

Subleading Logarithmic QED Initial State Corrections to $e^+ e^- \rightarrow \gamma^*/Z^*$

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TTP KARLSRUHE

[based on: Ablinger, Blümlein, De Freitas, Schönwald (Nucl. Phys. B955 (2020))]

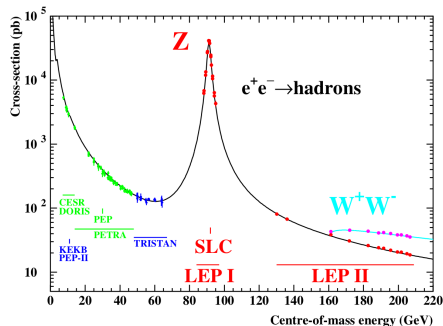
- 1 Motivation
- 2 The Method of Massive Operator Matrix Elements
- 3 Results
- 4 Conclusions

- Corrections due to initial state radiation (ISR) can be large, especially due to large logarithmic corrections

$$L = \ln(s/m_e^2) \approx 10.$$

- These corrections are important e.g.
 - for the prediction of the Z -boson peak
 - for $t\bar{t}$ production
 - associated Higgs production through $e^+e^- \rightarrow Z^*H^0$
 at future e^+e^- colliders.

- We extend the known $O(\alpha^2)$ ISR corrections up to $O(\alpha^6L^5)$, including the first three subleading logarithmic corrections at lower orders.



- 1988: First calculation to $O(\alpha^2)$ for the LEP analysis, through expansion of the phase space integrals.
[Berends, Burgers, van Neerven (Nucl. Phys. B297 (1988))]
- 2012: New calculation up to $O(\alpha^2)$ using the method of massive operator matrix elements.
[Blümlein, De Freitas, van Neerven (Nucl Phys. B855 (2012))]

⇒ Calculations **do not agree** at $O(\alpha^2 L^0)$!

- Errors in one of the calculations?
- Breakdown of factorization?

- We revisited the original calculation, doing the expansion in m_e at the latest stage
[Blümlein, De Freitas, Raab, KS (Nucl. Phys. B956 (2020))]
and found agreement with [Blümlein, De Freitas, van Neerven (Nucl Phys. B855 (2012))].

The initial state radiation factorizes from the born cross section:

$$\frac{d\sigma_{ij}}{ds'} = \frac{\sigma^{(0)}(s')}{s} \sum_{l,k} \Gamma_{li} \left(z, \frac{\mu^2}{m_e^2} \right) \otimes \tilde{\sigma}_{lk} \left(z, \frac{s'}{\mu^2} \right) \otimes \Gamma_{kj} \left(z, \frac{\mu^2}{m_e^2} \right) + \mathcal{O} \left(\frac{m_e^2}{s} \right) = \frac{\sigma^{(0)}(s')}{s} H_{ij} \left(z, \frac{s}{m_e^2} \right)$$

$$[f(z) \otimes g(z) = \int_0^1 dx_1 \int_0^1 dx_2 f(x_1) g(x_2) \delta(z - x_1 x_2), f(N) = \int_0^1 dz z^{N-1} f(z)]$$

with $z = s'/s$, μ the factorization scale, into:

- massless (Drell-Yan) cross sections $\tilde{\sigma}_{ij} \left(z, \frac{s'}{\mu^2} \right)$
 - [Hamberg, van Neerven, Matsuura (Nucl. Phys. B 359 (1991))]
 - [Harlander, Kilgore (Phys. Rev. Lett. 88 (2002))]
 - [Duhr, Dulat, Mistelberger (Phys. Rev. Lett. 125 (2020))]
- massive operator matrix elements $\Gamma_{ij} \left(z, \frac{\mu^2}{m_e^2} \right)$, which carry all mass dependence
 - [Blümlein, De Freitas, van Neerven (Nucl Phys. B855 (2012))]

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Massless cross sections and massive operator matrix elements obey renormalization group equations:

