^2;\boldsymbol{r})\,\mathcal{S}_{\mathcal{Y}}(\mu^2;\boldsymbol{r}) = \big(1\big|\mathcal{Y}(\mu^2;\boldsymbol{r})\,\mathcal{O}(\boldsymbol{r})\,\mathcal{S}(\mu^2)\,\mathcal{O}^{-1}(\boldsymbol{r})$$

with the help of the $[\cdot]_{\mathbb{P}}$ operation we can extract $\mathcal{S}_{\mathcal{Y}}(\mu^2;\boldsymbol{r})$ as

$$\mathcal{S}_{\mathcal{Y}}(\mu^2;\boldsymbol{r}) = \left[\mathcal{Y}(\mu^2;\boldsymbol{r})\,\mathcal{O}(\boldsymbol{r})\,\mathcal{S}(\mu^2)\,\mathcal{O}^{-1}(\boldsymbol{r})\right]_{\mathbb{P}} - \left[\mathcal{Y}(\mu^2;\boldsymbol{r}) - 1\right]\mathcal{S}_{\mathcal{Y}}(\mu^2;\boldsymbol{r})$$

and this can be solved **recursively** order by order (in powers of the shower generator $\mathcal{S}(\mu^2)$).

Observable dependent shower

We have **two equations and two unknowns**, so we can solve them recursively:

$$\mathcal{S}_{\mathcal{Y}}(\mu^2;\nu) = \sum_{k=1}^{\infty} \mathcal{S}_{\mathcal{Y}}^{[k]}(\mu^2;\nu)$$

$$\mathcal{Y}(\mu^2;\nu) = 1 + \sum_{k=1}^{\infty} \mathcal{Y}^{[k]}(\mu^2;\nu)$$

At **first order** level we have

$$\mathcal{S}_{\mathcal{Y}}^{[1]}(\mu^2;\boldsymbol{r}) = \left[\mathcal{O}(\boldsymbol{r})\,\mathcal{S}(\mu^2)\,\mathcal{O}^{-1}(\boldsymbol{r})\right]_{\mathbb{P}}$$

$$\mathcal{Y}^{[1]}(\mu^2;\boldsymbol{r}) = \int_{\mu_{\rm f}^2}^{\mu^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[\mathcal{O}(\boldsymbol{r})\,\mathcal{S}(\bar{\mu}^2)\,\mathcal{O}^{-1}(\boldsymbol{r})\right]_{\mathbb{P}}$$

The **second order** generator is a little bit more complicated:

$$\mathcal{S}_{\mathcal{Y}}^{[2]}(\mu^2;\boldsymbol{r}) = \int_{\mu_{\rm f}^2}^{\mu^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[\left[\mathcal{O}(\boldsymbol{r})\,\mathcal{S}(\bar{\mu}^2)\,\mathcal{O}^{-1}(\boldsymbol{r})\right]_{\mathbb{P}}\left[\mathcal{O}(\boldsymbol{r})\,\mathcal{S}(\mu^2)\,\mathcal{O}^{-1}(\boldsymbol{r})\right]_{1-\mathbb{P}}\right]_{\mathbb{P}}$$

Now the shower cross section (in a kind of analytical form) is

$$\sigma(\boldsymbol{r}) = \left(1\middle|\mathbb{T}\exp\left\{\int_{\mu_{\rm f}^2}^{\mu_{\rm H}^2} \frac{d\mu^2}{\mu^2}\left(\mathcal{S}_{\mathcal{Y}}^{[1]}(\mu^2,\boldsymbol{r}) + \sum_{k=2}^{\infty}\mathcal{S}_{\mathcal{Y}}^{[k]}(\mu^2,\boldsymbol{r})\right)\right\}\mathcal{V}(\mu_{\rm H}^2)\mathcal{O}(\boldsymbol{r})\,\mathcal{F}(\mu_{\rm H}^2)\,\middle|\rho_{\rm H}(\mu_{\rm H}^2)\right)$$

For many observables the **exponentiated single emission** operator provides NLL accuracy.

This should lead to only **subleading** logs (NNLL,…).

Thrust in e+e- annihilation

In this case the hard process at Born level is very simple, it is proportional to a single basis vector only with a quark-antiquark pair:

$$\big|\rho_{\rm H}(\mu_{\rm H}^2)\big) \propto \big|\{p,f,c,c\}_2\big)$$

This is always eigenvector of the exponent, thus the exponentiation is trivial:

$$\mathcal{S}_{\mathcal{Y}}^{[k]}(\mu^2;\nu)\big|\{p,f,c,c\}_2\big) = \lambda_{\mathcal{Y}}^{[k]}(\mu^2/Q^2;\nu)\big|\{p,f,c,c\}_2\big)$$

With this the cross section is rather simple,

This is the "**golden nugget**". The parton shower algorithm can agree with the analytic result.

$$\frac{\sigma(\boldsymbol{r})}{\sigma_0} = \exp\left\{\int_0^1 \frac{dx}{x}\left(\overbrace{\lambda_{\mathcal{Y}}^{[1]}(x,\nu)} + \underbrace{\sum_{k=2}^{\infty}\lambda_{\mathcal{Y}}^{[k]}(x,\nu)}\right)\right\} + \cdots$$

This is the shower generated "**junk**". This has to be subleading log contribution.

- We can study analytically the exponent when it is possible,

$$I^{[k]}(\nu) = \int_0^1 \frac{dx}{x}\lambda_{\mathcal{Y}}^{[k]}(x,\nu)$$

- When it is hard to test analytically, we can calculate the exponent numerically and test its large log behaviour in terms of $\log(\nu)$.

$$I^{[k]}(\nu) = \sum_{n=k}^{\infty}\left[\frac{\alpha_{\rm s}(Q^2/\nu)}{2\pi}\right]^n I_n^{[k]}(\nu)$$

- For NLL accuracy we should have

$$I_n^{[k]}(\nu) \sim \log^{n-1}(\nu)$$

for every $k > 1$.

- For LL accuracy we should have

$$I_n^{[k]}(\nu) \sim \log^{n}(\nu)$$

for every $k > 1$.

