On the non-cancellation of IR singularities in collisions with massive quarks

Davide Napoletano, 19/02/2021, CERN TH QCD Coffe

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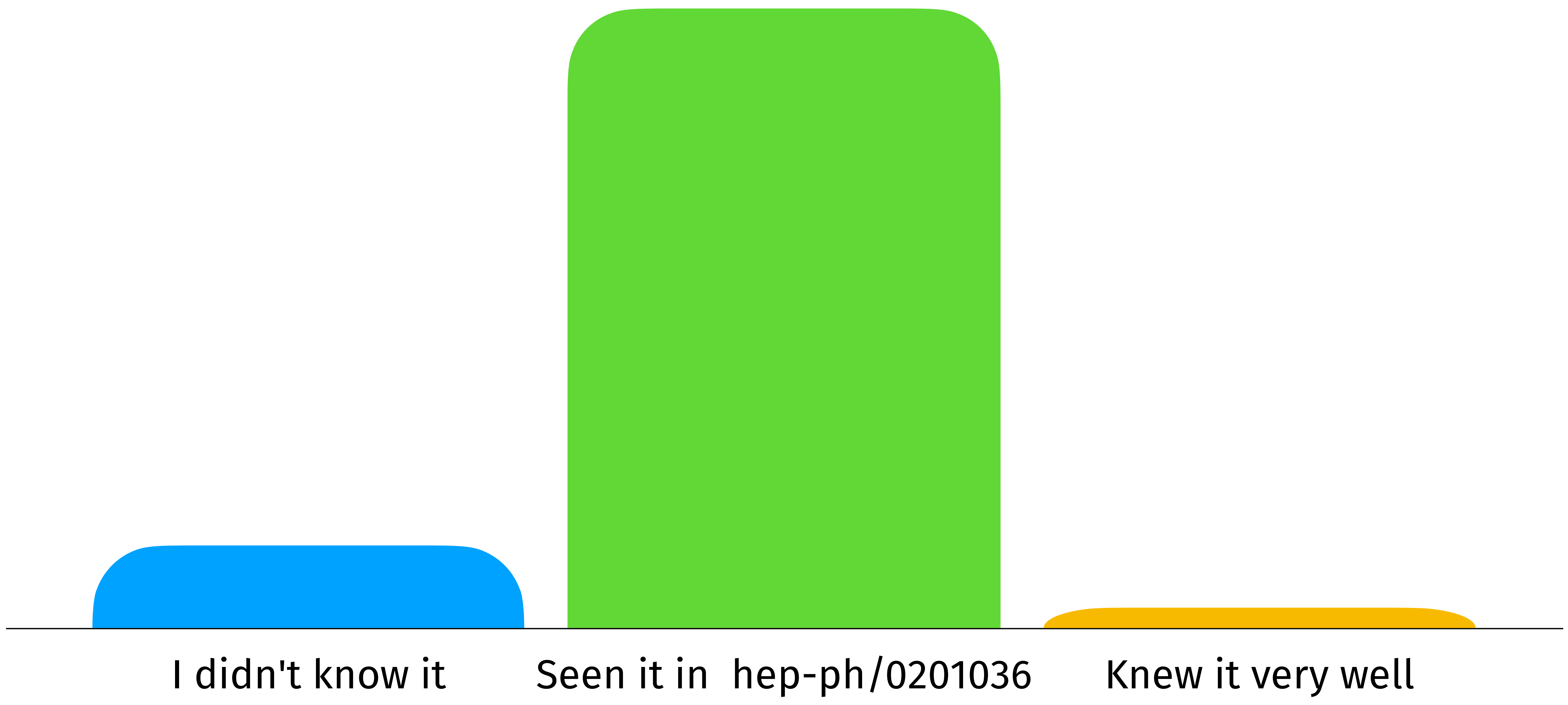
Based on ArXiv: 2011.04701

With Fabrizio Caola, Kirill Melnikov and Lorenzo Tancredi

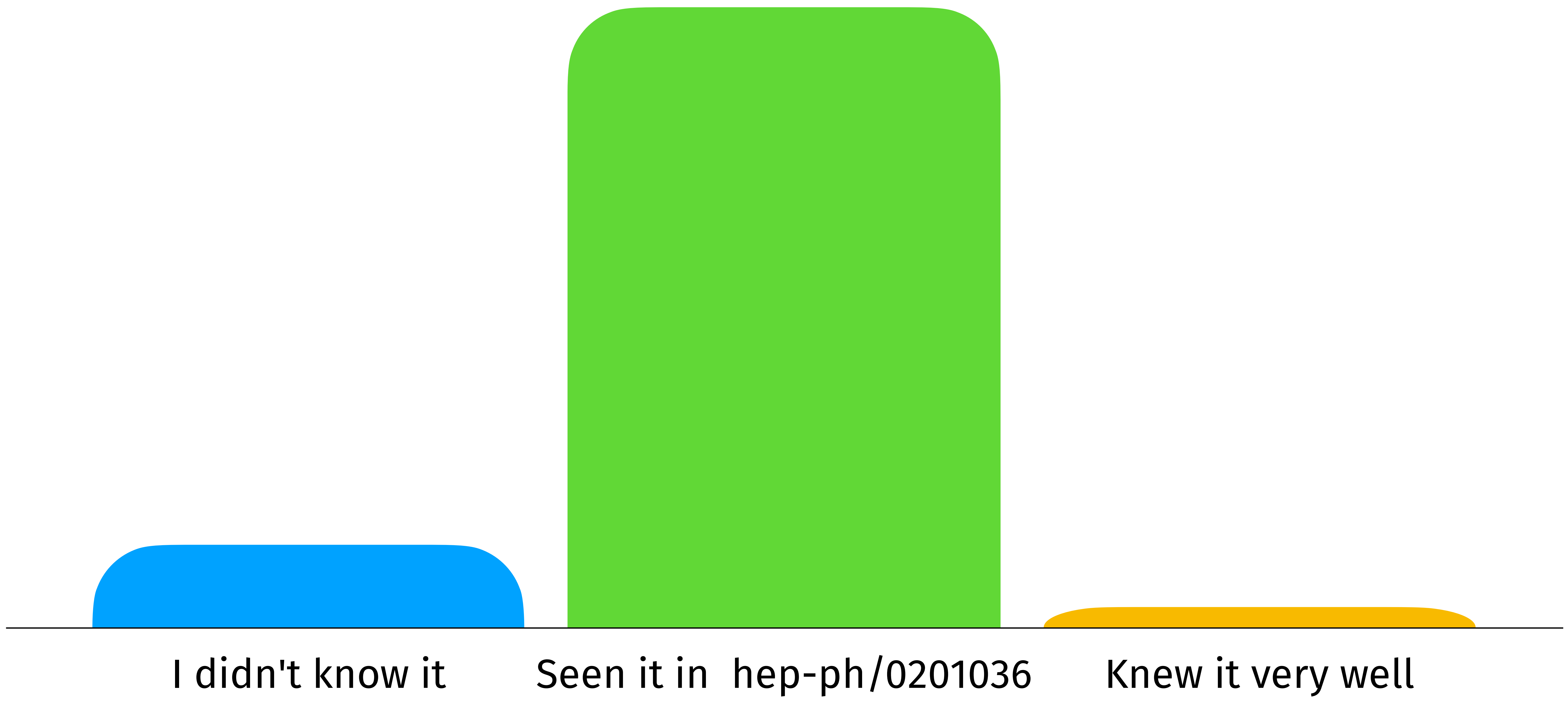
Introduction

- **If I were to play a social experiment…**



Introduction

- **If I were to play a social experiment…**



- **Not your fault!**

Introduction

- **Not your fault, as almost all papers on the subject …**

Counter Example to Nonabelian Bloch-Nordsieck Theorem
R. Doria (Oxford U.), J. Frenkel (Sao Paulo U. and Oxford U.), J.C. Taylor (Oxford U.)
Nucl.Phys.B 168 (1980) 93-110 • DOI: 10.1016/0550-3213(80)90278-3

A Counter Example to the Bloch-Nordsieck Theorem in Nonabelian Gauge Theories
C. Di'Lieto (Imperial Coll., London), S. Gendron (Imperial Coll., London), I.G. Halliday (Imperial Coll., London), Christopher T. Sachrajda (Southampton U.)
Nucl.Phys.B 183 (1981) 223-250 • DOI: 10.1016/0550-3213(81)90554-X

IS QUARK - ANTI-QUARK ANNIHILATION INFRARED SAFE AT HIGH-ENERGY?
J. Frenkel (Sao Paulo U.), J.G.M. Gatheral (Cambridge U.), J.C. Taylor (Cambridge U. and Sao Paulo U.)
Nucl.Phys.B 233 (1984) 307-335 • DOI: 10.1016/0550-3213(84)90418-8

NONCANCELLING INFRARED DIVERGENCES IN QCD COHERENT STATE
S. Catani (Florence U.), M. Ciafaloni (Florence U.), G. Marchesini (Parma U.)
Nucl.Phys.B 264 (1986) 588-620, Florence Univ. - DFF-85-15 (85,REC.AUG.) 56p, Nucl. Phys. B264 (1986) 588-620 and Florence Univ. - DFF-85-15 (85,REC.AUG.) 56p • DOI: 10.1016/0550-3213(86)90500-6

Violation of the Bloch-nordsieck Mechanism in General Nonabelian Theories and {SUSY} {QCD}
Stefano Catani (Florence U. and INFN, Florence)
Z.Phys.C 37 (1988) 357 • DOI: 10.1007/BF01578128

Soft Divergences in Perturbative {QCD}
A. Andrasi (Boskovic Inst., Zagreb and Oxford U.), M. Day (Oxford U.), R. Doria (Oxford U.), J. Frenkel (Sao Paulo U.), J.C. Taylor (Oxford U.)
Nucl.Phys.B 182 (1981) 104-124 • DOI: 10.1016/0550-3213(81)90460-0

Avoidance of Counter Example to Nonabelian Bloch-Nordsieck Conjecture by Using Coherent State Approach
C.A. Nelson (Fermilab)
Nucl.Phys.B 186 (1981) 187-204 • DOI: 10.1016/0550-3213(81)90099-7 • https://lss.fnal.gov/archive/1980/pub/Pub-80-082-T.pdf

Cancellation of Infrared Divergence and Initial Degenerate State in {QCD}
Ikuo Ito (Tokyo Inst. Tech.)
Prog.Theor.Phys. 65 (1981) 1466 • DOI: 10.1143/PTP.65.1466

Diagrammatical Display of the Counter Example to Nonabelian Bloch-nordsieck Conjecture
Nobuo Yoshida (Tokyo U.)
Prog.Theor.Phys. 66 (1981) 269 • DOI: 10.1143/PTP.66.269

Introduction

- **Plus they are either very obscure (but general)**

  Catani, Z.Phys.C 37 (1988) 357

Neither G'_μ nor G''_μ contribute to (3.9): G'_μ because it vanishes [11] after angular integration over $\hat{\mathbf{q}}$ and G''_μ due to its orthogonality to j^μ_{12}.

