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Collinear factorisation for e^+e^- collisions

Based on: 1909.03886 (SF), 1911.12040 (Bertone, Cacciari, SF, Stagnitto)
... and work in progress with MadGraph5_aMC@NLO (/w Zaro, Zhao)

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Such is the power of automation

While automated codes have been employed predominantly in hadronic collisions, they can work for e^+e^- too

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Take **MadGraph5_aMC@NLO**, widely used by both theorists and experimentalists. From 1405.0301: 

Process		Syntax	Cross section (pb)			
Heavy quarks and jets			LO 1 TeV		NLO 1 TeV	
i.1	$e^+e^- \rightarrow jj$	e+ e- > j j	$6.223 \pm 0.005 \cdot 10^{-1}$	+0.0% -0.0%	$6.389 \pm 0.013 \cdot 10^{-1}$	+0.2% -0.2%
i.2	$e^+e^- \rightarrow jjj$	e+ e- > j j j	$3.401 \pm 0.002 \cdot 10^{-1}$	+9.6% -8.0%	$3.166 \pm 0.019 \cdot 10^{-1}$	+0.2% -2.1%
i.3	$e^+e^- \rightarrow jjjj$	e+ e- > j j j j	$1.047 \pm 0.001 \cdot 10^{-1}$	+20.0% -15.3%	$1.090 \pm 0.006 \cdot 10^{-1}$	+0.0% -2.8%
i.4	$e^+e^- \rightarrow jjjjj$	e+ e- > j j j j j	$2.211 \pm 0.006 \cdot 10^{-2}$	+31.4% -22.0%	$2.771 \pm 0.021 \cdot 10^{-2}$	+4.4% -8.6%
i.5	$e^+e^- \rightarrow t\bar{t}$	e+ e- > t t~	$1.662 \pm 0.002 \cdot 10^{-1}$	+0.0% -0.0%	$1.745 \pm 0.006 \cdot 10^{-1}$	+0.4% -0.4%
i.6	$e^+e^- \rightarrow t\bar{t}j$	e+ e- > t t~ j	$4.813 \pm 0.005 \cdot 10^{-2}$	+9.3% -7.8%	$5.276 \pm 0.022 \cdot 10^{-2}$	+1.3% -2.1%
i.7*	$e^+e^- \rightarrow t\bar{t}jj$	e+ e- > t t~ j j	$8.614 \pm 0.009 \cdot 10^{-3}$	+19.4% -15.0%	$1.094 \pm 0.005 \cdot 10^{-2}$	+5.0% -6.3%