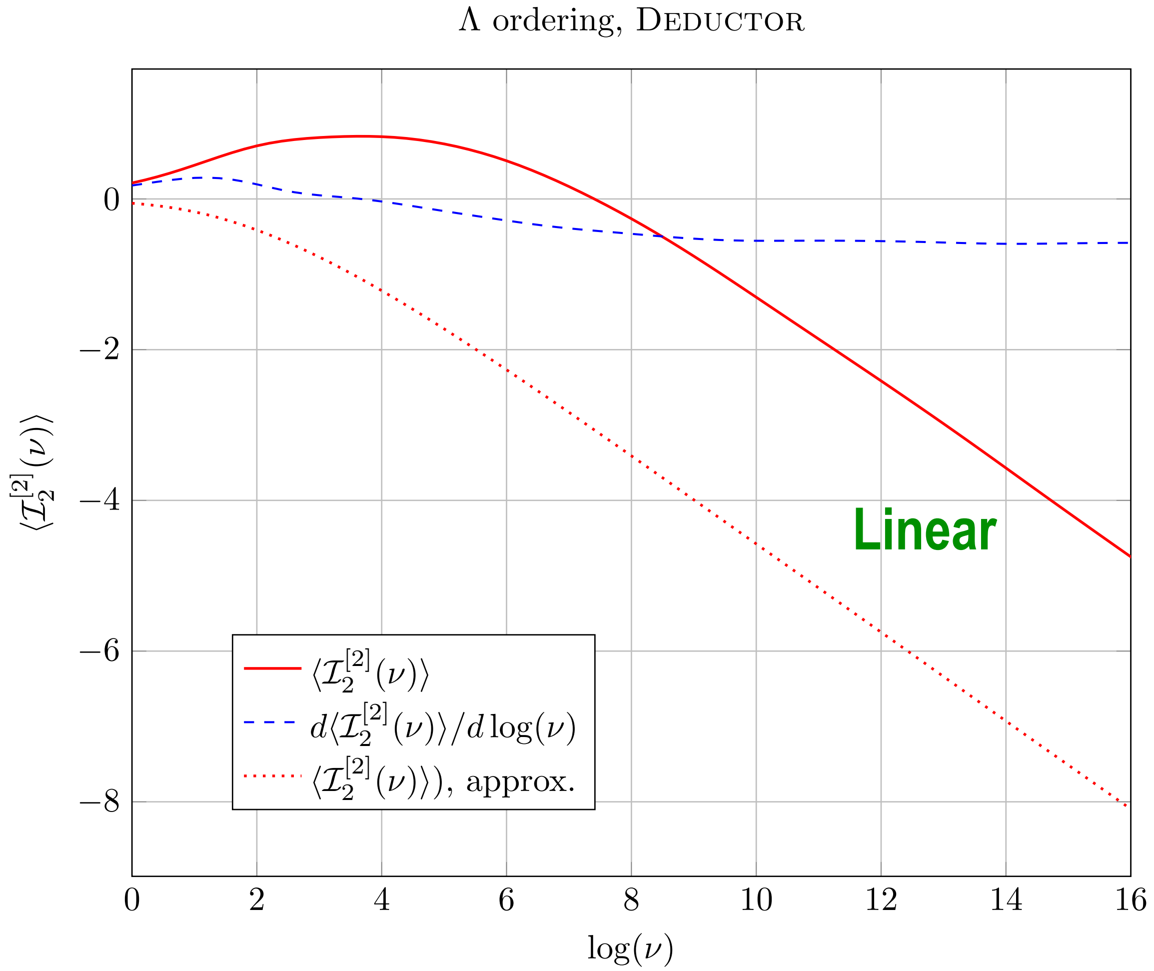
DEDUCTOR Λ-ordered

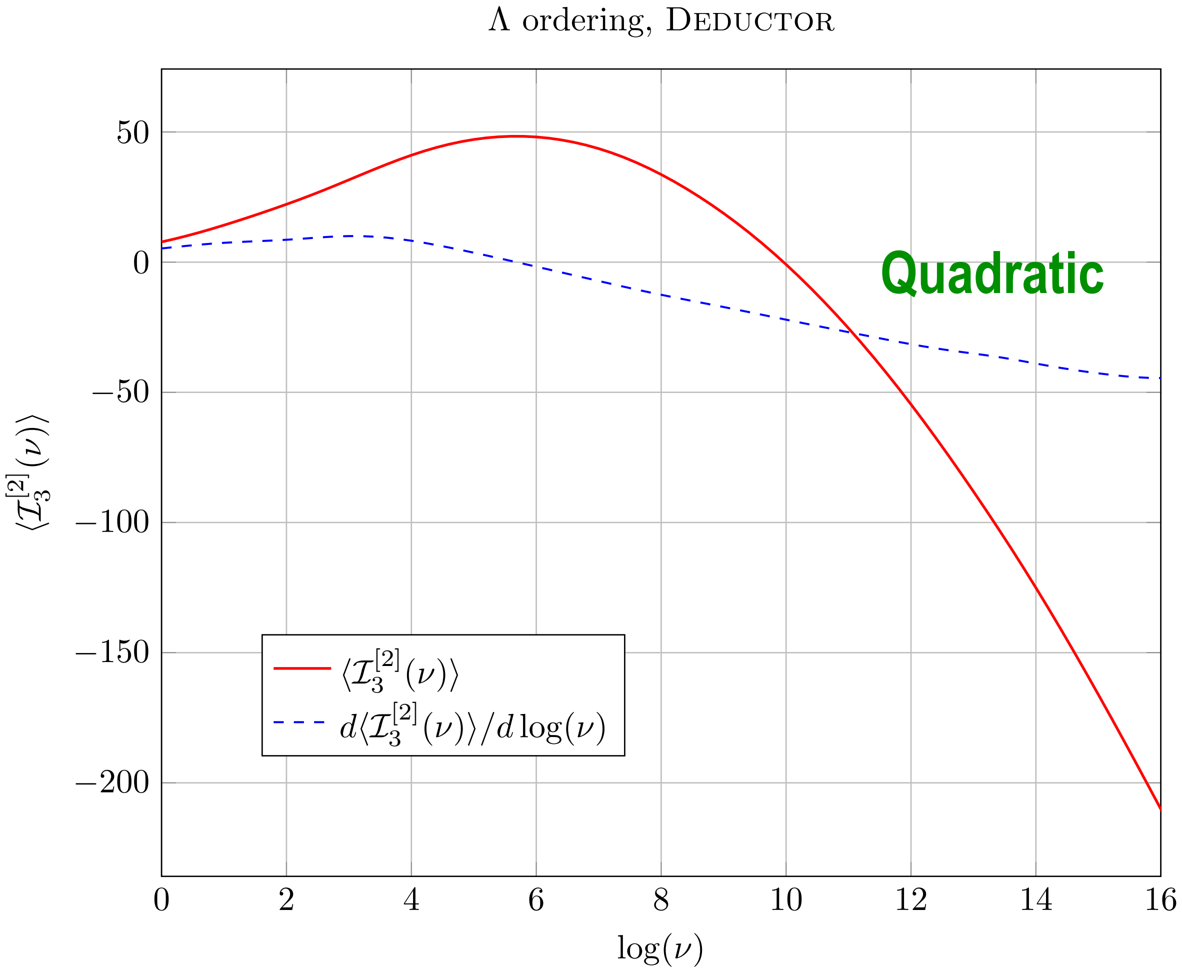
DEDUCTOR Lambda ordered shower

- The ordering variable is the virtuality divided by the mother parton energy

$$\Lambda^2 = \frac{(\hat{p}_l + \hat{p}_{m+1})^2}{2p_l \cdot Q} Q^2$$

- **Global** momentum mapping
- Proper soft gluon treatment with **full SU(3) colour** evolution at amplitude level
- In this case we **can prove analytically** that the shower sums up large logarithms at NLL level