Equation (3.11) shows that the non-cancelling effective potential V_R is IR finite and consequently the violation of the Bloch–Nordsieck mechanism is an IR subleading effect (i.e. of order $g^4 \log E/\lambda$) [8].

Given the kernels (3.11), it is straightforward to evaluate (3.9). One gets (Fig. 1)

$$\bar{\sigma} - \mathrm{Tr}\,\mathscr{M} = -\frac{1}{(4\pi)^2} C_A \log\frac{E}{\lambda} \sum_{j \neq k} \mathrm{Tr}(\mathscr{M} T^a_j T^a_k) \cdot (g_v^2 \Delta_v(v_{jk}) + g_y^2 \Delta_s(v_{jk}) - g_v^2)(g_v^2 F_v(v_{jk}) + g_y^2 F_s(v_{jk})), \tag{3.14}$$

where F_v and F_s are the vector and scalar bremsstrahlung functions of the relative velocity v_{jk}

$$2\pi^2 \left(\log\frac{E}{\lambda}\right)^{-1} \int_\lambda^E [dq]\, j^s_{12}(q) j^s_{12}(q) = F_s(v_{12}),$$

$$F_s(v) = 1 - \frac{\sqrt{1-v^2}}{2v} \log\frac{1+v}{1-v}, \tag{3.15}$$

$$-2\pi^2 \left(\log\frac{E}{\lambda}\right)^{-1} \int_\lambda^E [dq]\, j^\mu_{12}(q) j^\mu_{12}(q) = F_v(v_{12}),$$

$$F_v(v) = \frac{1}{2v} \log\frac{1+v}{1-v} - 1. \tag{3.16}$$

Equation (3.14) gives the magnitude of the two-loop Bloch–Nordsieck violating term for a general non abelian theory involving a vector and a scalar gauge bosons.

- **Or perform a direct calculation (still being a bit obscure)**

  Doria, Frenkel, Taylor Nucl.Phys.B 168 (1980) 93-110

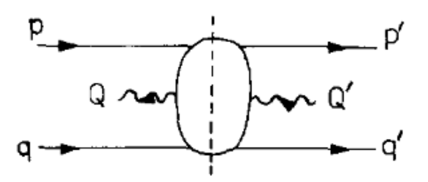


Fig. 2. The 'hard' amplitude quark + quark + γ^* → quark + quark + γ^*. The solid lines represent quarks, and the wavy lines photons. The 'blob' denotes the hard interaction, and the dashed line through the blob indicates the discontinuity is to be taken. In all subsequent diagrams the photons are not shown, and the 'cut' hard amplitude is represented by a small circle.

We remark that the final answer is much simpler than we have any right to expect from our technique. We have a strong suspicion that some more powerful cancellation technique, which is more faithful to the group structure, must exist.

Since we are summing over all possible intermediate states the inclusive cross section is given by Mueller's theorem [5],

$$\text{flux}\ \frac{d\sigma}{d^3Q} = \frac{\Delta_{\mathbb{S}}}{2i} A(S_L^+, \mathbb{S}, S_R^-), \tag{2.1}$$

where A is the forward scattering amplitude for quark + quark + virtual photon,

Introduction

- These results are nevertheless very important
- All of our calculations rely on QCD Factorisation
- As such it is important to understand its current limitations

Introduction

Introduction

KLN (Theorem) vs Bloch and Nordsieck (Conjecture)

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Divergences cancel if inclusive enough

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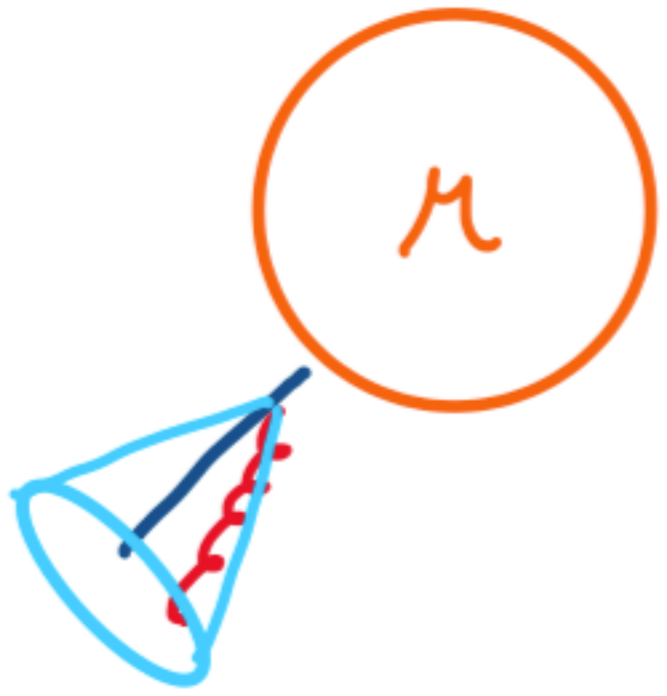
KLN (Theorem) vs Bloch and Nordsieck (Conjecture)

Divergences cancel if inclusive enough

In the IS, require Coherent states



Introduction

KLN (Theorem) vs Bloch and Nordsieck (Conjecture)

Divergences cancel if inclusive enough

Divergences cancel inclusively in the FS

In the IS, require Coherent states

Introduction

- **Why Bother if corrections scale like** $\mathcal{O}\left(\frac{m_q^2}{Q^2}\right)$ **?**

Introduction

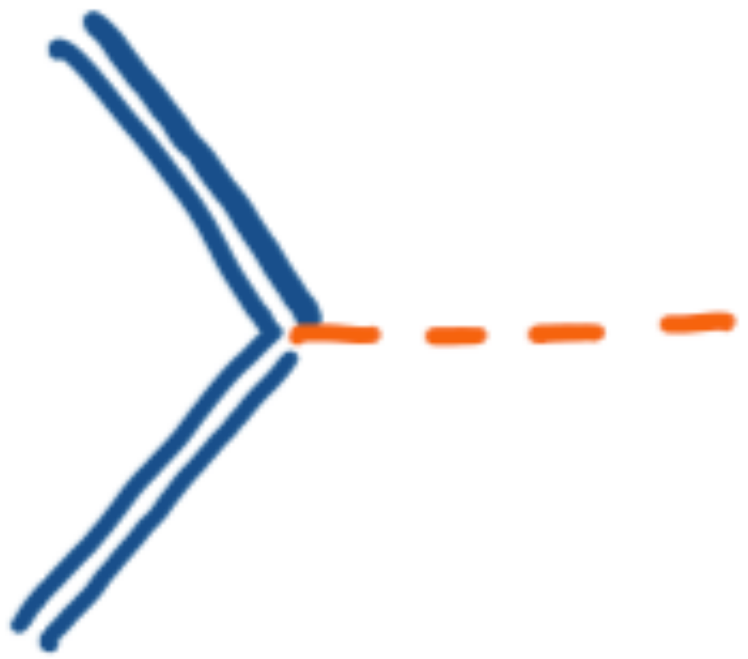
- **Why Bother if corrections scale like** $\mathcal{O}\left(\frac{m_q^2}{Q^2}\right)$ **?**

- **well, they don't! Even for inclusive observable they scale as** $\mathcal{O}\left(\frac{m_q^2}{x1\ x2\ S}\right)$

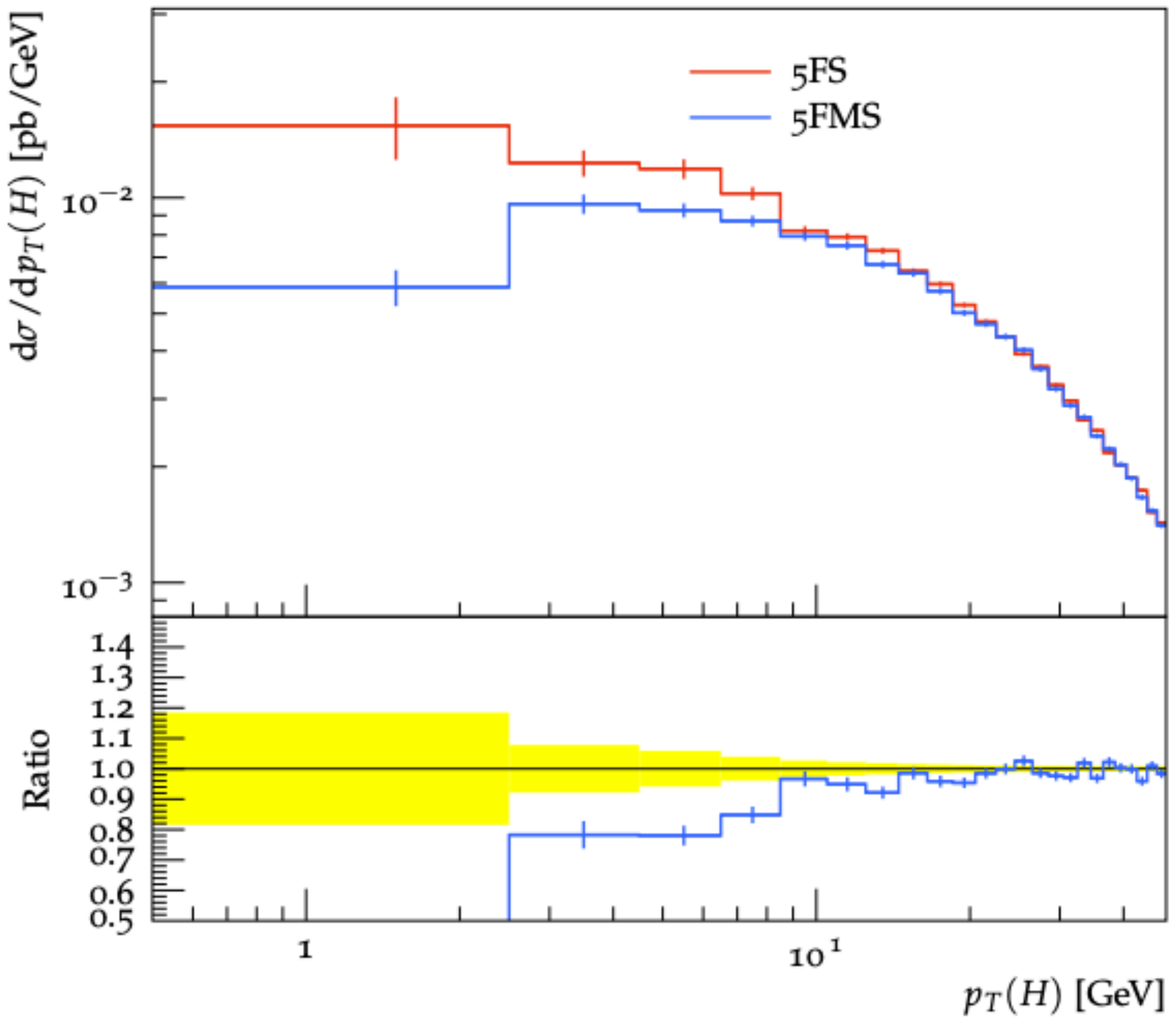
Introduction

- **Why Bother if corrections scale like** $\mathcal{O}\left(\frac{m_q^2}{Q^2}\right)$ **?**

