Fast Fourier and z-transforms with almost arbitrary frequency spacing

BLonD Meeting

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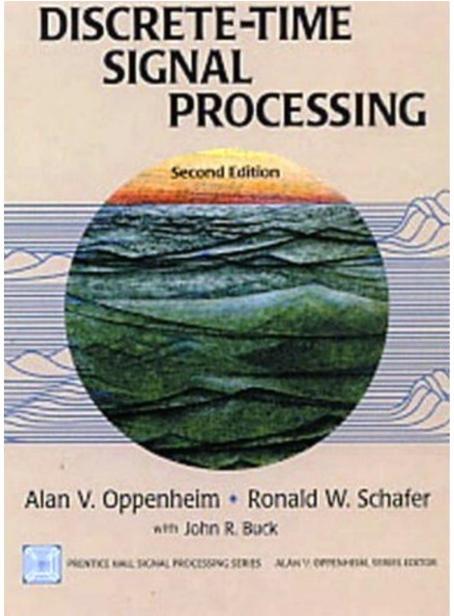
Outline

- Discrete Fourier and z-transforms.
- Linear / circular convolution and time aliasing.
- Chirp Transform Algorithm (CTA).
- Non-Fourier: chirp-z transform (CZT).



Setting the scene – assumptions

- Framework: discrete, finite length sequences.
- Exact calculation of the Fourier transform at almost arbitrary frequencies. (Focus: CTA.)
- Fast calculation, comparable to FFT.
- Potentially smaller size or higher resolution.

Terminology

- Non-uniform DFT: NUDFT
  - The DFT evaluated on non-uniformly sampled data. (Type I.)
  - The DFT evaluated with non-uniform frequency spacing. (Type II.)
  - Or both. (Type III.)
- Calculated using uniformly sampled FFTs → NUFFT.
- NUDFT can be expressed as a z-transform.
- z-transform with “N log N” performance, a special NUFFT:
  - Chirp Transform Algorithm (CTA): DFT with non-uniform frequency spacing. (Type II.)
- Not a Fourier transform: Chirp-z transform (CZT).

Discrete Fourier and z-transforms

Fourier transform:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

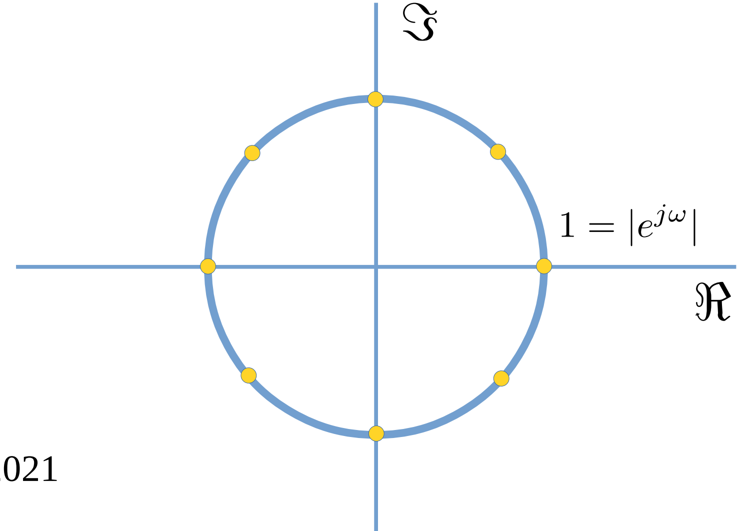
$z = e^{j\omega}$ ←

z-transform:

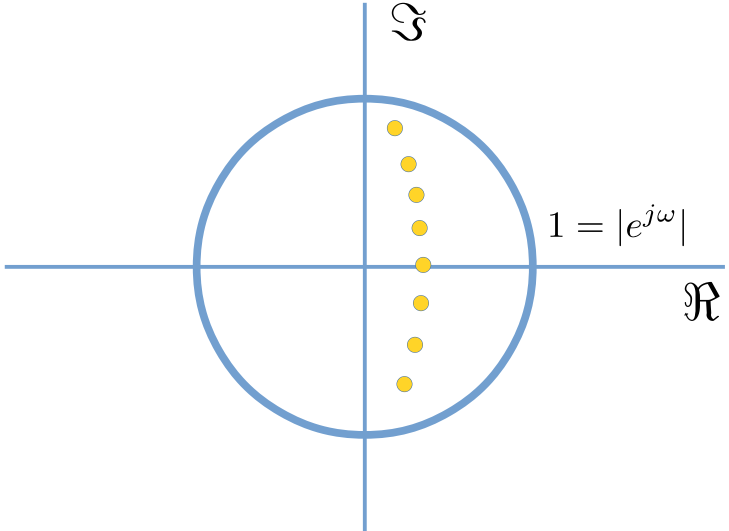
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Uniform sampling of the Fourier transform:

$$X(e^{j(2\pi/N)k}),\ 0 \leq k \leq N-1,\ N = 8$$



Sampling the z transform:



Convolutions in DTSP

Linked to cross-correlation and LTI system response

- Linear lengths N, M:

$$x_3[n] = \sum_{k=0}^{N+M-2} x_1[k] x_2[n-k] = x_1[n] * x_2[n]$$

Linked to Fourier transform

- Circular lengths N > M:

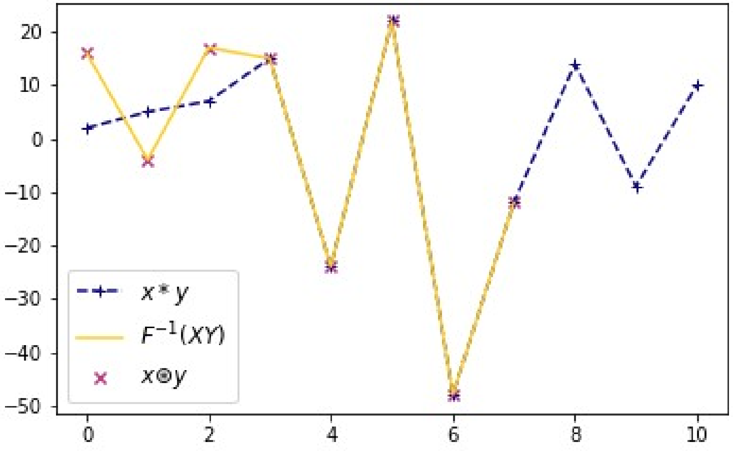
$$x_3[n] = \sum_{m=0}^{N-1} x_1[m] x_2[n-m] = x_1[n] \circledast x_2[n]$$

- Relationship between linear and circular convolutions:
  - They may not be equal except for N-M+1 samples.
  - Consequence: if FFT is used to compute the linear convolution, it must be at least N+M-1 long to avoid time aliasing.

Linear / circular convolution, FFT time aliasing example

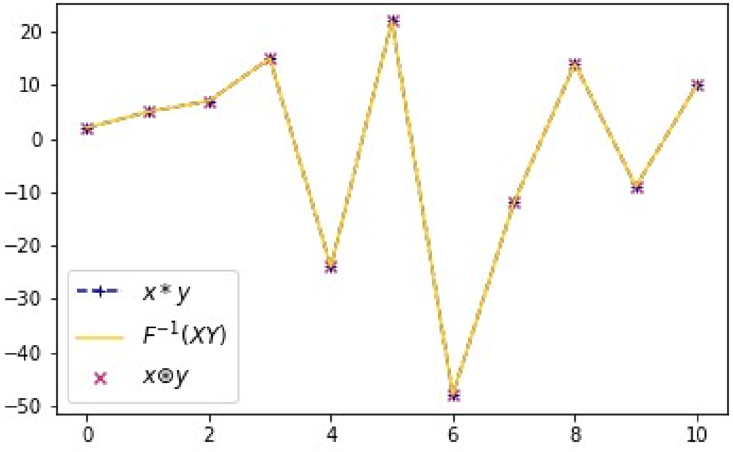
$$\mathrm{len}(\mathbf{x}) = 4 \quad \mathrm{len}(\mathbf{y}) = 8$$

Padding x to 8.