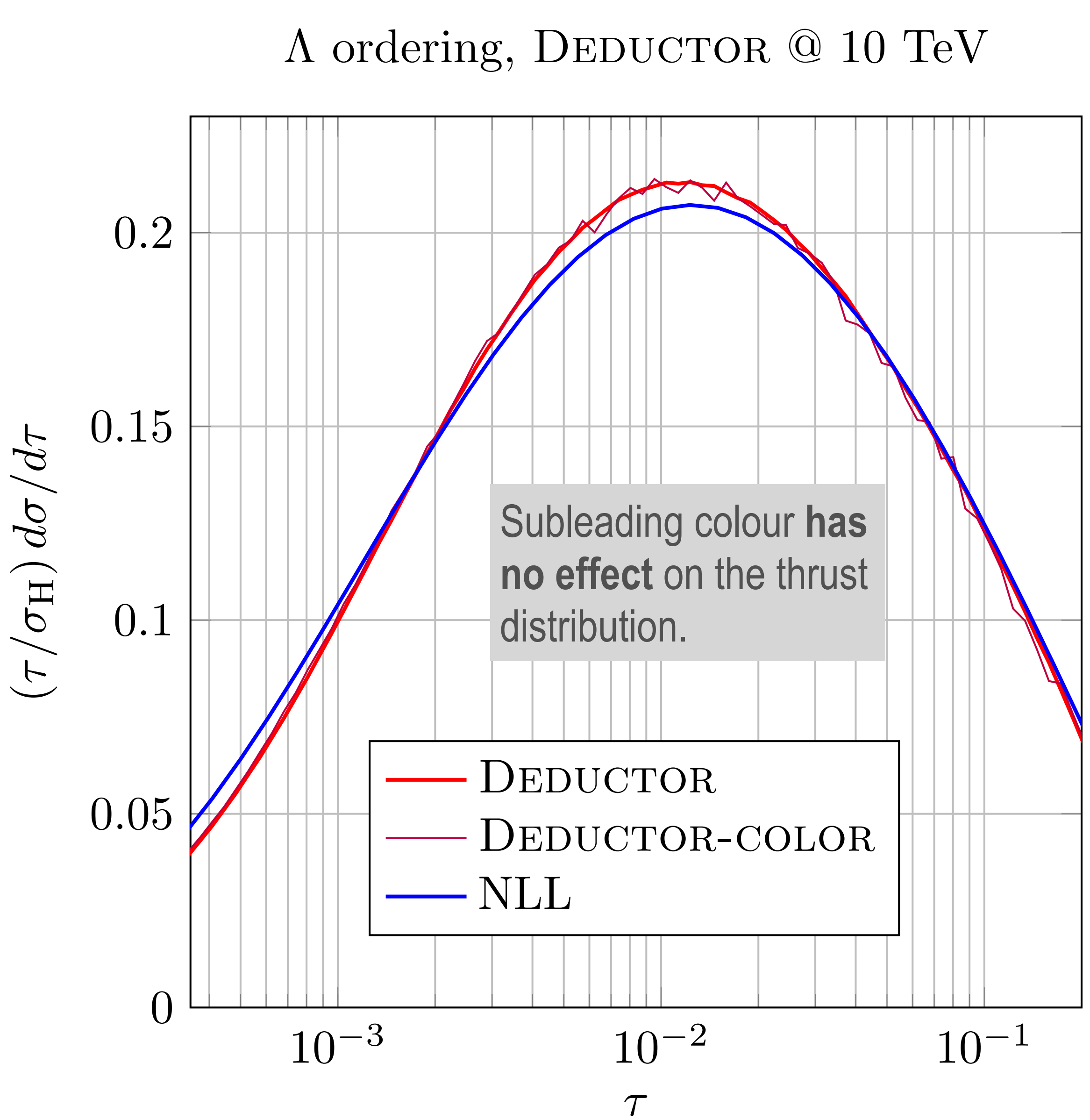


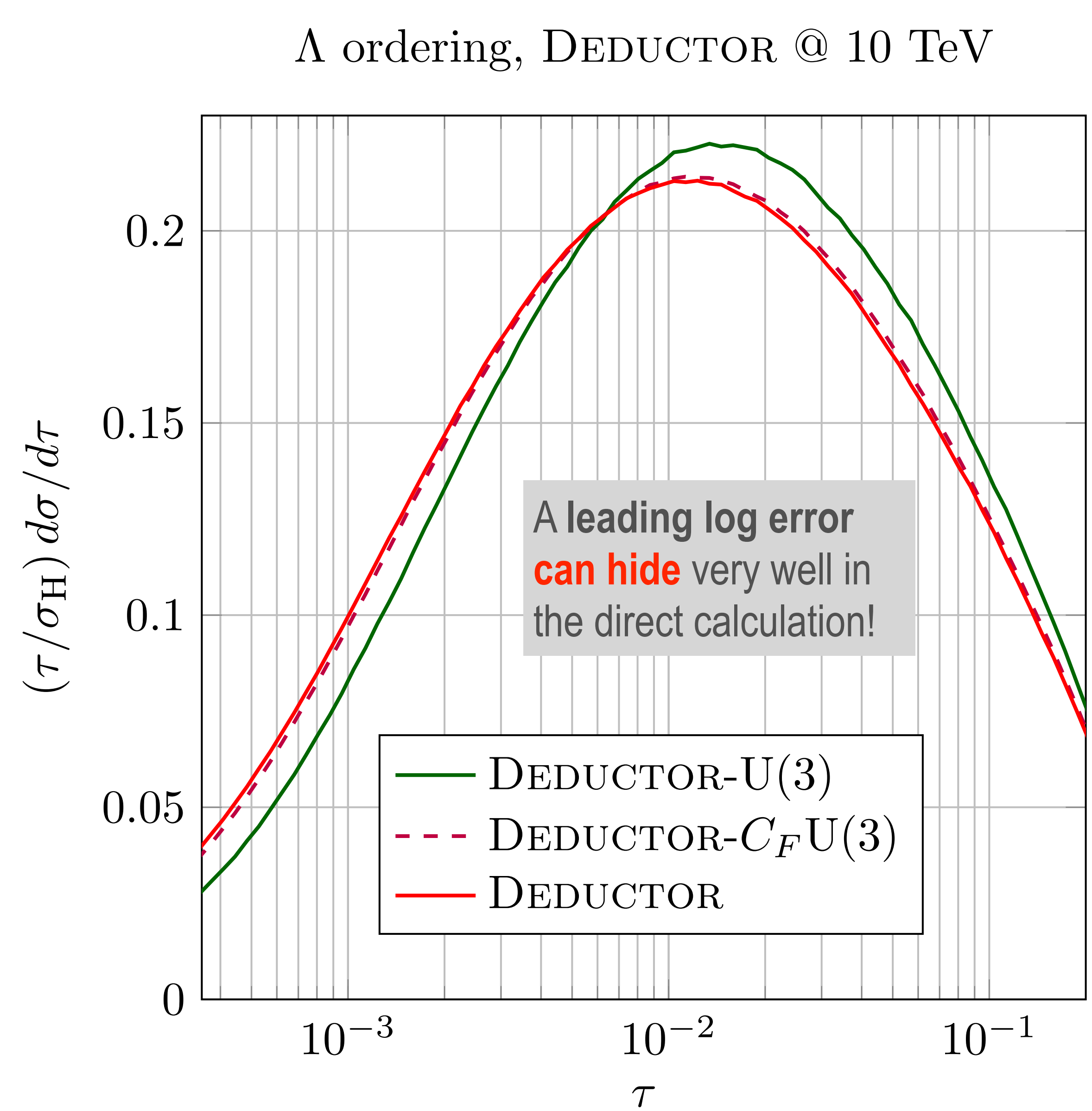


DEDUCTOR Λ-ordered

DEDUCTOR Lambda ordered shower

- **Direct shower cross section calculation**
- LC+ colour approximation with perturbative subleading colour improvement
- The first step of the shower is always exact in colour in e+e- annihilation.

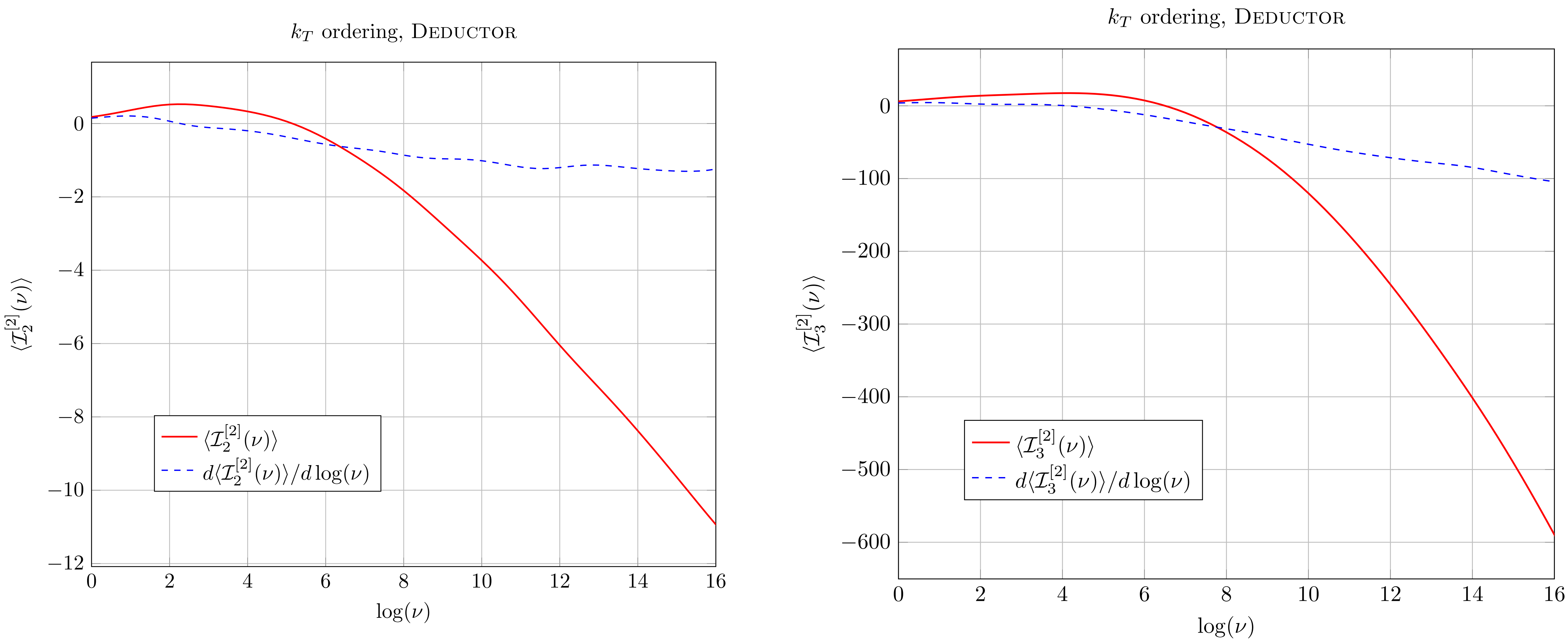




DEDUCTOR kT ordered

DEDUCTOR kT ordered shower

- The ordering variable is the **transverse momentum** of the splitting
- **Global** momentum mapping
- Proper soft gluon treatment with **full colour** evolution
- In this case we **cannot prove analytically** that the shower sums up large logarithms at NLL level
- We check numerically the first couple of $I_n^{[2]}(\nu)$ coefficients.
- It looks OK for k=2 and can be explained by real-virtual cancellation, but hard to see what happens for k > 2.