- **well, they don't! For exclusive observable the situation is even worse**

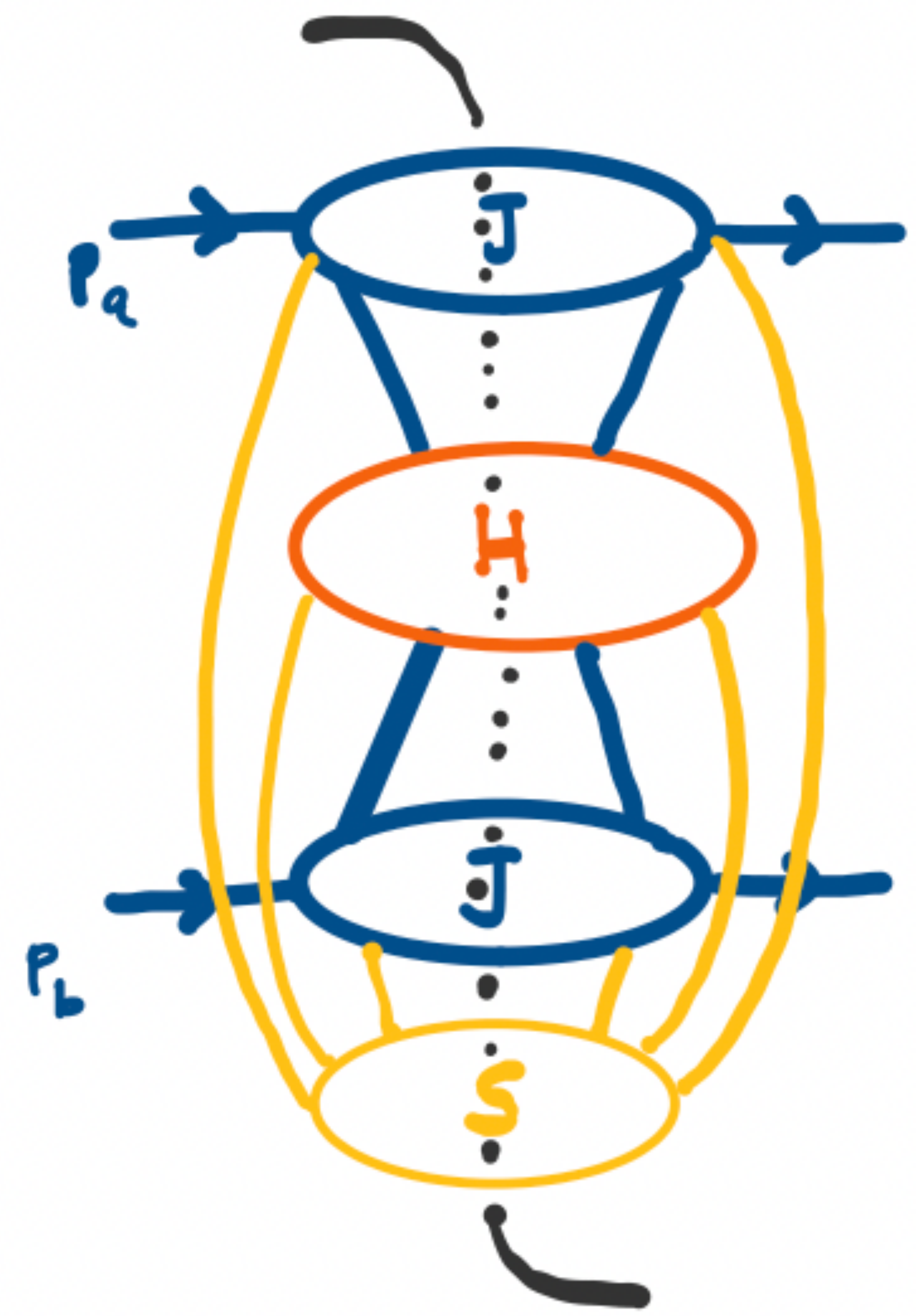


@ NLO



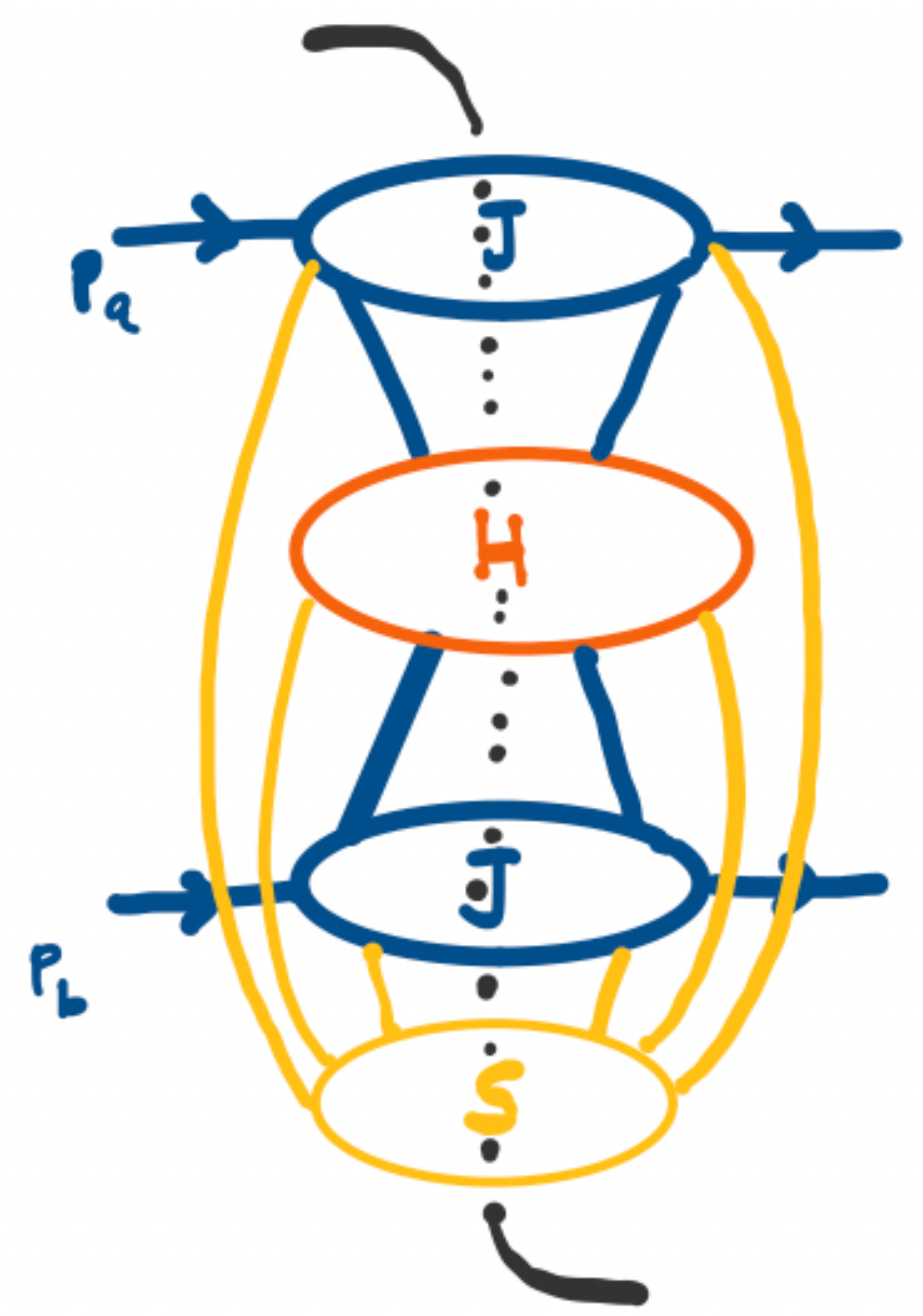
Introduction

- **Before I start, a little digression on Factorisation ⇔ Cancellation of IR Singularities:**



Introduction

- One of the main assumptions: each block may be divergent, but it cancel in the sum