- massless cross sections $\tilde{\sigma}_{ij}$

$$\left[\left(\frac{\partial}{\partial \lambda} - \beta(a) \frac{\partial}{\partial a} \right) \delta_{kl} \delta_{jm} + \frac{1}{2} \gamma_{kl}(N) \delta_{jm} + \frac{1}{2} \gamma_{jm}(N) \delta_{kl} \right] \tilde{\sigma}_{ij}(N) = 0$$

- massive operator matrix elements Γ_{ij}

$$\left[\left(\frac{\partial}{\partial \Lambda} + \beta(a) \frac{\partial}{\partial a} \right) \delta_{jl} + \frac{1}{2} \gamma_{kl}(N) \right] \Gamma_{ij}(N) = 0$$

with $\lambda = \ln(s'/\mu^2)$, $\Lambda = \ln(\mu^2/m_e^2)$, the QED β -function $\beta(a)$ and $a = \alpha/4$

- Here the splitting functions contribute:

$$P_{ij}(N) = \int_0^1 dz z^{N-1} P_{ij}(z) = -\gamma_{ij}(N)$$

The Method of Massive Operator Matrix Elements

$$\frac{d\sigma_{e^+e^-}}{ds'} = \frac{\sigma^{(0)}(s')}{s} H_{e^+e^-}(z, L) = \frac{\sigma^{(0)}(s')}{s} \sum_{i=0}^{\infty} \sum_{k=0}^i a^i L^k c_{i,k}$$

The radiators:

$$c_{1,1} = -\gamma_{ee}^{(0)},$$

$$c_{1,0} = \tilde{\sigma}_{ee}^{(0)} + 2\Gamma_{ee}^{(0)},$$

$$c_{2,2} = \frac{1}{2}\gamma_{ee}^{(0)2} + \frac{\beta_0}{2}\gamma_{ee}^{(0)} + \frac{1}{4}\gamma_{e\gamma}^{(0)}\gamma_{\gamma e}^{(0)},$$

...

$$\begin{aligned} c_{3,1} = & -\gamma_{ee}^{(2)} - 2\Gamma_{ee}^{(0)}\gamma_{ee}^{(1)} - \Gamma_{ee}^{(0)}\gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)} - \gamma_{e\gamma}^{(1)}\Gamma_{\gamma e}^{(0)} - \gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(1)} - \beta_1\tilde{\sigma}_{ee}^{(0)} - \gamma_{ee}^{(1)}\tilde{\sigma}_{ee}^{(0)} \\ & - \gamma_{e\gamma}^{(0)}\Gamma_{\gamma e}^{(0)}\tilde{\sigma}_{ee}^{(0)} - 2\Gamma_{ee}^{(0)}\gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} - \gamma_{\gamma e}^{(1)}\tilde{\sigma}_{e\gamma}^{(0)} - \Gamma_{\gamma e}^{(0)}\gamma_{\gamma\gamma}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} - \gamma_{\gamma e}^{(0)}\tilde{\sigma}_{\gamma e}^{(1)} + \beta_0 \left[-2\Gamma_{ee}^{(0)}\tilde{\sigma}_{ee}^{(0)} \right. \\ & \left. - 2\tilde{\sigma}_{ee}^{(1)} - 2\Gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} \right] - \gamma_{ee}^{(0)} \left[\Gamma_{ee}^{(0)2} + 2\Gamma_{ee}^{(1)} + 2\Gamma_{ee}^{(0)}\tilde{\sigma}_{ee}^{(0)} + \tilde{\sigma}_{ee}^{(1)} + \Gamma_{\gamma e}^{(0)}\tilde{\sigma}_{e\gamma}^{(0)} \right], \end{aligned}$$

...