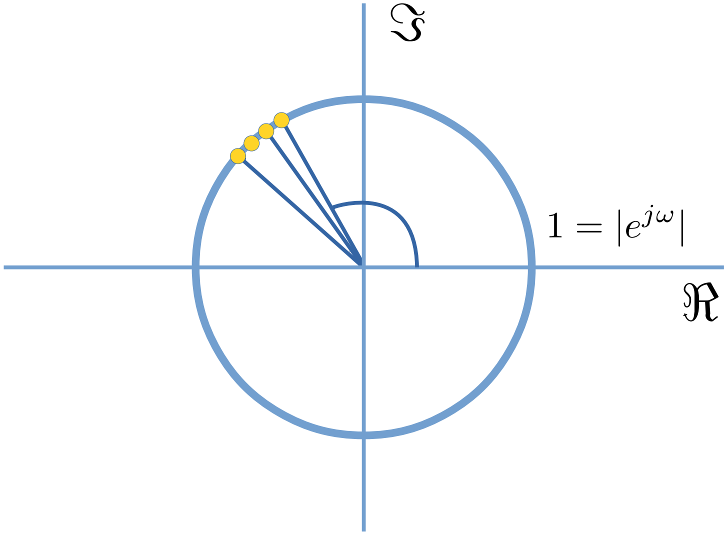
Time aliasing.

Padding x and y to 11.



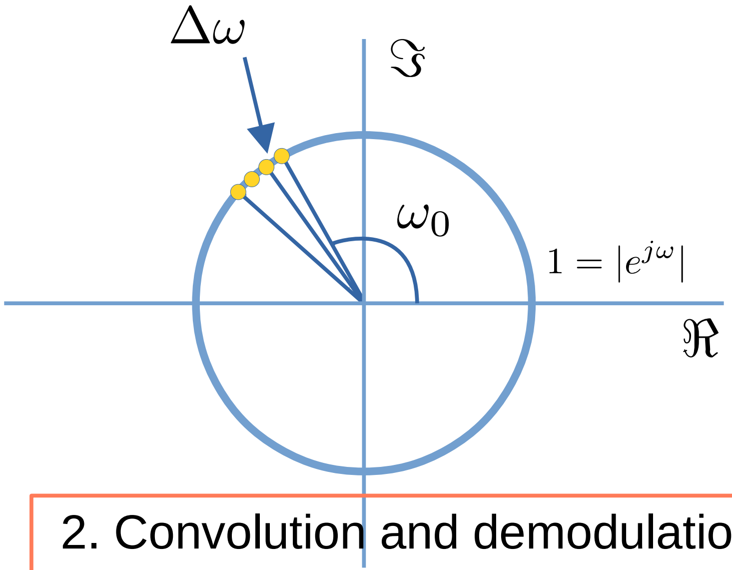
No time aliasing.

Chirp Transform Algorithm (CTA)



- Generalisation of the DFT on the unit circle.
- Partial coverage of the unit circle.
- Spacing can be chosen.
- Special case of the CZT.
- Calculated using fast convoltuion FFT.

Chirp Transform Algorithm (CTA)



$$\omega_k = \omega_0 + k\Delta\omega \quad k = 1, \ldots, M-1$$

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n} \qquad W = e^{-j\Delta\omega}$$

DFT: $\mathrm{M} = \mathrm{N}, \;\; \omega_0 = 0, \;\; \Delta\omega = 2\pi/N$

Substitution: $kn = 1/2\left[k^2 + n^2 - (k-n)^2\right]$

2. Convolution and demodulation

$$W^{k^2/2} \sum_{n=0}^{N-1} g[n] W^{-(k-n)^2/2},$$

Chirp

Circular conv.