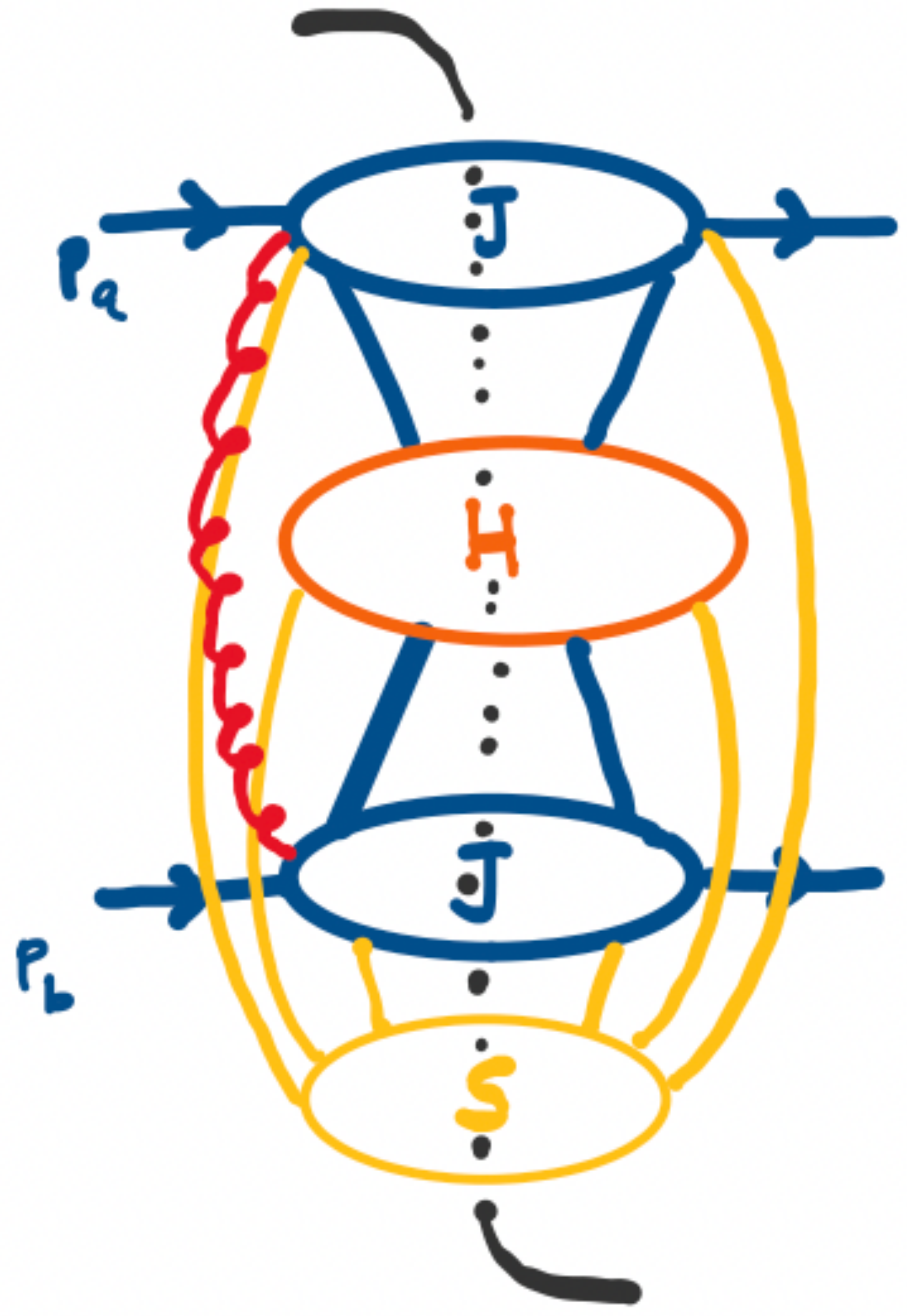


$$\Sigma_a^J + \Sigma_b^J + \Sigma^S + \Sigma^H = 0$$

(Σ = Singularities of...)

Introduction

- **Extra soft gluons connecting IS particles, if divergent and non cancelling, spoil Factorisation...**



$$\Sigma_a^J + \Sigma_b^J + \Sigma^S + \Sigma^H \neq 0$$

$$\Sigma_a^J + \Sigma_b^J + \Sigma^S + \Sigma^H + \Sigma^{\#} \stackrel{?}{=} 0$$

IR Finiteness: NLO

IR Finiteness: NLO Analysis

- **Given a process** $$d\sigma = d\sigma_{\mathrm{LO}} + [d\sigma_{\mathrm{V}} + d\sigma_{\mathrm{R}}] + \ldots$$

- **We can compute virtual and Real corrections**

$$d\sigma_{\mathrm{V}} = \frac{\alpha_s(\mu)}{2\pi}\left\{-\frac{2C_F}{\epsilon}\left[\frac{1}{2v}\ln\left(\frac{1-v}{1+v}\right)+1\right]\right\} d\sigma_{\mathrm{LO}} + d\sigma_{\mathrm{V,fin}}$$

IR Finiteness: NLO Analysis

- **We are only interested in the divergent part of real corrections**

$$\mathrm{d}\sigma_{\mathrm{R}} = \frac{1}{4J} \int \frac{\mathrm{d}E_g}{E_g^{1+2\epsilon}} \frac{\mathrm{d}\Omega_g^{(d-1)}}{2(2\pi)^{d-1}} F_g^{(d)} \left(p_1, p_2, p_V; p_g\right)$$

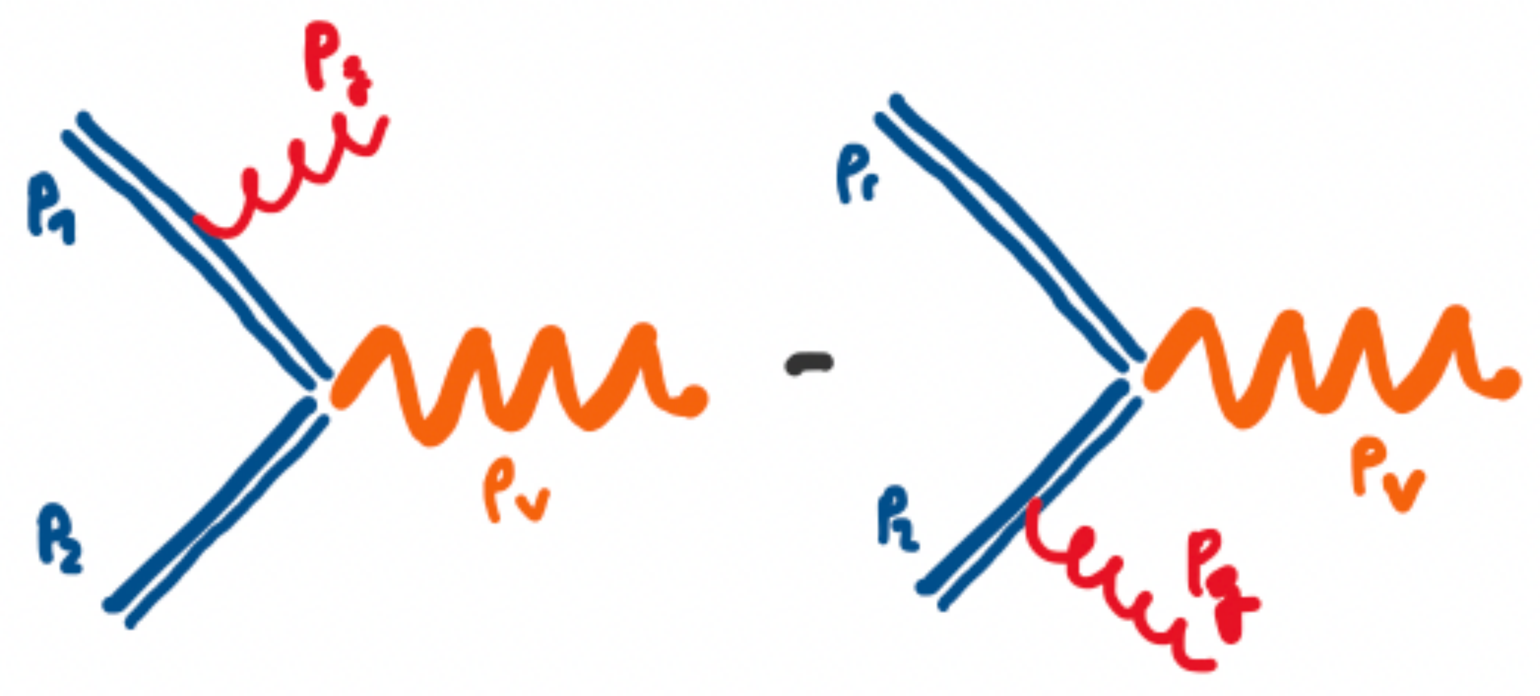
$$\begin{aligned}\mathrm{d}\sigma_{\mathrm{R}} &= \int_0^{\mathrm{E_{max}}} \frac{\mathrm{d}E_g}{E_g^{1+2\epsilon}} \frac{\mathrm{d}\Omega_g^{(3)}}{16\pi^3} \lim_{E_g \to 0} \left[F_g^{(4)} \left(p_1, p_2, p_V; p_g\right)\right] + \mathrm{d}\sigma_{\mathrm{R}}^{\mathrm{fin}} = \mathrm{d}\sigma_{\mathrm{R}}^{\mathrm{div}} + \ldots \\ &= -\frac{1}{2\epsilon} \int \frac{\mathrm{d}\Omega_g^{(3)}}{16\pi^3} \lim_{E_g \to 0} \left[F_g^{(4)} \left(p_1, p_2, p_V; p_g\right)\right] + \ldots\end{aligned}$$

$$\mathrm{d}\sigma^{\mathrm{div}} = \mathrm{Eik}_0 \times \mathrm{d}\sigma_{\mathrm{LO}}$$

$$\mathrm{Eik}_0 = -\frac{\alpha_s(\mu)}{2\pi} \frac{C_F}{\epsilon} \int \frac{\mathrm{d}\Omega_{3,g}}{4\pi} E_g^2 \left[\frac{2\,(p_1 \cdot p_2)}{(p_1 \cdot p_g)\,(p_2 \cdot p_g)} - \frac{m_q^2}{(p_1 \cdot p_g)^2} - \frac{m_q^2}{(p_2 \cdot p_g)^2}\right]$$

IR Finiteness: NLO Analysis

- **Given a process, we can compute the relevant virtual and real corrections**



$$d\sigma_V = \frac{\alpha_s(\mu)}{2\pi}\left\{-\frac{2C_F}{\epsilon}\left[\frac{1}{2v}\ln\left(\frac{1-v}{1+v}\right)+1\right]\right\}d\sigma_{LO} + d\sigma_{V,\mathrm{fin}}$$

$$d\sigma_R^{\mathrm{div}} = \frac{\alpha_s(\mu)}{2\pi}\times\frac{2C_F}{\epsilon}\left[\frac{1+\beta^2}{2\beta}\ln\left(\frac{1-\beta}{1+\beta}\right)+1\right]d\sigma_{LO}$$

$$v = \frac{2\beta}{1+\beta^2}$$

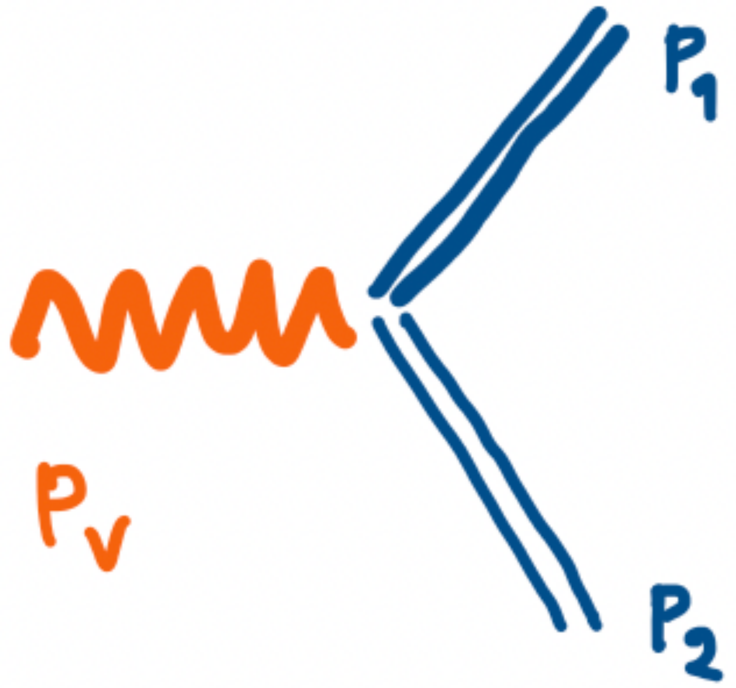
$$d\sigma = d\sigma_{LO} + [\mathrm{finite}] + \ldots$$