i.8*	$e^+e^- \rightarrow t\bar{t}jjj$	e+ e- > t t~ j j j	$1.044 \pm 0.002 \cdot 10^{-3}$	+30.5% -21.6%	$1.546 \pm 0.010 \cdot 10^{-3}$	+10.6% -11.6%
i.9*	$e^+e^- \rightarrow t\bar{t}t\bar{t}$	e+ e- > t t~ t t~	$6.456 \pm 0.016 \cdot 10^{-7}$	+19.1% -14.8%	$1.221 \pm 0.005 \cdot 10^{-6}$	+13.2% -11.2%
i.10*	$e^+e^- \rightarrow t\bar{t}t\bar{t}j$	e+ e- > t t~ t t~ j	$2.719 \pm 0.005 \cdot 10^{-8}$	+29.9% -21.3%	$5.338 \pm 0.027 \cdot 10^{-8}$	+18.3% -15.4%
i.11	$e^+e^- \rightarrow b\bar{b}$ (4f)	e+ e- > b b~	$9.198 \pm 0.004 \cdot 10^{-2}$	+0.0% -0.0%	$9.282 \pm 0.031 \cdot 10^{-2}$	+0.0% -0.0%
i.12	$e^+e^- \rightarrow b\bar{b}j$ (4f)	e+ e- > b b~ j	$5.029 \pm 0.003 \cdot 10^{-2}$	+9.5% -8.0%	$4.826 \pm 0.026 \cdot 10^{-2}$	+0.5% -2.5%
i.13*	$e^+e^- \rightarrow b\bar{b}jj$ (4f)	e+ e- > b b~ j j	$1.621 \pm 0.001 \cdot 10^{-2}$	+20.0% -15.3%	$1.817 \pm 0.009 \cdot 10^{-2}$	+0.0% -3.1%
i.14*	$e^+e^- \rightarrow b\bar{b}jjj$ (4f)	e+ e- > b b~ j j j	$3.641 \pm 0.009 \cdot 10^{-3}$	+31.4% -22.1%	$4.936 \pm 0.038 \cdot 10^{-3}$	+4.8% -8.9%
i.15*	$e^+e^- \rightarrow b\bar{b}b\bar{b}$ (4f)	e+ e- > b b~ b b~	$1.644 \pm 0.003 \cdot 10^{-4}$	+19.9% -15.3%	$3.601 \pm 0.017 \cdot 10^{-4}$	+15.2% -12.5%
i.16*	$e^+e^- \rightarrow b\bar{b}b\bar{b}j$ (4f)	e+ e- > b b~ b b~ j	$7.660 \pm 0.022 \cdot 10^{-5}$	+31.3% -22.0%	$1.537 \pm 0.011 \cdot 10^{-4}$	+17.9% -15.3%
i.17*	$e^+e^- \rightarrow t\bar{t}b\bar{b}$ (4f)	e+ e- > t t~ b b~	$1.819 \pm 0.003 \cdot 10^{-4}$	+19.5% -15.0%	$2.923 \pm 0.011 \cdot 10^{-4}$	+9.2% -8.9%
i.18*	$e^+e^- \rightarrow t\bar{t}b\bar{b}j$ (4f)	e+ e- > t t~ b b~ j	$4.045 \pm 0.011 \cdot 10^{-5}$	+30.5% -21.6%	$7.049 \pm 0.052 \cdot 10^{-5}$	+13.7% -13.1%