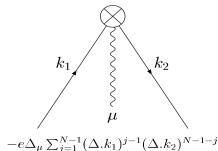
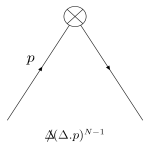
For the first three logarithmic orders we need the following ingredients:

- splitting functions γ_{ij} up to three-loop order
[E.G. Floratos, D.A. Ross, C.T. Sachrajda (Nucl. Phys. B129 (1977))]
[A. Gonzalez-Arroyo, C. Lopez, F.J. Yndurain (Nucl. Phys. B153 (1979))]
...
[S. Moch, J. Vermaseren, A. Vogt (Nucl.Phys.B 688/691 (2004))]
- massless (Drell-Yan) cross sections $\tilde{\sigma}_{ij}$ up to two-loop order
[Hamberg, van Neerven, Matsuura (Nucl. Phys. B 359 (1991))]
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- massive operator matrix elements Γ_{ij} up to two-loop order¹
[Blümlein, De Freitas, van Neerven (Nucl. Phys. B855 (2012))]

⇒ $\Gamma_{\gamma e}$ was only considered up to one-loop order

¹In the case of massless external states massive operator matrix elements have been considered in the context of DIS.
[Buza, Matiounine, Smith, Mignerone, van Neerven (Nucl. Phys. B472 (1996)),
Bierenbaum, Blümlein, Klein (Nucl. Phys. B820 (2009)), ...]

The Missing Operator Matrix Element $\Gamma_{\gamma e}$



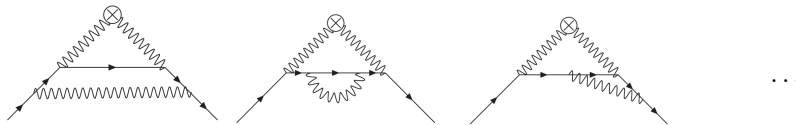
$$\frac{1 + (-1)^N}{2} (\Delta \cdot p)^{N-2} [g_{\mu\nu} (\Delta \cdot p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu) \Delta \cdot p + p^2 \Delta_\mu \Delta_\nu]$$

- We have to compute on-shell 2-point functions with local operator insertions ($\Delta^2 = 0$).
- The operator can be resummed into a propagator like term:

$$\sum_{N=0}^{\infty} t^N (\Delta \cdot k)^N = \frac{1}{1 - t \Delta \cdot k}$$

- The calculation can now follow standard techniques:

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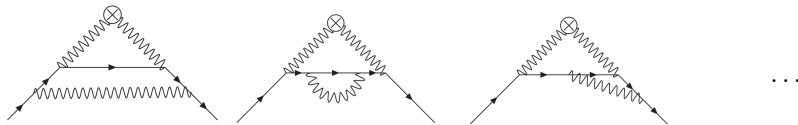
– Feynman diagrams to reduction to master integrals

– Calculation of the master integrals using the package `PySecDec` [1]

– Use of the Mellin-Barnes representation for propagator denominators and Mellin-Barnes

- For the calculations we make use of the packages `Sigma` [C. Schneider (Sem. Lothar. Combin.56 (2007))] and `HarmonicSums` [J. Ablinger et al. (arXiv:1011.1176)] .

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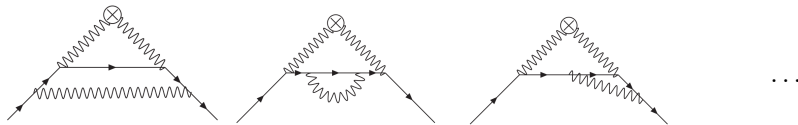


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 - Integration-By-Parts reduction to master integrals.
 - Calculation of the master integrals via differential equations in the resummation variable t .
 - We find the Mellin-space expression by symbolically computing the N -th derivative.
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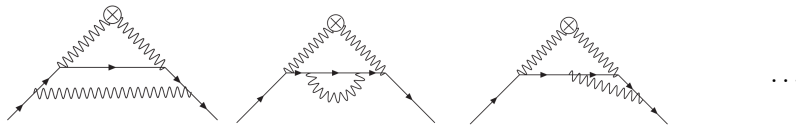