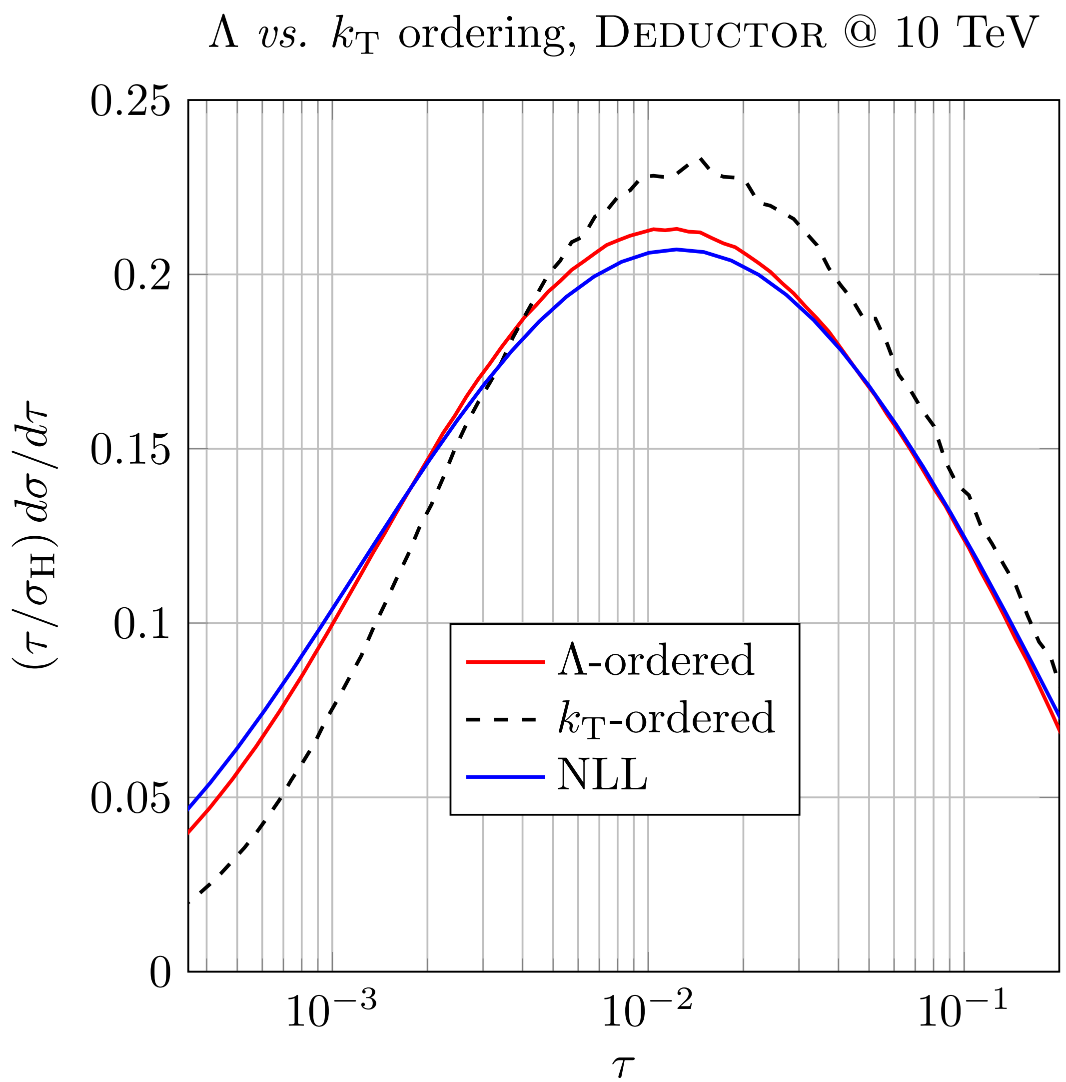


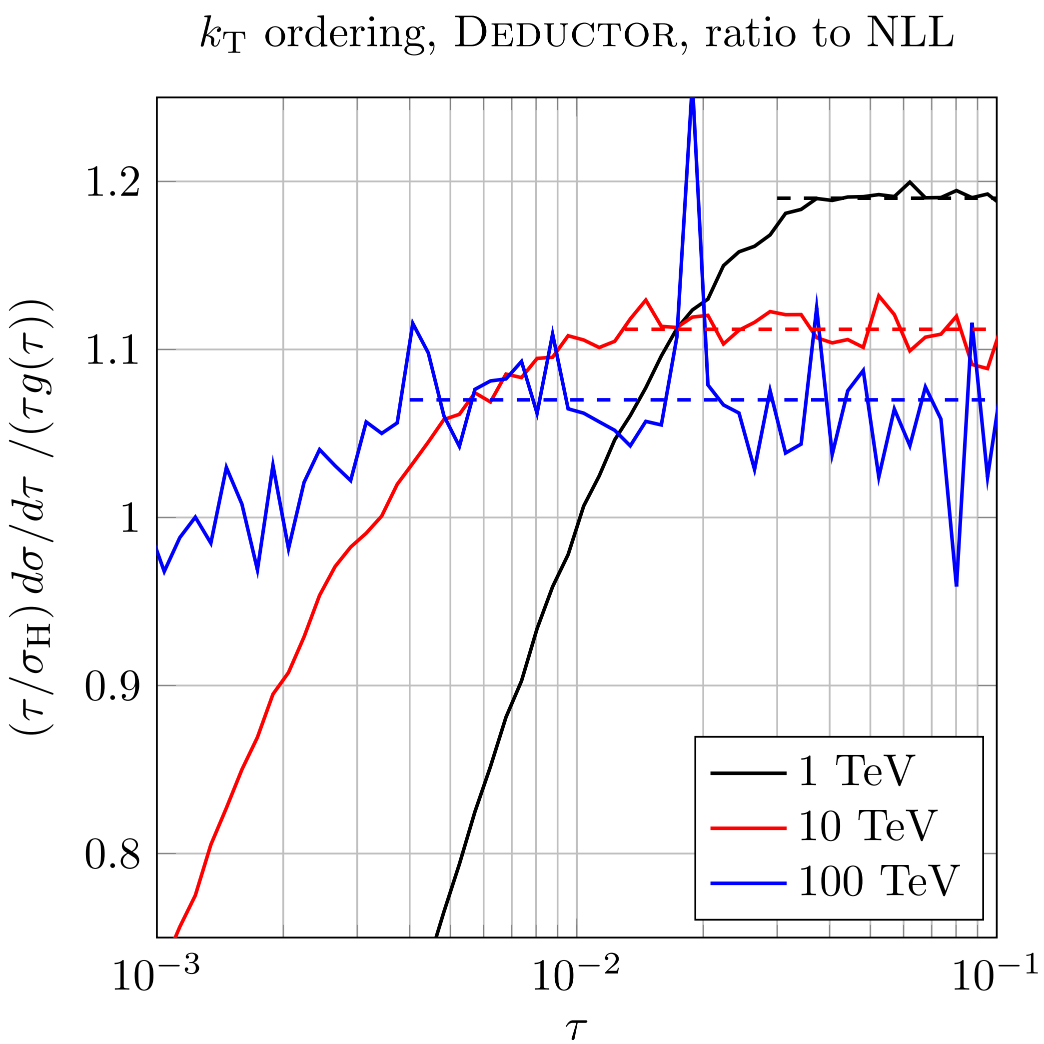
DEDUCTOR kT ordered

DEDUCTOR kT ordered shower

- **Direct shower cross section calculation**
- Compared to analytical result at various collider energy
- It looks good…

This is the strategy of Dasgupta *et al.*, ***Phys.Rev.Lett.* 125** (2020) 5, 052002.