IR Finiteness: NLO Part 2

- **This derivation is already cumbersome and intrasparent at NLO**

- **We can get a broader perspective if we take a different path:**

$$\mathrm{d}\sigma_{\mathrm{NLO}}^{\mathrm{decay}} = \mathrm{d}\sigma_{V}^{\mathrm{decay}} + \mathrm{d}\sigma_{R}^{\mathrm{decay}} \qquad\qquad \mathrm{d}\sigma_{\mathrm{NLO}}^{\mathrm{prod}} = \mathrm{d}\sigma_{V}^{\mathrm{prod}} + \mathrm{d}\sigma_{R}^{\mathrm{prod}}$$



$$\{p_V, p_1, p_2\} \leftrightarrow \{-p_V, -p_1, -p_2\}$$

IR Finiteness: NLO Part 2 (V)

- **For this need to look at the analytical structure of virtual corrections:**

(Catani et al. hep-ph/0011222)

$$d\sigma_V^{\text{decay}} = \boldsymbol{I}_m^{\text{RS}} \times d\sigma_{\text{LO}}^{\text{decay}} + \ldots$$

$$\boldsymbol{I}_m^{\text{RS}}\left(\epsilon, \mu^2; \{p_i, m_i\}\right) = \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left\{ q\frac{1}{2}\left(\frac{\beta_0}{\epsilon} - \tilde{\beta}_0^{\text{R.S}}\right) \right.$$
$$+ \sum_{\substack{j,k=1 \\ k\neq j}}^{m} \boldsymbol{T}_j \cdot \boldsymbol{T}_k \left(\frac{\mu^2}{|s_{jk}|}\right)^\epsilon \left[\mathcal{V}_{jk}^{(\text{cc})}\left(s_{jk}; m_j, m_k; \epsilon\right) + \frac{1}{v_{jk}}\left(\frac{1}{\epsilon}\mathrm{i}\pi - \frac{\pi^2}{2}\right)\Theta\left(s_{jk}\right)\right]$$
$$\left. - \sum_{j=1}^{m} \Gamma_j^{\text{R.S}}\left(\mu, m_j; \epsilon\right) \right\}$$

which is symmetric under the simultaneous exchange of the initial states

$$d\sigma_V^{\text{decay}} = d\sigma_V^{\text{prod}} + \ldots$$

IR Finiteness: NLO Part 2 (R)

- **The real emission case is even simpler, in fact the Eikonal term**

$$J_\mu^{i,a,(0)} = T_i^a \frac{p_{i,\mu}}{p_i \cdot p_g}$$

Is homogeneous in the hard momenta, and so must be Eik_0

$$\mathrm{d}\sigma_{\mathrm{R}}^{\mathrm{decay}} = \mathrm{Eik}_0 \times \mathrm{d}\sigma_{\mathrm{LO}}^{\mathrm{decay}} = \mathrm{d}\sigma_{\mathrm{R}}^{\mathrm{prod}} + \cdots$$

IR Finiteness: NLO Part 2 (V+R)

- **Thus, the NLO xs is IR finite**

$$d\sigma = d\sigma_{\mathrm{LO}} + [\text{finite}] + \ldots$$

IR Finiteness: NNLO

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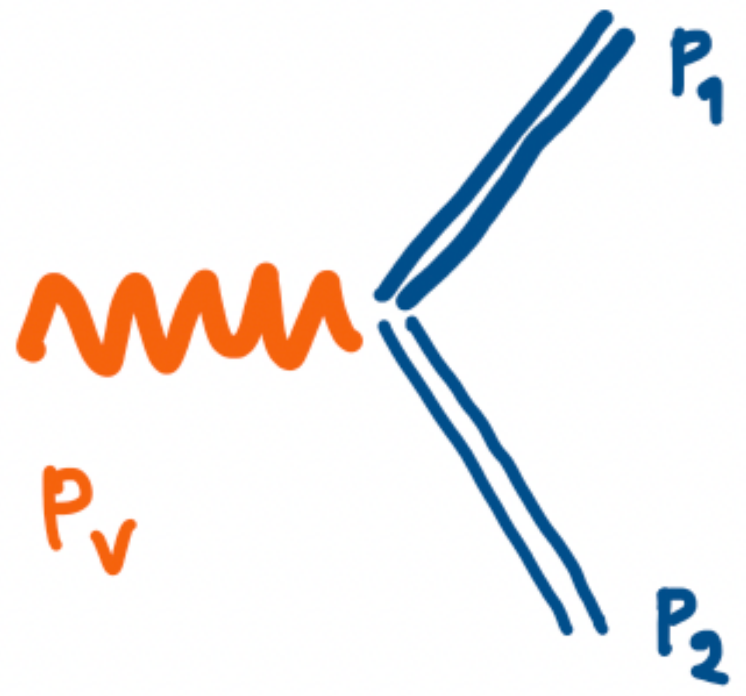
- **We can easily extend the argument at NNLO:**

$$d\sigma^{\text{decay}}_{\text{NNLO}} = d\sigma^{\text{decay}}_{\text{VV}} + d\sigma^{\text{decay}}_{\text{RR}} + d\sigma^{\text{decay}}_{\text{RV}}$$

$$d\sigma^{\text{prod}}_{\text{NNLO}} = d\sigma^{\text{prod}}_{\text{VV}} + d\sigma^{\text{prod}}_{\text{RR}} + d\sigma^{\text{prod}}_{\text{RV}}$$

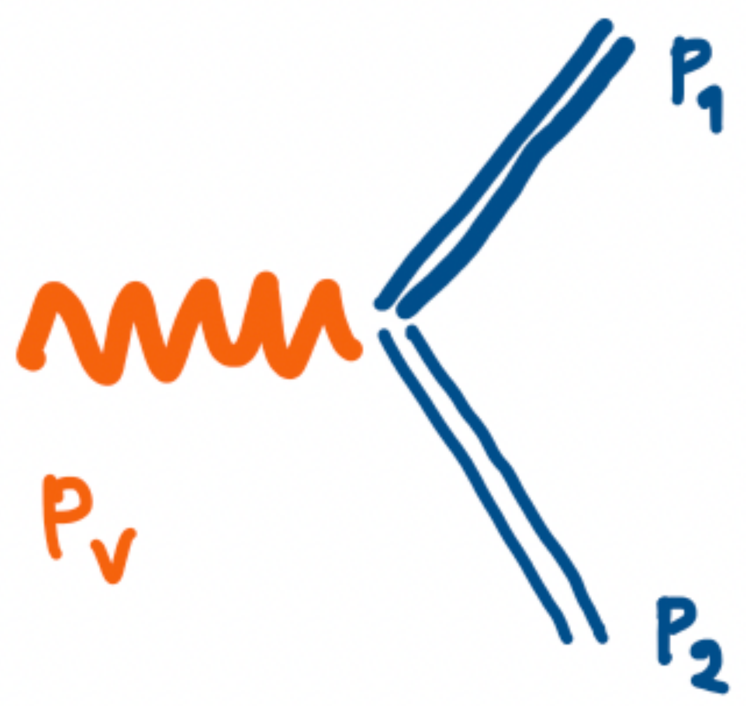


$$\{p_V, p_1, p_2\} \leftrightarrow \{-p_V, -p_1, -p_2\}$$

IR Finiteness: NNLO

- **We can easily extend the argument at NNLO:**

$$d\sigma^{\text{decay}}_{\text{NNLO}} = d\sigma^{\text{decay}}_{\text{VV}} + d\sigma^{\text{decay}}_{\text{RR}} + d\sigma^{\text{decay}}_{\text{RV}} \qquad d\sigma^{\text{prod}}_{\text{NNLO}} = d\sigma^{\text{prod}}_{\text{VV}} + d\sigma^{\text{prod}}_{\text{RR}} + d\sigma^{\text{prod}}_{\text{RV}}$$



$$\{p_V, p_1, p_2\} \leftrightarrow \{-p_V, -p_1, -p_2\}$$

The argument here is only strictly valid for the coulourless FS!

IR Finiteness: NNLO (VV)

- Double virtual corrections behave similarly to the NLO case:
  It's a form factor that has a simple dependence on the two hard momenta,
  and as such it's the same for both decay and production.