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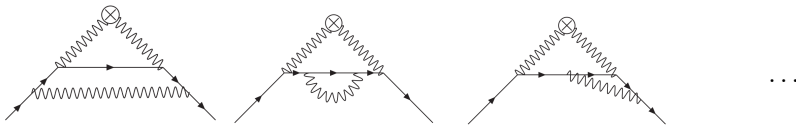


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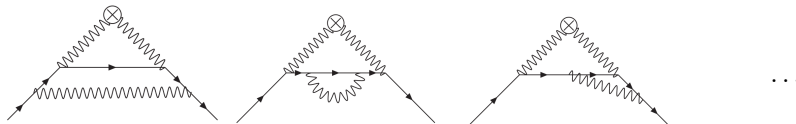


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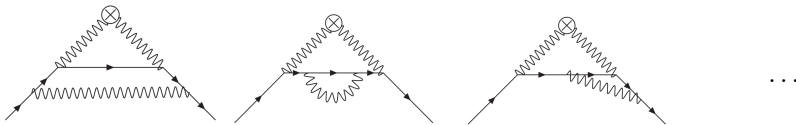


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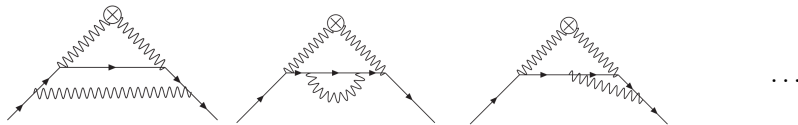


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The Missing Operator Matrix Element $\Gamma_{\gamma e}$

$$\begin{aligned}
 \Gamma_{\gamma e}^{(1)}(N) = & \frac{P_8}{27(N-4)(N-3)(N-2)(N-1)N^4(N+1)^4} + \left(\frac{2P_7}{9(N-4)(N-3)(N-2)(N-1)N^3(N+1)^3} + \frac{2(N^2+N+2)}{(N-1)N(N+1)} S_2 \right) S_1 \\
 & + \frac{P_3}{3(N-2)(N-1)N(N+1)^2} S_1^2 + \frac{2(N^2+N+2)}{3(N-1)N(N+1)} S_1^3 + \frac{P_6}{3(N-2)(N-1)N^2(N+1)^2} S_2 + \frac{4(N^2+N+2)}{3(N-1)N(N+1)} S_3 \\
 & + \frac{3 \cdot 2^{6+N}}{(N-2)(N+1)^2} S_{1,1} \left(\frac{1}{2}, 1 \right) + \frac{2^{6-N} P_5}{3(N-3)(N-2)(N-1)^2 N^2} \left(S_2(2) + S_1 S_1(2) - S_{1,1}(1,2) - S_{1,1}(2,1) \right) \\
 & - \frac{32(N^2+N+2)}{(N-1)N(N+1)} \left[S_1(2) S_{1,1} \left(\frac{1}{2}, 1 \right) + S_{1,2} \left(\frac{1}{2}, 2 \right) - S_{1,1,1} \left(\frac{1}{2}, 1, 2 \right) - S_{1,1,1} \left(\frac{1}{2}, 2, 1 \right) - \frac{\zeta_2}{2} S_1(2) \right] \\
 & - \frac{48(N^2+N+2)}{(N-1)N(N+1)} S_{2,1} + \frac{4P_4}{(N-2)(N-1)N^2(N+1)^2} \zeta_2
 \end{aligned}$$

harmonic sums:

$$S_{a,\vec{b}} = S_{a,\vec{b}}(N) = \sum_{i=1}^N \frac{\text{sgn}(a)^i}{i^a} S_{\vec{b}}(i)$$

generalized harmonic sums:

$$S_{a,\vec{b}}(c, \vec{d}) = S_{a,\vec{b}}(c, \vec{d}; N) = \sum_{i=1}^N \frac{(\text{sgn}(a) \cdot c)^i}{i^a} S_{\vec{b}}(\vec{d}; i)$$