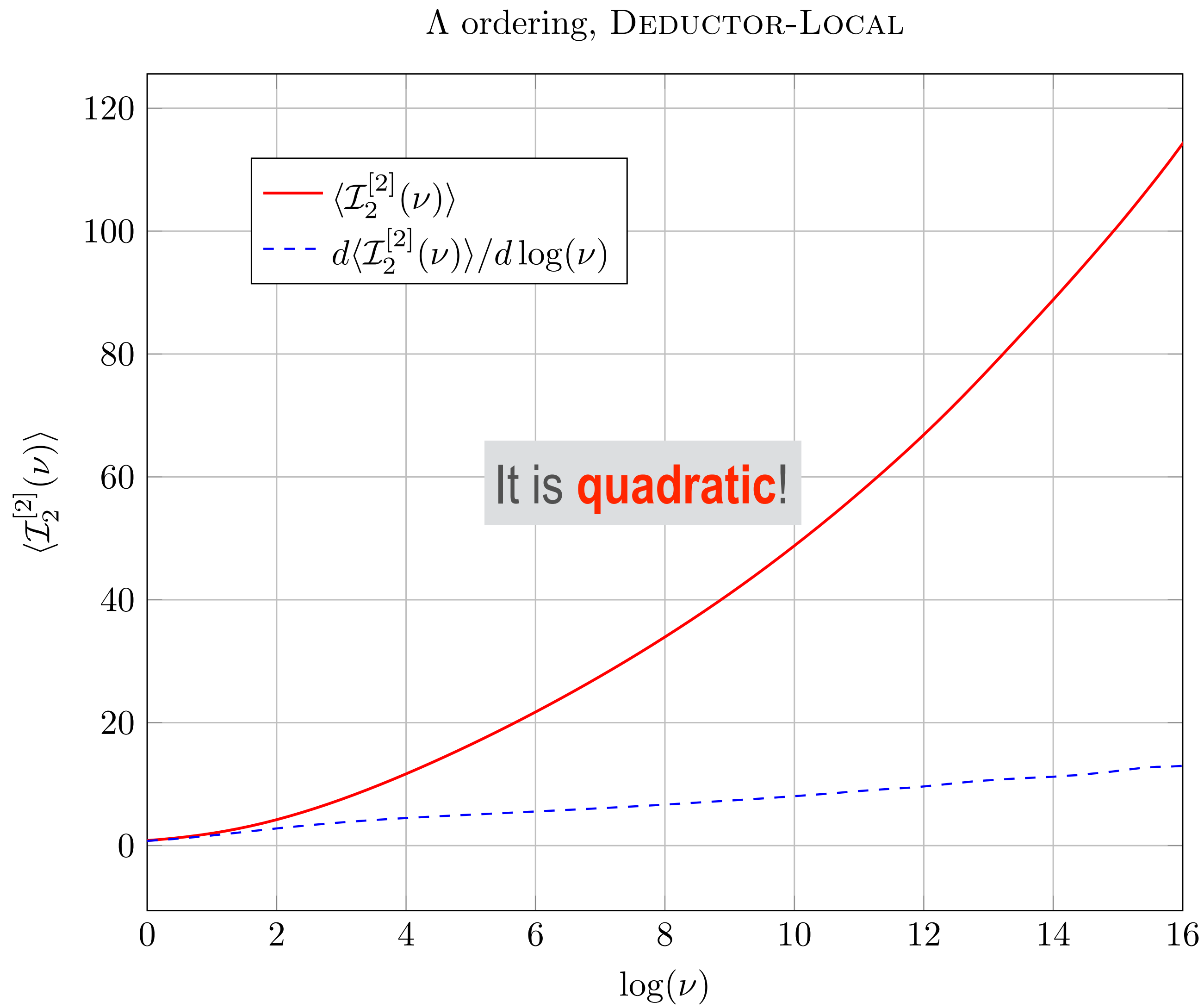
Deductor Λ-ordered (local mapping)

DEDUCTOR Lambda ordered shower

- The ordering variable is the virtuality divided by the mother parton energy

$$\Lambda^2 = \frac{(\hat{p}_l + \hat{p}_{m+1})^2}{2p_l \cdot Q} Q^2$$

- **Local** momentum mapping (Catani-Seymour mapping)
- Proper soft gluon treatment with **full colour** evolution
- **Only LL** accuracy can be achieved.