(Becher, Neubert hep-ph/0904.1021)

(Mitov et al. hep-ph/0903.3241)

IR Finiteness: NNLO (RR)

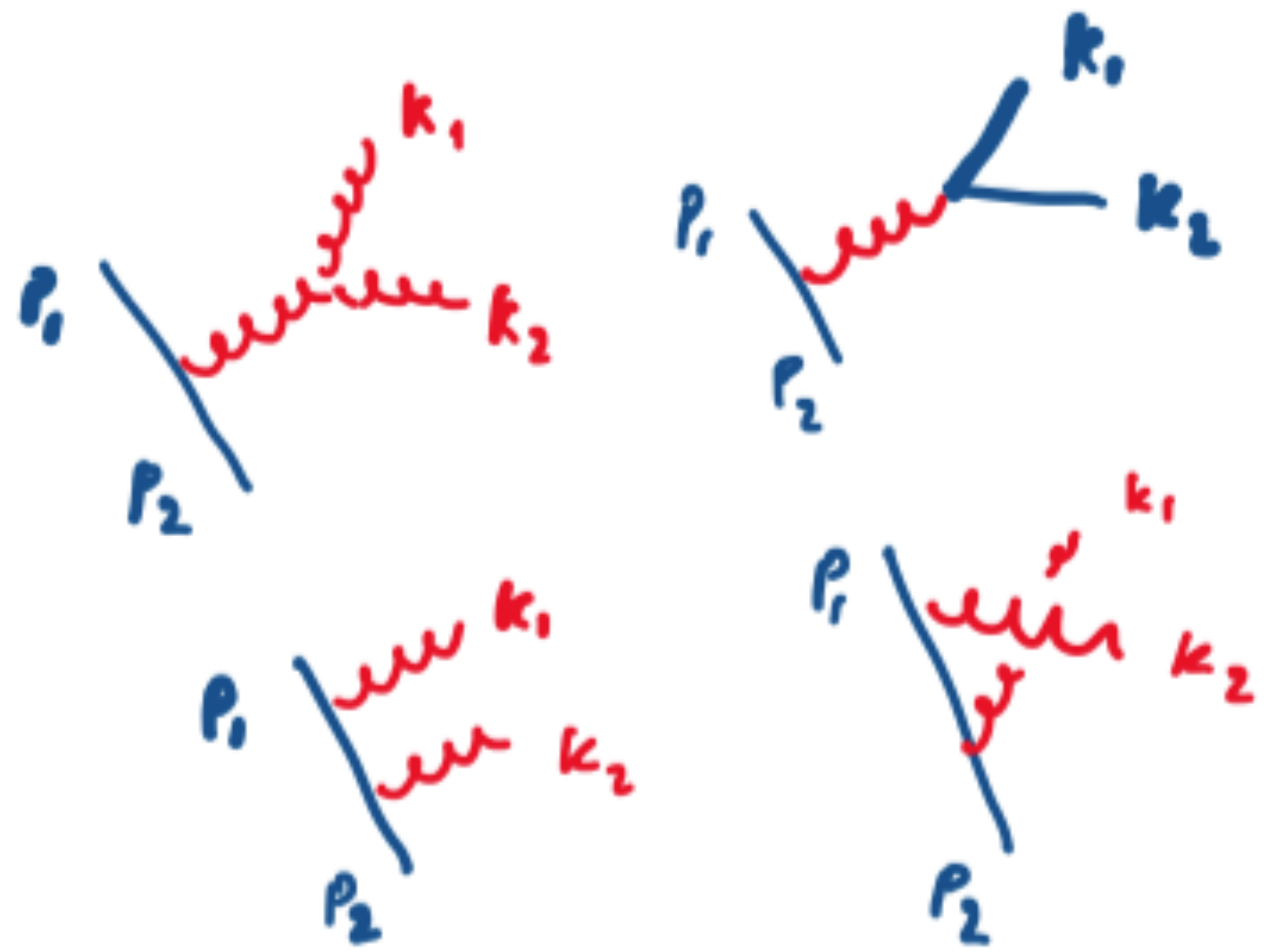
- **Double Real corrections are described by either one soft gluon:**

$$J_\mu^{i,a,(0)} = T_i^a \frac{p_{i,\mu}}{p_i \cdot p_g}$$

IR Finiteness: NNLO (RR)

- **Or by the emission of two soft partons**

$$J^{\mu_1\mu_2}_{a_1 a_2}(p_1,p_2) = \frac{1}{2}\left\{J^{\mu_1}_{a_1}(p_1), J^{\mu_2}_{a_2}(p_2)\right\}$$
$$+ \quad i f_{a_1 a_2 a} \sum_{i=1}^{n} T_i^a \left\{ \frac{k_i^{\mu_1} p_1^{\mu_2} - k_i^{\mu_2} p_2^{\mu_1}}{(p_1\cdot p_2)[k_i\cdot(p_1+p_2)]} - \frac{k_i\cdot(p_1-p_2)}{2[k_i\cdot(p_1+p_2)]}\left[\frac{k_i^{\mu_1}k_i^{\mu_2}}{(k_i\cdot p_1)(k_i\cdot p_2)} + \frac{g^{\mu_1\mu_2}}{p_1\cdot p_2}\right]\right\}$$



IR Finiteness: NNLO (RR)

- **Double Real corrections are described by either one soft gluon:**

$$J_{\mu}^{i,a,(0)} = T_i^a \, \frac{p_{i,\mu}}{p_i \cdot p_g}$$

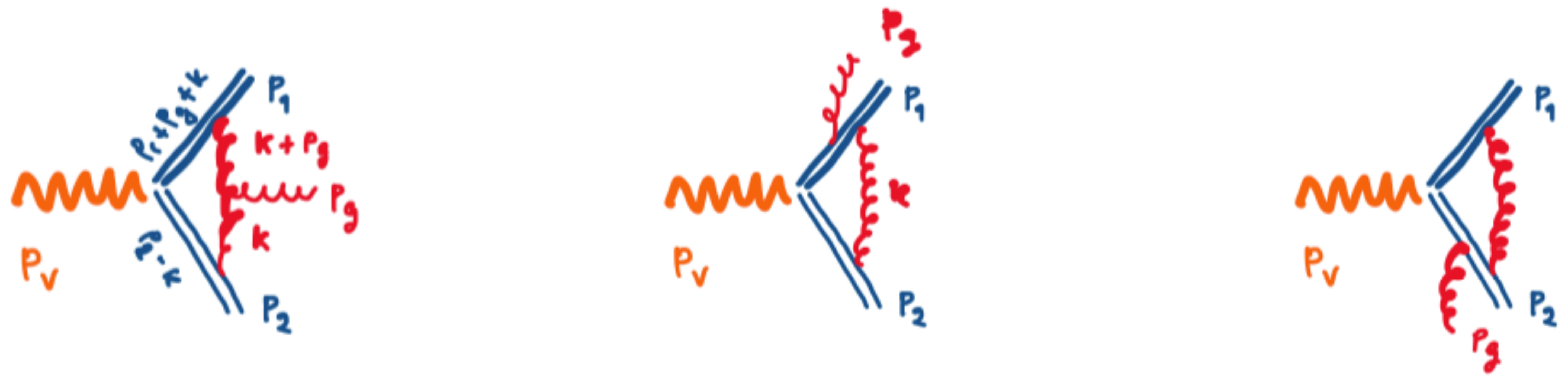
- **Or by the emission of two soft partons**

$$J_{a_1 a_2}^{\mu_1 \mu_2}(p_1, p_2) = \frac{1}{2}\left\{J_{a_1}^{\mu_1}(p_1), J_{a_2}^{\mu_2}(p_2)\right\}$$
$$+ \quad i f_{a_1 a_2 a} \sum_{i=1}^{n} T_i^a \left\{ \frac{k_i^{\mu_1} p_1^{\mu_2} - k_i^{\mu_2} p_2^{\mu_1}}{(p_1 \cdot p_2)[k_i \cdot (p_1 + p_2)]} - \frac{k_i \cdot (p_1 - p_2)}{2[k_i \cdot (p_1 + p_2)]} \left[\frac{k_i^{\mu_1} k_i^{\mu_2}}{(k_i \cdot p_1)(k_i \cdot p_2)} + \frac{g^{\mu_1 \mu_2}}{p_1 \cdot p_2} \right] \right\}$$

Both these functions are again homogeneous in the external momenta

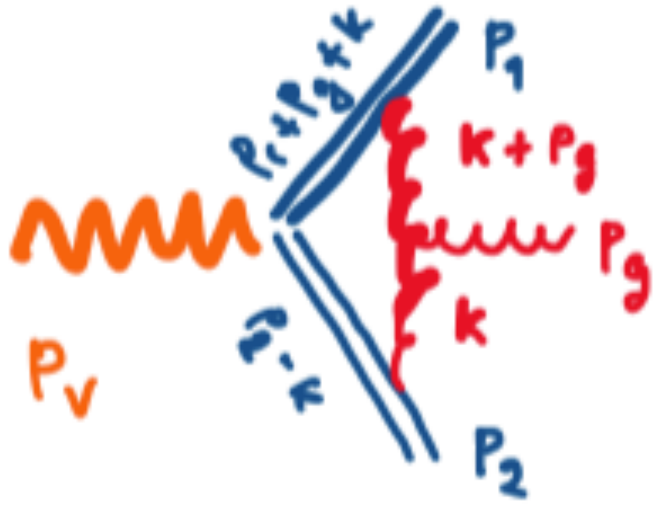
IR Finiteness: NNLO (RV)

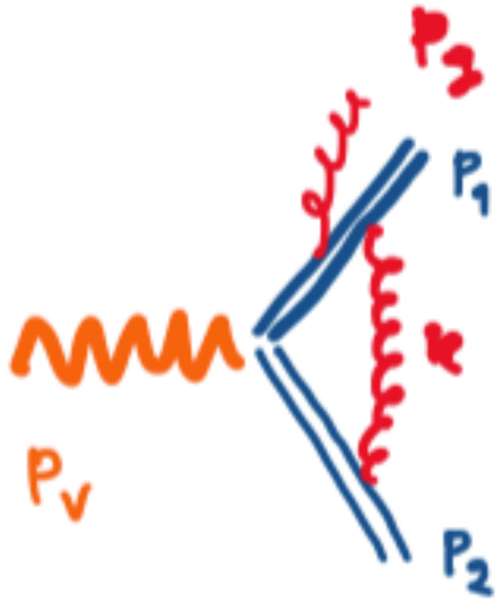
- Real Virtual corrections have both explicit $\frac{1}{\epsilon}$ and phase-space divergences

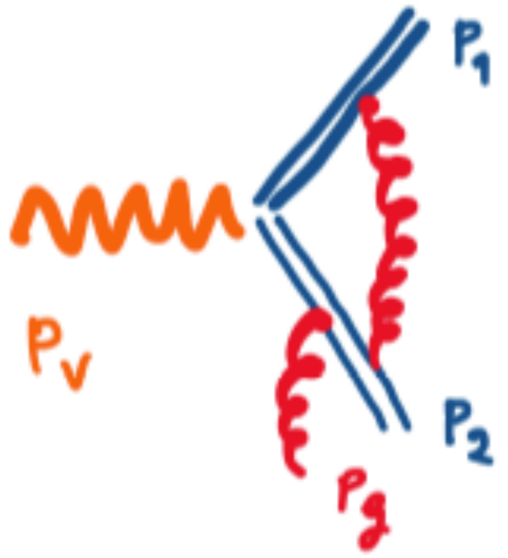


IR Finiteness: NNLO (RV)

- **Real Virtual corrections have both explicit $\frac{1}{\epsilon}$ and phase-space divergences**