The missing operator matrix element $\Gamma_{\gamma e}$

Analytic Mellin-inversion with HarmonicSums:

$$\begin{aligned}
 \Gamma_{\gamma e}^{(1)}(z) = & \frac{P_9}{135z^3} - \frac{320 - 335z + 231z^2}{15z} H_0 + \frac{12 + 23z}{6} H_0^2 + \frac{2 - z}{3} H_0^3 + 32(2 - z) \left(\frac{(2 - z)^2}{3z^2} - H_0 \right) (\tilde{H}_{-1} \tilde{H}_0 - \tilde{H}_{0,-1}) \\
 & - 8(2 - z) H_{0,0,1} - \frac{96 - 190z + 118z^2 - 41z^3}{3z^2} H_1^2 - 32(2 - z) (\tilde{H}_{-1} \tilde{H}_0 - \tilde{H}_{0,-1}) \tilde{H}_1 \\
 & - \left(\frac{2(32 - 48z + 36z^2 - 13z^3)}{3z^2} + 4(2 - z) H_0 \right) H_{0,1} - \left(\frac{2P_{10}}{45z^4} - \frac{2(32 - 48z + 12z^2 + 7z^3)}{3z^2} H_0 \right) H_1 \\
 & + \frac{2(2 - 2z + z^2)}{z} \left(\frac{H_3}{3} + 8H_1 H_{0,1} + 16\tilde{H}_0 \tilde{H}_{0,-1} - 32\tilde{H}_{0,0,-1} - 16H_{0,1,1} + 8\tilde{H}_0 \zeta_2 \right) + \left(\frac{4(32 - 48z + 24z^2 - 3z^3)}{3z^2} \right. \\
 & \left. - 8(2 - z) (H_0 + 2\tilde{H}_1) \right) \zeta_2 + \frac{8(12 - 10z + 5z^2)}{z} \zeta_3
 \end{aligned}$$

harmonic polylogarithms of argument z and $1 - z$ ($\tilde{H}(z) = H(1 - z)$):

$$H_{a,b} = H_{a,b}(z) = \int_0^1 d\tau f_a(\tau) H_b(\tau), \quad \text{with} \quad f_0(\tau) = \frac{1}{\tau}, \quad f_1(\tau) = \frac{1}{1 - \tau}, \quad f_{-1}(\tau) = \frac{1}{1 + \tau}$$

$$\frac{d\sigma_{e^+e^-}}{ds'} = \frac{\sigma^{(0)}(s')}{s} H_{e^+e^-}(z, L) = \frac{\sigma^{(0)}(s')}{s} \sum_{i=0}^{\infty} \sum_{k=0}^i a^i L^k c_{i,k}$$

- The radiators do not depend on the factorization scale, i.e. no collinear singularities for massive electrons.
- The analytic structures directly translate from the different ingredients.
- Radiators are distributions in z -space:

$$c_{i,j}(z) = c_{i,j}^{\delta} \delta(1-z) + c_{i,j}^+ + c_{i,j}^{\text{reg}}$$

$$c_{3,3}^{\delta} = \frac{572}{9} - \frac{704}{3} \zeta_2 + \frac{512}{3} \zeta_3,$$

$$c_{3,3}^+ = \left(\frac{5744}{27} - 256 \zeta_2 \right) \mathcal{D}_0 + \frac{1408}{3} \mathcal{D}_1 + 256 \mathcal{D}_2,$$

$$\mathcal{D}_k = \left(\frac{\ln^k(1-z)}{1-z} \right)_+,$$

$$c_{3,3}^{\text{reg}} = \left\{ \frac{16H_0 P_{104}}{9(z-1)} - \frac{4P_{131}}{27z} + \frac{8(3-19z^2)H_0^2}{3(z-1)} \right. \\ \left. + \left[\frac{16P_{105}}{9z} - \frac{128(1+z^2)H_0}{z-1} \right] H_1 - 128(1+z)H_1^2 \right. \\ \left. - \frac{352}{3}(1+z)H_{0,1} + \frac{736}{3}(1+z)\zeta_2 \right\}$$