Process		Syntax	Cross section (pb)			
Top quarks + bosons			LO 1 TeV		NLO 1 TeV	
j.1	$e^+e^- \rightarrow t\bar{t}H$	$e^+ e^- > t t\sim h$	$2.018 \pm 0.003 \cdot 10^{-3}$	+0.0% -0.0%	$1.911 \pm 0.006 \cdot 10^{-3}$	+0.4% -0.5%
j.2*	$e^+e^- \rightarrow t\bar{t}Hj$	$e^+ e^- > t t\sim h j$	$2.533 \pm 0.003 \cdot 10^{-4}$	+9.2% -7.8%	$2.658 \pm 0.009 \cdot 10^{-4}$	+0.5% -1.5%
j.3*	$e^+e^- \rightarrow t\bar{t}Hjj$	$e^+ e^- > t t\sim h j j$	$2.663 \pm 0.004 \cdot 10^{-5}$	+19.3% -14.9%	$3.278 \pm 0.017 \cdot 10^{-5}$	+4.0% -5.7%
j.4*	$e^+e^- \rightarrow t\bar{t}\gamma$	$e^+ e^- > t t\sim a$	$1.270 \pm 0.002 \cdot 10^{-2}$	+0.0% -0.0%	$1.335 \pm 0.004 \cdot 10^{-2}$	+0.5% -0.4%
j.5*	$e^+e^- \rightarrow t\bar{t}\gamma j$	$e^+ e^- > t t\sim a j$	$2.355 \pm 0.002 \cdot 10^{-3}$	+9.3% -7.9%	$2.617 \pm 0.010 \cdot 10^{-3}$	+1.6% -2.4%
j.6*	$e^+e^- \rightarrow t\bar{t}\gamma jj$	$e^+ e^- > t t\sim a j j$	$3.103 \pm 0.005 \cdot 10^{-4}$	+19.5% -15.0%	$4.002 \pm 0.021 \cdot 10^{-4}$	+5.4% -6.6%
j.7*	$e^+e^- \rightarrow t\bar{t}Z$	$e^+ e^- > t t\sim z$	$4.642 \pm 0.006 \cdot 10^{-3}$	+0.0% -0.0%	$4.949 \pm 0.014 \cdot 10^{-3}$	+0.6% -0.5%
j.8*	$e^+e^- \rightarrow t\bar{t}Zj$	$e^+ e^- > t t\sim z j$	$6.059 \pm 0.006 \cdot 10^{-4}$	+9.3% -7.8%	$6.940 \pm 0.028 \cdot 10^{-4}$	+2.0% -2.6%
j.9*	$e^+e^- \rightarrow t\bar{t}Zjj$	$e^+ e^- > t t\sim z j j$	$6.351 \pm 0.028 \cdot 10^{-5}$	+19.4% -15.0%	$8.439 \pm 0.051 \cdot 10^{-5}$	+5.8% -6.8%
j.10*	$e^+e^- \rightarrow t\bar{t}W^\pm jj$	$e^+ e^- > t t\sim wpm j j$	$2.400 \pm 0.004 \cdot 10^{-7}$	+19.3% -14.9%	$3.723 \pm 0.012 \cdot 10^{-7}$	+9.6% -9.1%
j.11*	$e^+e^- \rightarrow t\bar{t}HZ$	$e^+ e^- > t t\sim h z$	$3.600 \pm 0.006 \cdot 10^{-5}$	+0.0% -0.0%	$3.579 \pm 0.013 \cdot 10^{-5}$	+0.1% -0.0%
j.12*	$e^+e^- \rightarrow t\bar{t}\gamma Z$	$e^+ e^- > t t\sim a z$	$2.212 \pm 0.003 \cdot 10^{-4}$	+0.0% -0.0%	$2.364 \pm 0.006 \cdot 10^{-4}$	+0.6% -0.5%
j.13*	$e^+e^- \rightarrow t\bar{t}\gamma H$	$e^+ e^- > t t\sim a h$	$9.756 \pm 0.016 \cdot 10^{-5}$	+0.0% -0.0%	$9.423 \pm 0.032 \cdot 10^{-5}$	+0.3% -0.4%
j.14*	$e^+e^- \rightarrow t\bar{t}\gamma\gamma$	$e^+ e^- > t t\sim a a$	$3.650 \pm 0.008 \cdot 10^{-4}$	+0.0% -0.0%	$3.833 \pm 0.013 \cdot 10^{-4}$	+0.4% -0.4%
j.15*	$e^+e^- \rightarrow t\bar{t}ZZ$	$e^+ e^- > t t\sim z z$	$3.788 \pm 0.004 \cdot 10^{-5}$	+0.0% -0.0%	$4.007 \pm 0.013 \cdot 10^{-5}$	+0.5% -0.5%
j.16*	$e^+e^- \rightarrow t\bar{t}HH$	$e^+ e^- > t t\sim h h$	$1.358 \pm 0.001 \cdot 10^{-5}$	+0.0% -0.0%	$1.206 \pm 0.003 \cdot 10^{-5}$	+0.9% -1.1%
j.17*	$e^+e^- \rightarrow t\bar{t}W^+W^-$	$e^+ e^- > t t\sim w^+ w^-$	$1.372 \pm 0.003 \cdot 10^{-4}$	+0.0% -0.0%	$1.540 \pm 0.006 \cdot 10^{-4}$	+1.0% -0.9%