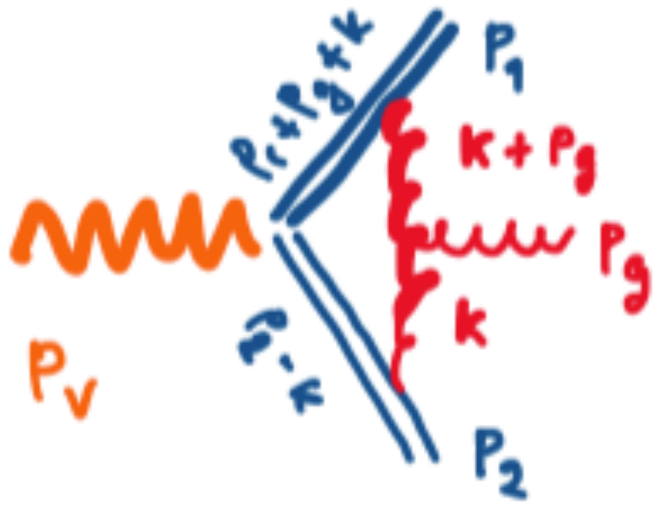


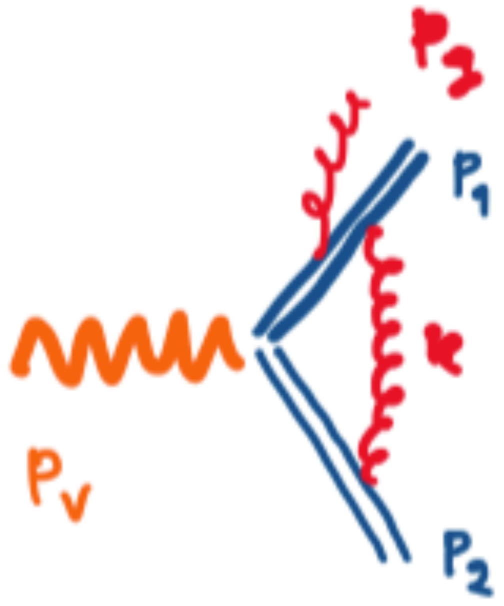


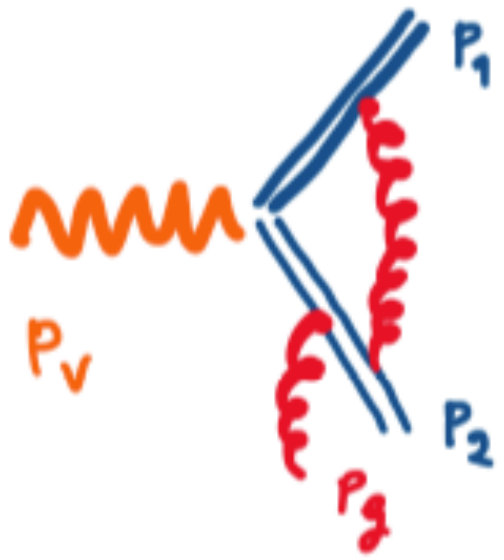


$$\mathcal{M}_1\left(p_V;p_1,p_2,p_g\right) \approx g_s^2 \varepsilon^{\mu}\left[J_{\mu}^{a,(0)}\mathcal{M}_1\left(p_V;p_1,p_2\right)+g_s^2 J_{\mu}^{a,(1)}\mathcal{M}_0\left(p_V;p_1,p_2\right)\right]$$

IR Finiteness: NNLO (RV)







$$\mathcal{M}_1\left(p_V;p_1,p_2,p_g\right) \approx g_s^2 \varepsilon^\mu \left[\cancel{J_\mu^{a,(0)} \mathcal{M}_1\left(p_V;p_1,p_2\right)} + g_s^2 J_\mu^{a,(1)} \mathcal{M}_0\left(p_V;p_1,p_2\right)\right]$$

$$\begin{aligned} J^{a,(1),\mu}\left(p_1,p_2;p_g\right) &= i f_{abc} \sum_{\substack{i,j=1\\ i\neq j}}^{2} T_i^b T_j^c \left(\frac{p_i^\mu}{p_i\cdot p_g} - \frac{p_j^\mu}{p_j\cdot p_g}\right) g_{ij}^{(1)}\left(\epsilon,p_g;p_i,p_j\right) \\ &= g_{12}^{(1)}\left(\epsilon,p_g;p_1,p_2\right) C_A J^{a,(0),\mu}\left(p_1,p_2;p_g\right) \end{aligned}$$

(Catani, Grazzini Nucl. Phys. B 591 (2000))

The one-loop Soft Current

$$J^{a,(1),\mu}\left(p_1, p_2; p_g\right) = g_{12}^{(1)}\left(\epsilon, p_g; p_1, p_2\right) C_A J^{a,(0),\mu}\left(p_1, p_2; p_g\right)$$

- **Purely non-Abelian, and thus we recover the well-know fact that in Abelian theories this process is IR-finite, even with massive quarks.**

The one-loop Soft Current

$$J^{a,(1),\mu}\left(p_1, p_2; p_g\right) = g_{12}^{(1)}\left(\epsilon, p_g; p_1, p_2\right) C_A \cancel{J^{a,(0),\mu}\left(p_1, p_2; p_g\right)}$$

- **J, as we have already discussed is invariant in the crossing**

The one-loop Soft Current

- **Let's take the massless case first**

$$g_{12}^{(1)}\left(\epsilon, p_g; p_1, p_2\right) = -\frac{1}{16\pi^2}\frac{1}{\epsilon^2}\frac{\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{(-s_{12}-i\delta)}{\left(-s_{1g}-i\delta\right)\left(-s_{2g}-i\delta\right)}\right]^{\epsilon}$$

- **And, given that we are only interested in Re(g) at NNLO**

$$\Re\left[g_{12}^{(1)}\left(\epsilon, p_g; -p_1, -p_2\right)\right] = \Re\left[g_{12}^{(1)}\left(\epsilon, p_g; p_1, p_2\right)\right]$$

The one-loop Soft Current

- **Let's take the massless case first**

$$\Re\left[g_{12}^{(1)}\left(\epsilon, p_g; -p_1, -p_2\right)\right] = \Re\left[g_{12}^{(1)}\left(\epsilon, p_g; p_1, p_2\right)\right]$$

We conclude that in the massless case there are no additional singularities at this order

The one-loop Soft Current

(I. Bierenbaum, M. Czakon and A. Mitov, Nucl. Phys. B 856 (2012))

- **In the massive case**

$$g_{12}^{(1)}\left(\epsilon, p_g; p_1, p_2\right) = \sum_{i=1}^{3} f_i\left(p_g; p_1, p_2\right) M_i\left(\epsilon, p_g; p_1, p_2\right)$$

The one-loop Soft Current

(I. Bierenbaum, M. Czakon and A. Mitov, Nucl. Phys. B 856 (2012))

- **In the massive case**

$$g_{12}^{(1)}\left(\epsilon, p_g; p_1, p_2\right) = \cancel{\sum_{i=1}^{3} f_i\left(p_g; p_1, p_2\right)} M_i\left(\epsilon, p_g; p_1, p_2\right)$$

$$M_1\left(\epsilon, p_g; p_1, p_2\right) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left[k^2 + i\delta\right]\left[\left(k + p_g\right)^2 + i\delta\right]\left[-2p_2 \cdot k + i\delta\right]},$$

$$M_2\left(\epsilon, p_g; p_1, p_2\right) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left[k^2 + i\delta\right]\left[2p_1 \cdot k + 2p_1 \cdot p_g + i\delta\right]\left[-2p_2 \cdot k + i\delta\right]}$$

$$M_3\left(\epsilon, p_g; p_1, p_2\right) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left[k^2 + i\delta\right]\left[\left(k + p_g\right)^2 + i\delta\right]\left[2p_1 \cdot k + 2p_2 \cdot p_g + i\delta\right]\left[-2p_2 \cdot k + i\delta\right]}$$

The one-loop Soft Current (M1)

$$M_1\left(\epsilon, p_g; p_1, p_2\right) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left[k^2 + i\delta\right]\left[\left(k+p_g\right)^2 + i\delta\right]\left[-2p_2 \cdot k + i\delta\right]}$$

Feynman parametrisation

$$M_1\left(\epsilon, p_g; p_1, p_2\right) = -G_1(\epsilon) \prod_{i=1}^{2} \int_0^\infty dx_i \frac{\left(x_1 + x_2\right)^{-1+2\epsilon}}{\left[m^2 - s_{2g} x_2 - i\delta\right]^{1+\epsilon}}$$

Rescaling

$$x_i \to \frac{x_i}{-s_{2g} - i\delta}$$

$$M_1\left(\epsilon, p_g; p_1, p_2\right) \propto \left(-s_{2g} - i\delta\right)^{-1-2\epsilon} = -\left|s_{2g}\right|^{-1-2\epsilon} e^{2i\pi\epsilon}$$

The one-loop Soft Current (M1)

$$M_1\left(\epsilon, p_g; p_1, p_2\right) \propto \left(-s_{2g} - i\delta\right)^{-1-2\epsilon} = -\left|s_{2g}\right|^{-1-2\epsilon} e^{2i\pi\epsilon}$$

Perform the crossing $\left\{s_{1g}, s_{2g}\right\} \to \left\{-s_{1g}, -s_{2g}\right\}$ **and take the soft limit**

$$M_1\left(\epsilon, p_g; -p_1, -p_2\right) = -M_1\left(\epsilon, p_g; p_1, p_2\right) e^{-2i\pi\epsilon} \propto -E_g^{-2\epsilon} e^{-2i\pi\epsilon}$$

The one-loop Soft Current (M2/3)

- **A similar analysis yields**

$$M_{2,3}\left(\epsilon, p_g; -p_1, -p_2\right) = M_{2,3}\left(\epsilon, p_g; p_1, p_2\right) e^{-2i\pi\epsilon} \propto E_g^{-2\epsilon} e^{-2i\pi\epsilon}$$

The one-loop Soft Current

$$g_{12}^{(1)}\left(\epsilon, p_g; -p_1, -p_2\right) = e^{-2i\epsilon\pi} g_{12}^{(1)}\left(\epsilon, p_g; p_1, p_2\right) = e^{-2i\epsilon\pi} \frac{\alpha_s}{2\pi} E_g^{-2\epsilon} \sum_{k=-2}^{\infty} \left[\mathfrak{r}_k + i \cdot \mathfrak{i}_k\right] \epsilon^k$$

- **Expand and plug this back in the ME expressions and match it to CS**

$$\Re\left[g_{12}^{(1)}(\epsilon, p_g; -p_1, -p_2)\right] = \Re\left[g_{12}^{(1)}(\epsilon, p_g; p_1, p_2)\right] + \frac{\alpha_s}{2\pi} E_g^{-2\epsilon} \left[\left(\frac{1-v}{v}\right)\pi^2 + \mathcal{O}(\epsilon)\right]$$