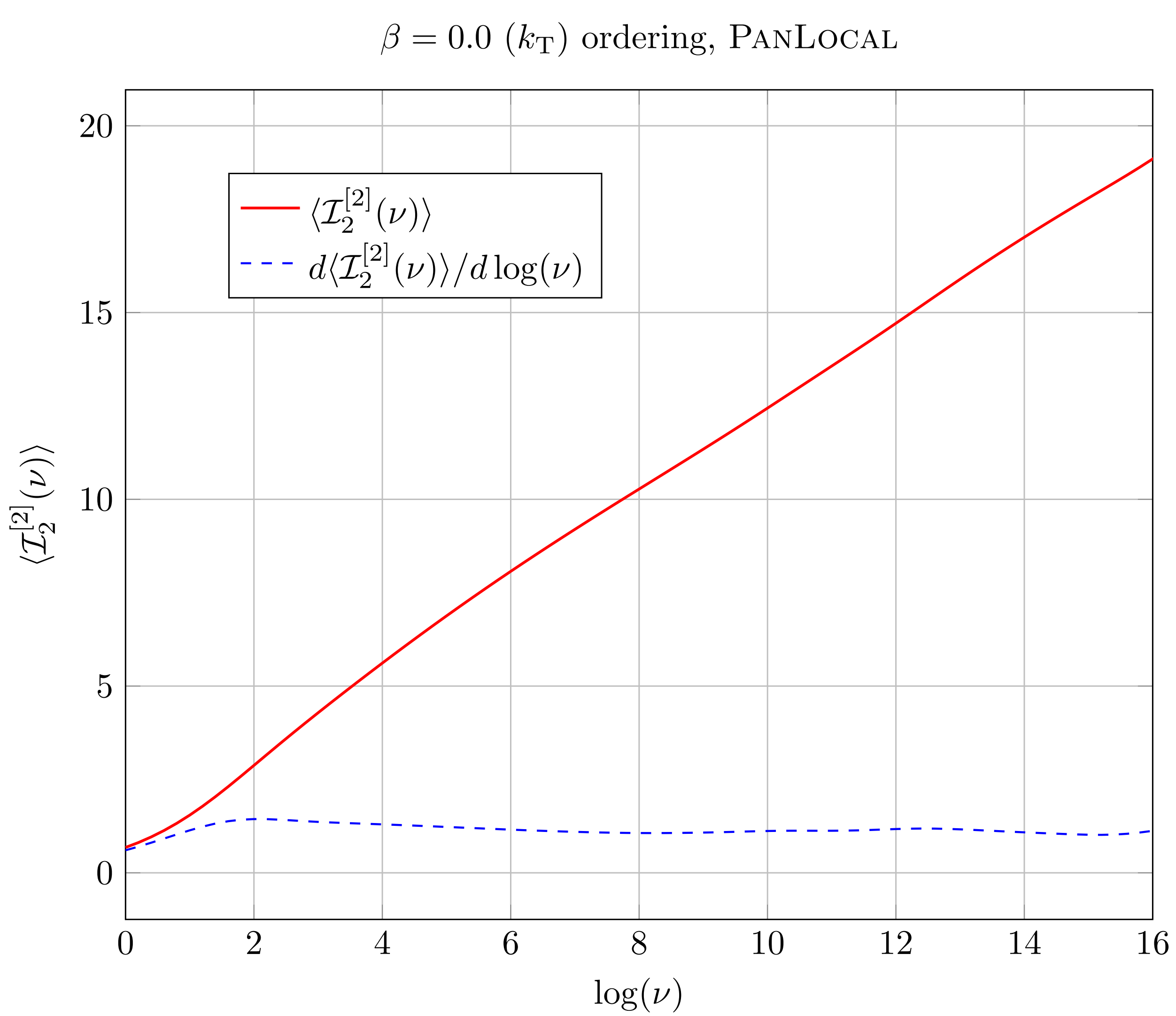


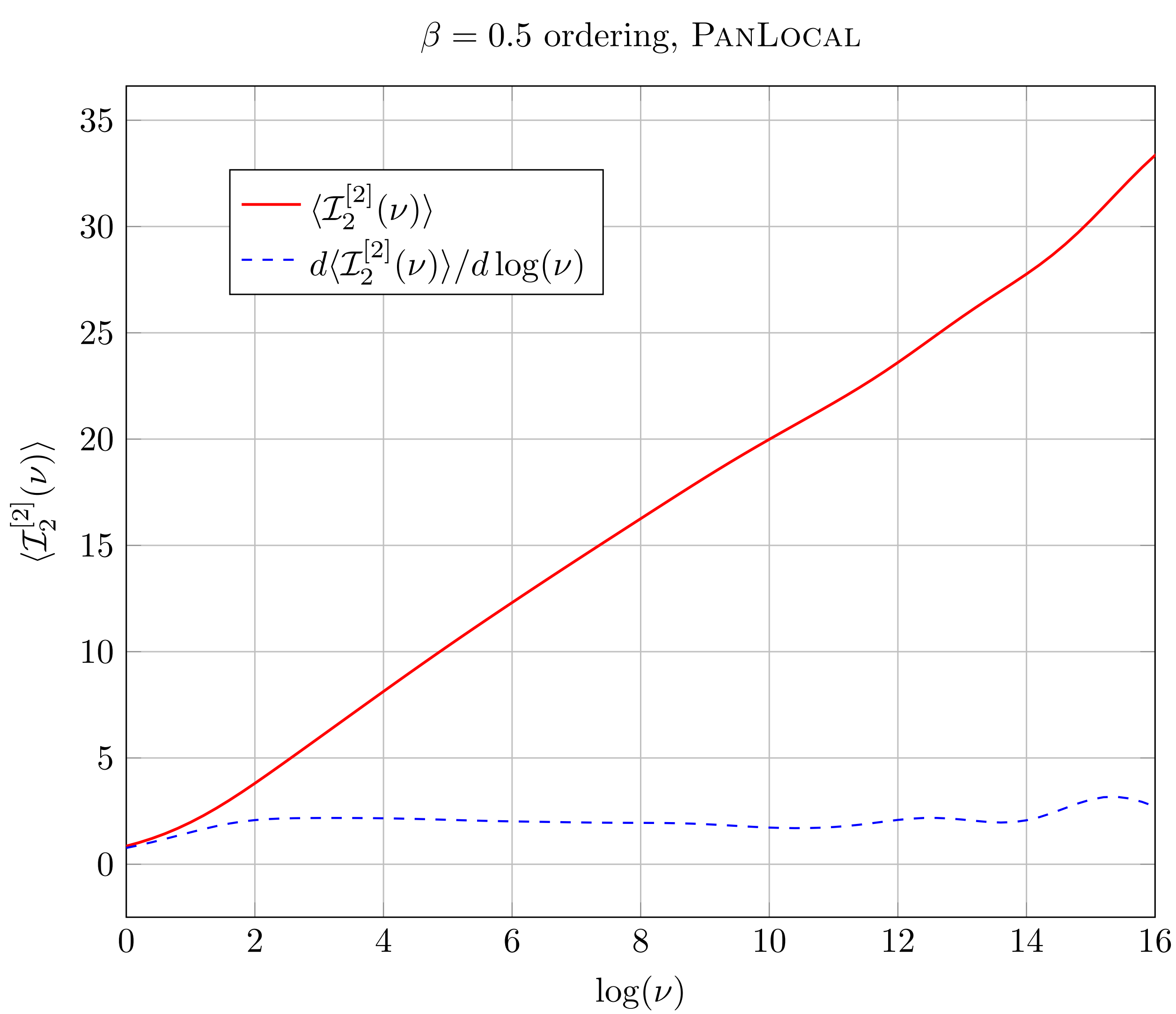
PanLocal Shower with full colour

Dasgupta *et al.*, ***Phys.Rev.Lett.* 125** (2020) 5, 052002

PanLocal shower $\beta = 0, 0.5$

- The ordering variable is transverse momentum based
- **Local** momentum mapping (it is Catani-Seymour mapping)
- Proper soft gluon treatment with **full colour** evolution (this is not in the original definition)
- It works similarly like the **Deductor** kT ordered shower for $\beta = 0, 0.5$, but fails for $\beta = 1$ (only LL accuracy).