So are we done?

Not quite. In those results:

- ▶ No beamstrahlung
- ▶ NLO was in α_s , not α
- ▶ No description of all-order electron-mass factorisable effects
(i.e. collider energy \equiv collision energy)

Consider the production of a system X at an e^+e^- collider:

$$e^+(P_{e^+}) + e^-(P_{e^-}) \longrightarrow X$$

Its cross section is written as follows:

$$d\Sigma_{e^+e^-}(P_{e^+}, P_{e^-}) = \sum_{kl} \int dy_+ dy_- \mathcal{B}_{kl}(y_+, y_-) d\sigma_{kl}(y_+ P_{e^+}, y_- P_{e^-})$$

To be definite, let's stipulate that:

$$k \in \{e^+, \gamma\}, \quad l \in \{e^-, \gamma\}$$

which is immediate to generalise, if need be. Then:

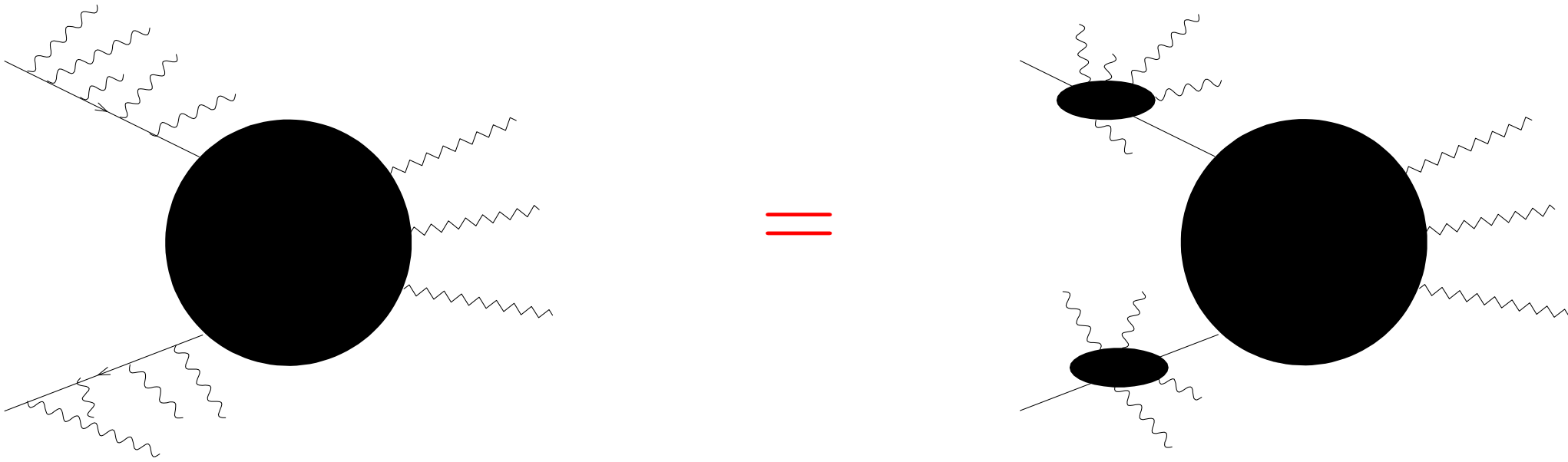
- ◆ $d\Sigma_{e^+e^-}$: the collider-level cross section
- ◆ $d\sigma_{kl}$: the particle-level cross section
- ◆ $\mathcal{B}_{kl}(y_+, y_-)$: describes beam dynamics (mainly beamstrahlung)
- ◆ e^+, e^- on the lhs: the beams
- ◆ e^+, e^-, γ on the rhs: the particles

The particle-level cross section embeds all that is not beam dynamics

The NLO bit has been addressed in [1804.10017](#): full automation of NLO computations in α (as well as for any combination $\alpha_s^k \alpha^p$). This solves once and for all the problem at the level of short-distance cross sections

As for electron-mass factorisable effects, use a factorisation approach (I'll concentrate here on ISR. Analogous formulae hold for FSR)

Factorisation



$$\sigma = \text{PDF} \star \text{PDF} \star \hat{\sigma}$$

PDFs collect (universal) small-angle dynamics

$$d\sigma_{kl}(p_k, p_l) = \sum_{ij=e^+, e^-, \gamma} \int dz_+ dz_- \Gamma_{i/k}(z_+, \mu^2, m^2) \Gamma_{j/l}(z_-, \mu^2, m^2) \\ \times d\hat{\sigma}_{ij}(z_+ p_k, z_- p_l, \mu^2) + \mathcal{O}\left(\left(\frac{m^2}{s}\right)^p\right)$$

where one calculates Γ and $d\hat{\sigma}$ to predict $d\sigma$

- ◆ $k, l = e^+, e^-, \gamma$ on the lhs: the particles that emerge from beamstrahlung
- ◆ $i, j = e^+, e^-, \gamma$ on the rhs: the partons
- ◆ $d\sigma_{kl}$: the particle-level (ie observable) cross section
- ◆ $d\hat{\sigma}_{ij}$: the subtracted parton-level cross section.
Generally with $m = 0 \implies$ power-suppressed terms in $d\sigma$ discarded
- ◆ $\Gamma_{i/k}$: the PDF of parton i inside particle k
- ◆ μ : the hard scale, $m^2 \ll \mu^2 \sim s$