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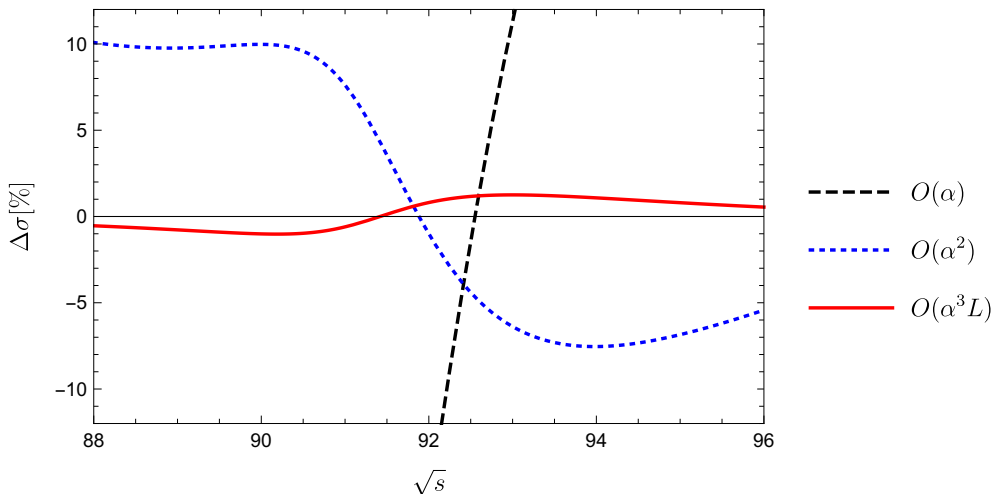
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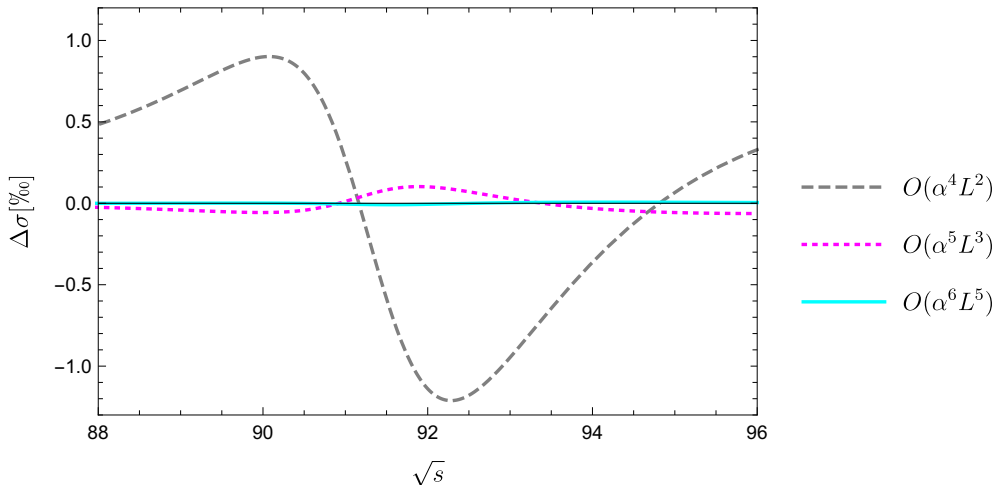
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- $\Delta\sigma$ is the change in the total cross section from one order to the other for $z_0 = 4m_\tau^2$



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Conclusions:

- We calculated the ISR corrections to the process $e^+e^- \rightarrow \gamma^*/Z^*$ up to $O(\alpha^6 L^5)$ including the first three logarithmic terms at lower orders.
- The corrections can become important at future e^+e^- machines running at high luminosities at or close to the Z -pole.
- The radiators can be used for other processes like $e^+e^- \rightarrow t\bar{t}$ and $e^+e^- \rightarrow ZH$.

Outlook:

- The massless Drell-Yan cross sections are known up to $O(\alpha^3)$
⇒ An extension to the first four logarithmic orders is possible, but needs the calculation the operator matrix elements up to $O(\alpha^3)$.
- The technique can be extended to A_{FB} [Blümlein, De Freitas, KS (arXiv:2102.12237)].
- The method can be extended to QCD to study e.g. the heavy-quark initiated Drell-Yan process.