DEDUCTOR angular ordered

DEDUCTOR angular ordered shower

- The ordering variable is emission angle
- Deductor's global momentum mapping
- Proper soft gluon treatment with **full colour** evolution
- Even the **LL summation fails**

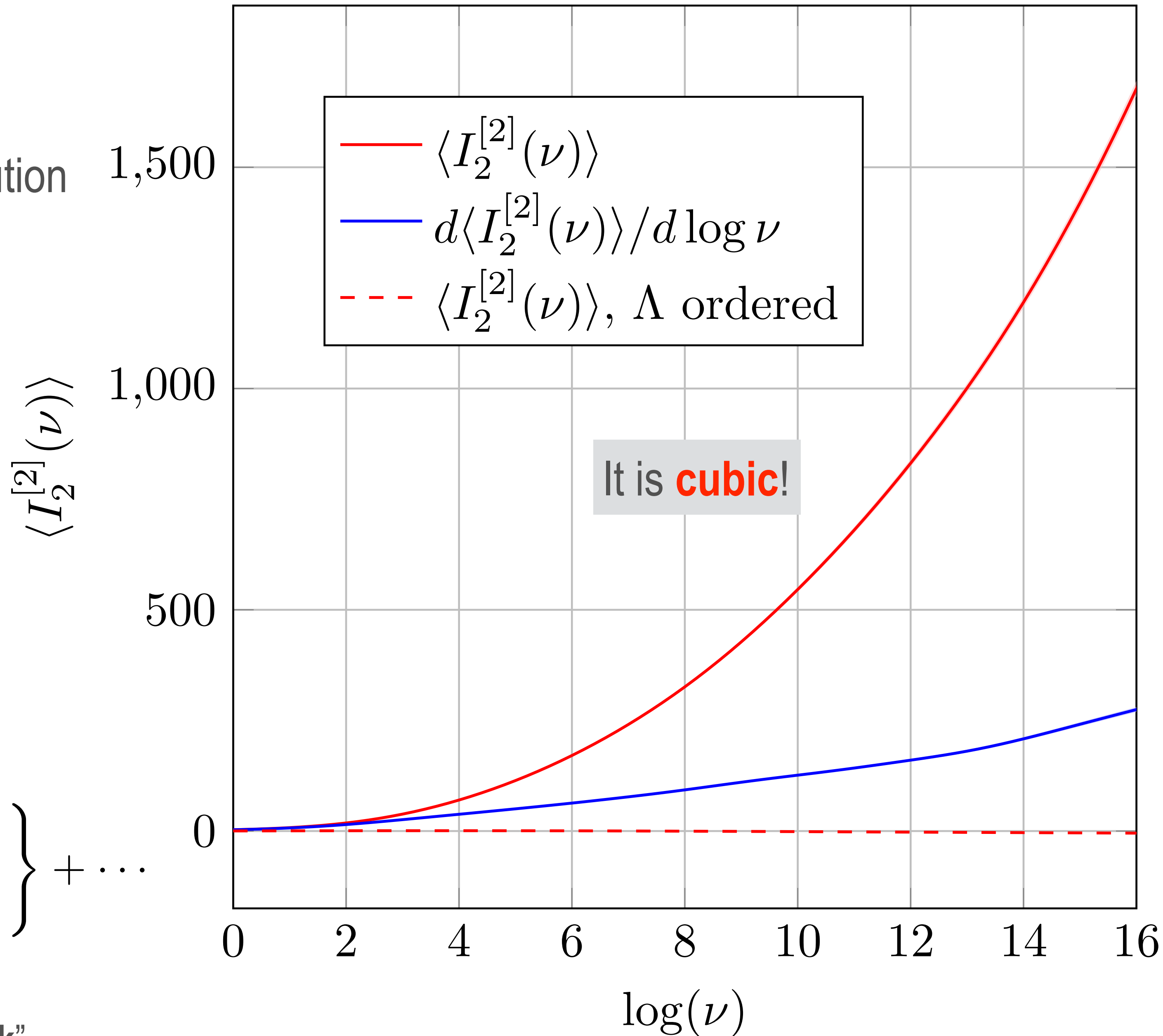
What goes wrong?

This term **agrees** with the analytic result.

$$\frac{\sigma(\boldsymbol{r})}{\sigma_0} = \exp\left\{\int_0^1 \frac{dx}{x}\left(\overbrace{\lambda_{\mathcal{Y}}^{[1]}(x,\nu)} + \underbrace{\sum_{k=2}^{\infty}\lambda_{\mathcal{Y}}^{[k]}(x,\nu)}\right)\right\} + \cdots$$

The shower generated "**junk**" **spoils** even the **LL summation**.

Angular ordering



Conclusion

- General and unified scheme for fixed order and parton shower calculation.
  - After all the parton shower is a lot of linear algebra and renormalisation group.
  - It is important to make clear the difference between **systematical approximation** and "**bending the theory**". (e.g.: LC+ vs. LC)
- We managed to reformulate the shower cross section in such a way to be able to compare with analytical calculations.
- As long as we do all order calculation, all the three approaches lead to the same cross section.
  - Fixed order calculations are truncated in $\alpha_s(\mu^2)$ at **cross section** level.
  - Parton shower formulas are truncated in $\alpha_s(\mu^2)$ in the **shower exponent.**
  - The "shower resummation formulae" is truncated in $\alpha_s(\mu^2)L$ in the "Sudakov" exponent.
- We extensively studied the thrust distribution in e+e- annihilation.
  - We were able to prove analytically the NLL summation property only in lambda ordered **DEDUCTOR**.
  - With other shower schemes we showed numerically that $I^{[2]}(\nu)$ is only a subleading log contribution. We did not say anything about the higher order contributions.

Outlook

- We want to test more observables
  - Jet rates in e+e- annihilation
  - Drell-Yan kT distribution with and without threshold logarithm
  - …
- Check the $I^{[3]}(\nu)$ operator numerically for kT, and PanLocal showers, and/or do the full analytical proof.
- Our shower scheme is still **not general enough**. It cannot accommodate the **angular ordered shower** correctly and systematically.
- In the recent years there have been lots of progress on NNLO fixed order calculations. This is a good base to start to think about **NLO parton shower**. Parton shower is not just "stitched" DGLAP evolutions, beyond the first order it is even more serious linear algebra. *It will be painful…*