Very similar to QCD, with some notable differences:

- ◆ PDFs and power-suppressed terms can be computed perturbatively
- ◆ An object (e.g. e^-) may play the role of both particle and parton

As in QCD, a particle is a physical object, a parton is not

As I have said, parton-level cross section computations are highly automated, and can be carried out at the NLO in both α and α_s with MadGraph5_aMC@NLO

Conversely, until recently PDFs were only available at the LO+LL, which is insufficient in the context of NLO simulations



z -space LO+LL PDFs $(\alpha \log(E/m))^k$:

~ 1992

- ▶ $0 \leq k \leq \infty$ for $z \simeq 1$ (Gribov, Lipatov)
- ▶ $0 \leq k \leq 3$ for $z < 1$ (Skrzypek, Jadach; Cacciari, Deandrea, Montagna, Nicosini; Skrzypek)
- ▶ matching between these two regimes

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z -space NLO+NLL PDFs $(\alpha \log(E/m))^k + \alpha (\alpha \log(E/m))^{k-1}$:

~ 1909.03886, 1911.12040

- ▶ $0 \leq k \leq \infty$ for $z \simeq 1$
- ▶ $0 \leq k \leq 3$ for $z < 1 \iff \mathcal{O}(\alpha^3)$
- ▶ matching between these two regimes
- ▶ for e^+ , e^- , and γ
- ▶ both numerical and analytical

Main tool: the solution of PDFs evolution equations

Henceforth, I consider the dominant production mechanism at an e^+e^- collider, namely that associated with partons inside an electron*

Simplified notation:

$$\Gamma_i(z, \mu^2) \equiv \Gamma_{i/e^-}(z, \mu^2)$$

*The case of the positron is identical, at least in QED, and will be understood

NLO initial conditions (1909.03886)

Conventions for the perturbative coefficients:

$$\Gamma_i = \Gamma_i^{[0]} + \frac{\alpha}{2\pi} \Gamma_i^{[1]} + \mathcal{O}(\alpha^2)$$

Results:

$$\Gamma_i^{[0]}(z, \mu_0^2) = \delta_{ie} - \delta(1-z)$$

$$\Gamma_{e^-}^{[1]}(z, \mu_0^2) = \left[\frac{1+z^2}{1-z} \left(\log \frac{\mu_0^2}{m^2} - 2 \log(1-z) - 1 \right) \right]_+ + K_{ee}(z)$$

$$\Gamma_\gamma^{[1]}(z, \mu_0^2) = \frac{1+(1-z)^2}{z} \left(\log \frac{\mu_0^2}{m^2} - 2 \log z - 1 \right) + K_{\gamma e}(z)$$

$$\Gamma_{e^+}^{[1]}(z, \mu_0^2) = 0$$

Note:

- ▶ Meaningful only if $\mu_0 \sim m$
- ▶ In $\overline{\text{MS}}$, $K_{ij}(z) = 0$; in general, these functions *define* an IR scheme

NLL evolution (1911.12040)

General idea: solve the evolution equations starting from the initial conditions computed previously

$$\frac{\partial \Gamma_i(z, \mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi} [P_{ij} \otimes \Gamma_j](z, \mu^2) \iff \frac{\partial \Gamma(z, \mu^2)}{\partial \log \mu^2} = \frac{\alpha(\mu)}{2\pi} [\mathbb{P} \otimes \Gamma](z, \mu^2),$$

Done conveniently in terms of non-singlet, singlet, and photon

Two ways:

- ◆ Mellin space: suited to both numerical solution and all-order, large- z analytical solution (called *asymptotic solution*)
- ◆ Directly in z space in an integrated form: suited to fixed-order, all- z analytical solution (called *recursive solution*)