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Backup

- Usually it is more convenient to work in Mellin space:

$$M[f(z)](N) = \int_0^1 dz z^{N-1} f(z)$$

- Here the convolution \otimes

$$f(z) \otimes g(z) = \int_0^1 dz_1 \int_0^1 dz_2 f(z_1) g(z_2) \delta(z - z_1 z_2)$$

factorizes:

$$M[f(z) \otimes g(z)](N) = M[f(z)](N) \cdot M[g(z)](N)$$

	Fixed width		s dep. width	
	Peak (MeV)	Width (MeV)	Peak (MeV)	Width (MeV)
$O(\alpha)$ correction	185.638	539.408	181.098	524.978
$O(\alpha^2 L^2)$:	-96.894	-177.147	-95.342	-176.235
$O(\alpha^2 L)$:	6.982	22.695	6.841	21.896
$O(\alpha^2)$:	0.176	-2.218	0.174	-2.001
$O(\alpha^3 L^3)$:	23.265	38.560	22.968	38.081
$O(\alpha^3 L^2)$:	-1.507	-1.888	-1.491	-1.881
$O(\alpha^3 L)$:	-0.152	0.105	-0.151	-0.084
$O(\alpha^4 L^4)$:	-1.857	0.206	-1.858	0.146
$O(\alpha^4 L^3)$:	0.131	-0.071	0.132	-0.065
$O(\alpha^4 L^2)$:	0.048	-0.001	0.048	0.001
$O(\alpha^5 L^5)$:	0.142	-0.218	0.144	-0.212
$O(\alpha^5 L^4)$:	-0.000	0.020	-0.001	0.020
$O(\alpha^5 L^3)$:	-0.008	0.009	-0.008	0.008
$O(\alpha^6 L^6)$:	-0.007	0.027	-0.007	0.027
$O(\alpha^6 L^5)$:	-0.001	0.000	-0.001	0.000

Table 1: Shifts in the Z-mass and the width due to the different contributions to the ISR QED radiative corrections for a fixed width of $\Gamma_Z = 2.4952$ GeV and s-dependent width using $M_Z = 91.1876$ GeV and $s_0 = 4m_e^2$.

	L^6	L^5	L^4	L^3	L^2	L	L^0
$O(\alpha)$						✓	✓
$O(\alpha^2)$					✓	✓	✓
$O(\alpha^3)$				✓	✓	✓	-
$O(\alpha^4)$			✓	✓	✓	-	-
$O(\alpha^5)$		✓	✓	✓	-	-	-
$O(\alpha^6)$	✓	✓	-	-	-	-	-

- The technique can also be used for A_{FB} : [Böhm et al. (LEP Physics Workshop 1989, p.203–234)]

$$A_{FB}(s) = \frac{1}{\sigma_T(s)} \int_{z_0}^1 dz \frac{4z}{(1+z)^2} H_{FB}(z) \sigma_{FB}^{(0)}(zs)$$

- Due to the angle dependence the integrand does not factorize with the Mellin-transform:

$$H_{FB}^{LL} = \int_0^1 dx_1 \int_0^1 dx_2 \frac{(1+z)^2}{(x_1+x_2)^2} \Gamma_{ee}^{LL}(x_1) \Gamma_{ee}^{LL}(x_2) \delta(z - x_1 x_2).$$

- At leading-logarithmic order the integrals can be solved in terms of cyclotomic harmonic polylogarithms with the additional letters:

$$f_{\{4,0\}}(\tau) = \frac{1}{1+\tau^2}, \quad f_{\{4,1\}}(\tau) = \frac{\tau}{1+\tau^2}$$

- At sub-leading log the angle dependence of the Wilson coefficients will most likely introduce more structures.

- ΔA_{FB} is the change of the forward-backward asymmetry from one order to the other for $z_0 = 4m_\tau^2$