The Divergent remainder

- **The soft current at one-loop is proportional to the tree-level one**

- **Thus, our result is proportional to that obtained at NLO**

$$\mathrm{d}\sigma_{\mathrm{RV}}^{\mathrm{decay}} = \mathrm{Eik}_1(p_1, p_2) \times \mathrm{d}\sigma_{\mathrm{LO}}^{\mathrm{decay}} + \cdots$$

$$\mathrm{d}\sigma_{\mathrm{RV}} = \mathrm{Eik}_1(-p_1, -p_2) \times \mathrm{d}\sigma_{\mathrm{LO}} = \mathrm{Eik}_1(p_1, p_2) \times \mathrm{d}\sigma_{\mathrm{LO}} + \Delta[\mathrm{d}\sigma_{\mathrm{RV}}^{\mathrm{div}}] + \cdots$$

The Divergent remainder

- **The soft current at one-loop is proportional to the tree-level one**
- **Thus, our result is proportional to that obtained at NLO**

$$\Delta[\mathrm{d}\sigma_{\mathrm{RV}}^{\mathrm{div}}] = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \frac{2C_A C_F\ \pi^2}{\epsilon}\left[\frac{1}{2v}\ln\left(\frac{1-v}{1+v}\right)+1\right]\left(\frac{1-v}{v}\right)\mathrm{d}\sigma_{\mathrm{LO}}.$$

The Divergent remainder

$$\Delta[\mathrm{d}\sigma_{\mathrm{RV}}^{\mathrm{div}}] = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \frac{2C_A C_F\ \pi^2}{\epsilon} \left[\frac{1}{2v}\ln\left(\frac{1-v}{1+v}\right) + 1\right]\left(\frac{1-v}{v}\right)\mathrm{d}\sigma_{\mathrm{LO}}.$$

$$\lim_{v\to 1} \Delta[\mathrm{d}\sigma_{\mathrm{RV}}^{\mathrm{div}}] \sim 1 - v \sim \frac{m_1^2\ m_2^2}{\hat{s}^2} \longrightarrow 0$$

The divergence disappears if one of the two quarks is massless

The Divergent remainder

$$\Delta[\mathrm{d}\sigma_{\mathrm{RV}}^{\mathrm{div}}] = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \frac{2C_A C_F\ \pi^2}{\epsilon}\left[\frac{1}{2v}\ln\left(\frac{1-v}{1+v}\right)+1\right]\left(\frac{1-v}{v}\right)\mathrm{d}\sigma_{\mathrm{LO}}.$$

$$\lim_{v\to 1}\Delta[\mathrm{d}\sigma_{\mathrm{RV}}^{\mathrm{div}}] \sim 1-v \sim \frac{m_1^2\, m_2^2}{\hat{s}^2} \longrightarrow 0$$

Is formally of higher twist (as expected from previous analyses)

The Divergent remainder

$$\Delta[\mathrm{d}\sigma_{\mathrm{RV}}^{\mathrm{div}}] = \left[\frac{\alpha_s(\mu)}{2\pi}\right]^2 \frac{2C_A C_F\ \pi^2}{\epsilon} \left[\frac{1}{2v} \ln\left(\frac{1-v}{1+v}\right) + 1\right] \left(\frac{1-v}{v}\right) \mathrm{d}\sigma_{\mathrm{LO}}.$$

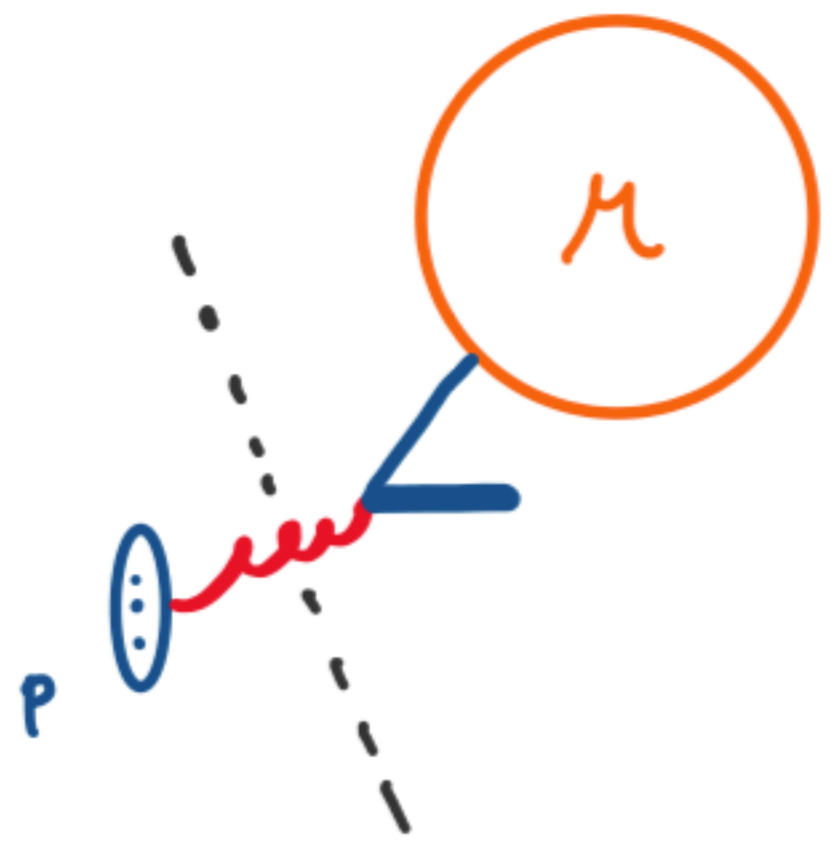
$$\lim_{v \to 1} \Delta[\mathrm{d}\sigma_{\mathrm{RV}}^{\mathrm{div}}] \sim 1 - v \sim \frac{m_1^2\, m_2^2}{\hat{s}^2} \longrightarrow 0$$

Is formally of higher twist (as expected from previous analyses)

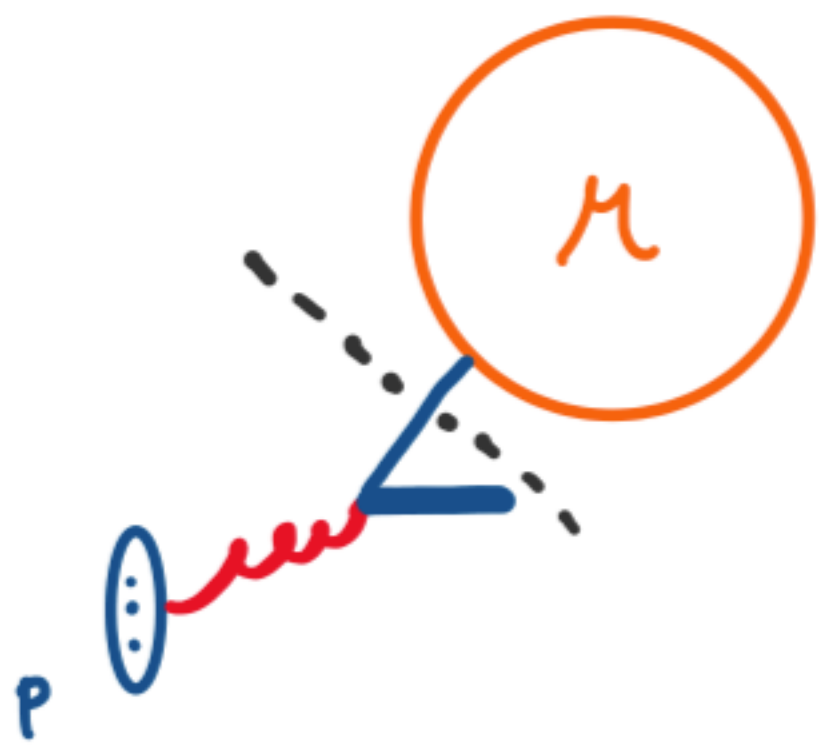
Thus it's not really divergent, it's just that our framework for factorisation doesn't work

Conclusions

- **Off-Shell: Finite**

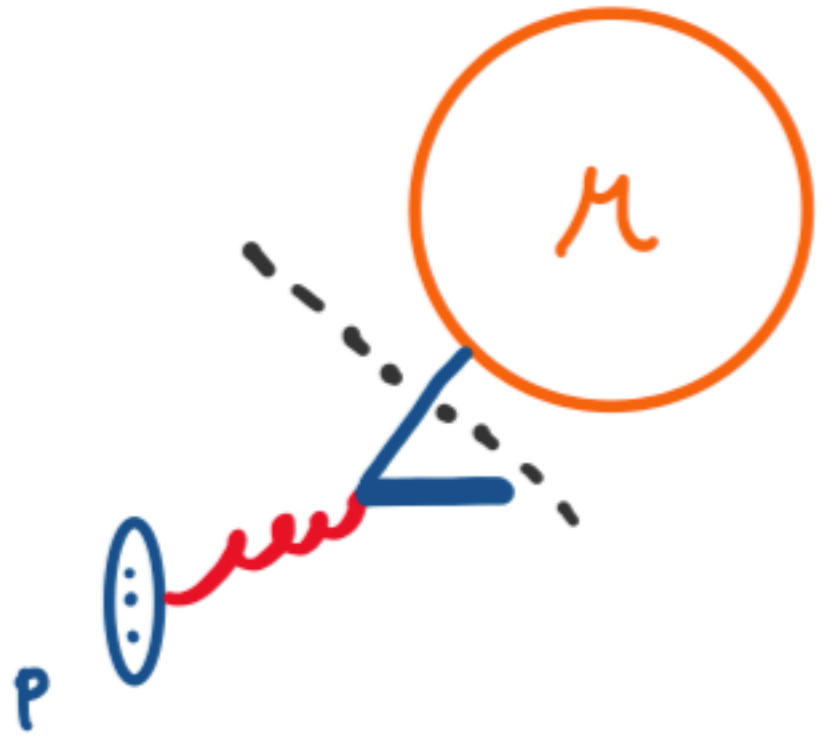


- **On-Shell: divergent**



Conclusions

- **On-Shell: divergent**



- **The scaling suggests that one could use a formalism**

  were we neglect terms of order $\dfrac{m_q^4}{\hat{s}^2}$

  while retaining those of order $\dfrac{m_q}{p_\perp} \to 1$

(Pietrulewicz et al. 1703.09702)