Asymptotic solution

Non-singlet \equiv singlet; photon is more complicated

$$\Gamma_{\text{NLL}}(z, \mu^2) = \frac{e^{-\gamma_E \xi_1} e^{\hat{\xi}_1}}{\Gamma(1 + \xi_1)} \xi_1 (1 - z)^{-1 + \xi_1} \times \left\{ 1 + \frac{\alpha(\mu_0)}{\pi} \left[\left(\log \frac{\mu_0^2}{m^2} - 1 \right) \left(A(\xi_1) + \frac{3}{4} \right) - 2B(\xi_1) + \frac{7}{4} + \left(\log \frac{\mu_0^2}{m^2} - 1 - 2A(\xi_1) \right) \log(1 - z) - \log^2(1 - z) \right] \right\}$$

where:

$$A(\kappa) = -\gamma_E - \psi_0(\kappa)$$

$$B(\kappa) = \frac{1}{2} \gamma_E^2 + \frac{\pi^2}{12} + \gamma_E \psi_0(\kappa) + \frac{1}{2} \psi_0(\kappa)^2 - \frac{1}{2} \psi_1(\kappa)$$

with:

$$\begin{aligned}
\xi_1 &= 2t - \frac{\alpha(\mu)}{4\pi^2 b_0} \left(1 - e^{-2\pi b_0 t}\right) \left(\frac{20}{9} n_F + \frac{4\pi b_1}{b_0}\right) \\
&= 2t + \mathcal{O}(\alpha t) = \eta_0 + \dots \\
\hat{\xi}_1 &= \frac{3}{2} t + \frac{\alpha(\mu)}{4\pi^2 b_0} \left(1 - e^{-2\pi b_0 t}\right) \left(\lambda_1 - \frac{3\pi b_1}{b_0}\right) \\
&= \frac{3}{2} t + \mathcal{O}(\alpha t) = \lambda_0 \eta_0 + \dots \\
\lambda_1 &= \frac{3}{8} - \frac{\pi^2}{2} + 6\zeta_3 - \frac{n_F}{18} (3 + 4\pi^2)
\end{aligned}$$

and:

$$\begin{aligned}
t &= \frac{1}{2\pi b_0} \log \frac{\alpha(\mu)}{\alpha(\mu_0)} \\
&= \frac{\alpha(\mu)}{2\pi} L - \frac{\alpha^2(\mu)}{4\pi} \left(b_0 L^2 - \frac{2b_1}{b_0} L\right) + \mathcal{O}(\alpha^3), \quad L = \log \frac{\mu^2}{\mu_0^2}.
\end{aligned}$$

Recursive solution

Too involved to be reported here. For the record, the (previously unknown) recursive NLL equations are:

$$\begin{aligned}\mathcal{J}_k^{\text{LL}} &= \mathbb{P}^{[0]} \overline{\otimes} \mathcal{J}_{k-1}^{\text{LL}} \\ \mathcal{J}_k^{\text{NLL}} &= (-)^k (2\pi b_0)^k \mathcal{F}^{[1]}(\mu_0^2) \\ &\quad + \sum_{p=0}^{k-1} (-)^p (2\pi b_0)^p \left(\mathbb{P}^{[0]} \overline{\otimes} \mathcal{J}_{k-1-p}^{\text{NLL}} + \mathbb{P}^{[1]} \overline{\otimes} \mathcal{J}_{k-1-p}^{\text{LL}} \right. \\ &\quad \left. - \frac{2\pi b_1}{b_0} \mathbb{P}^{[0]} \overline{\otimes} \mathcal{J}_{k-1-p}^{\text{LL}} \right)\end{aligned}$$

Integrated PDFs expanded on the basis of the \mathcal{J}^{LL} and \mathcal{J}^{NLL} functions with known coefficients

We have computed these for $k \leq 3$ (\mathcal{J}^{LL}) and $k \leq 2$ (\mathcal{J}^{NLL}), ie to $\mathcal{O}(\alpha^3)$

Results in 1911.12040 and its ancillary files

A remarkable fact

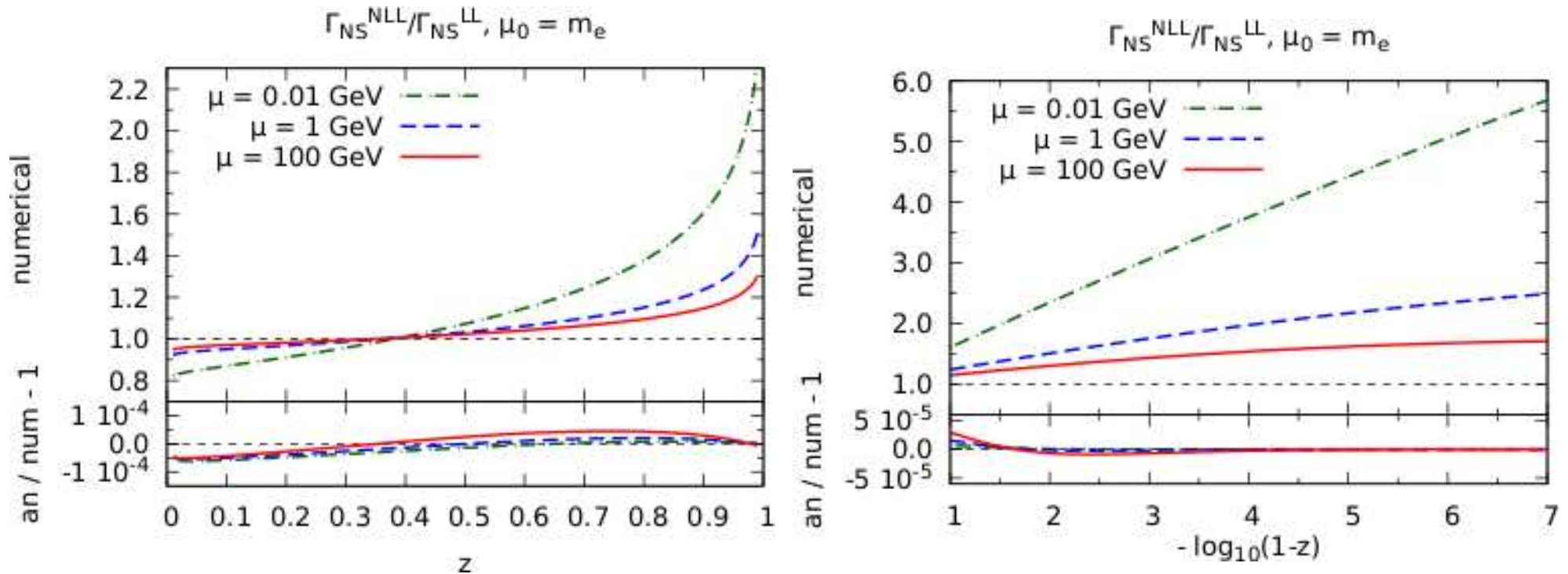
Our asymptotic solutions, expanded in α , feature *all* of the terms:

$$\frac{\log^q(1-z)}{1-z} \quad \text{singlet, non-singlet}$$
$$\log^q(1-z) \quad \text{photon}$$

of our recursive solutions. This ensures a smooth matching

Non-trivial; stems from keeping subleading terms (at $z \rightarrow 1$) in the AP kernels

Sample results



NLL vs LL, non-singlet. The insets show the double ratio, ie numerical vs analytical

This *does not mean* NLO and LO cross sections will differ by a large factor:
PDFs are unphysical

Take-home message

We can easily exploit the enormous amount of work on automated computations of the past decade to port tools which are part of the LHC lore to e^+e^- collider simulations

The easiest way to do so is to exploit the similarities of collinear-factorisation formulae in QCD and QED

The popularity of MadGraph5_aMC@NLO at the LHC with both theorists and experimentalists stems from:

- ◆ Its flexibility
- ◆ The possibility for the user to extend its physics scope (by providing Lagrangians)

MadGraph5_aMC@NLO has been able to perform short-distance NLO EW computations since 2018

At that time, there were thus two missing ingredients for fully-fledged e^+e^- simulations:

◆ QED PDFs of matching accuracy (NLO+NLL)

→ Solved in 1909.03886 and 1911.12040; ongoing work on alternative IR schemes

◆ Implementation of beamstrahlung and QED collinear factorisation

→ Completed, and being stress-tested

We plan to release the first public version this spring