1. Chirp modulation

Input series

$$g[n] = x[n] e^{-j\omega_0 n} W^{n^2/2}$$

Frequency offset

Chirp

Illustration 1

- 64/128 frequency sinusoid, amplitude 1. Signal length: 256 samples.
- Rectangular window + FFT.
- 41 point CTA around the peak (same 256 samples).
- Rectangular window transformed around 64.

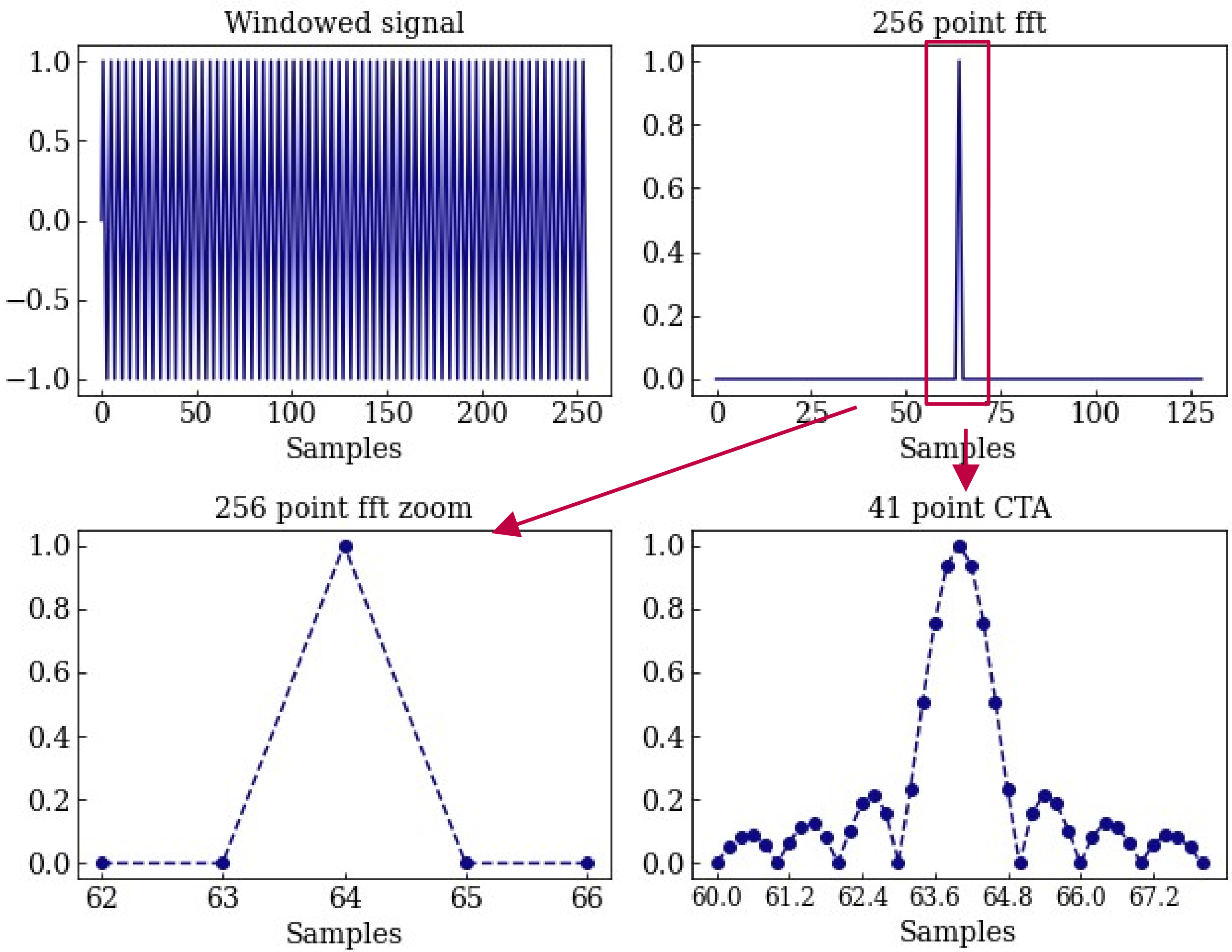
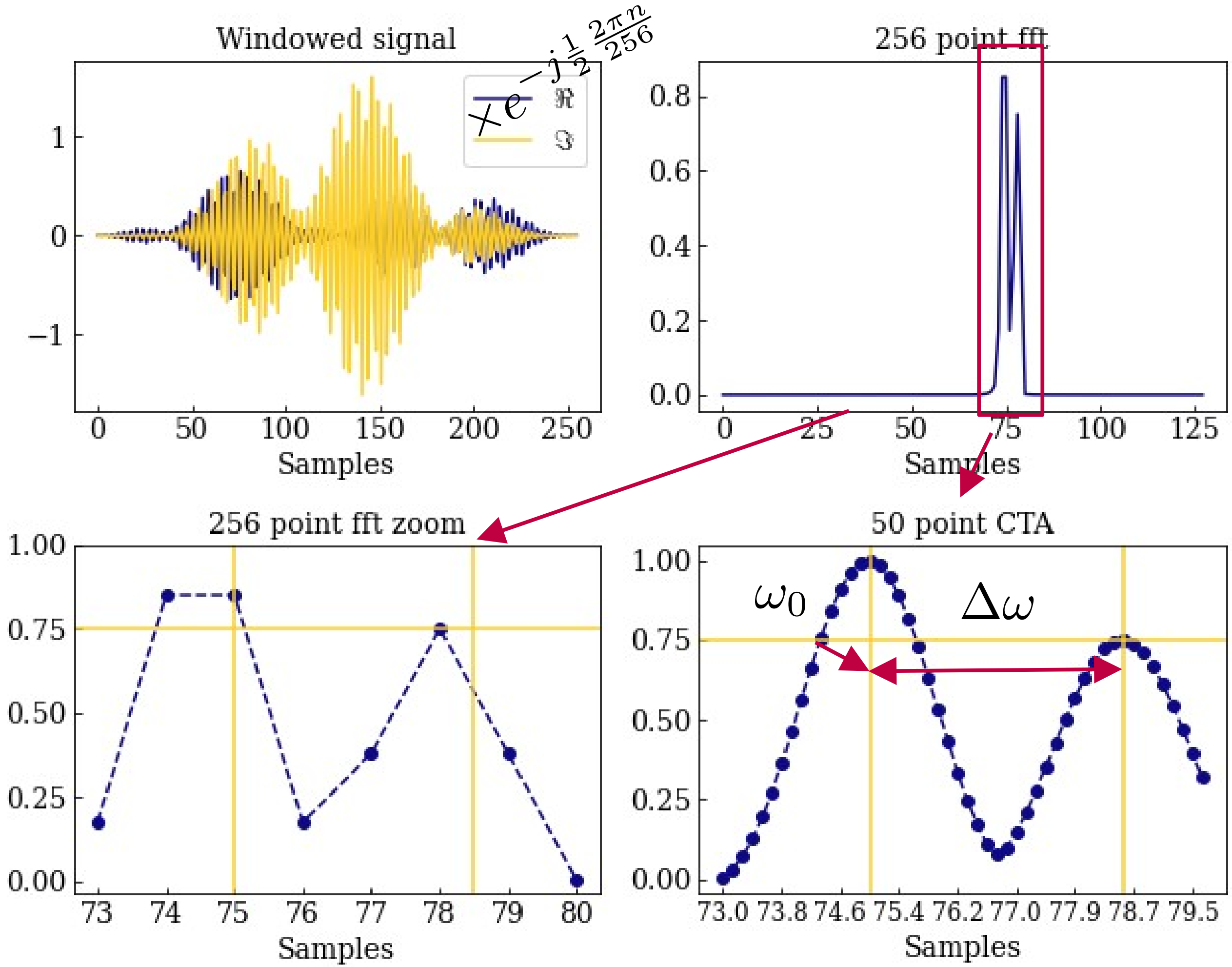


Illustration 2

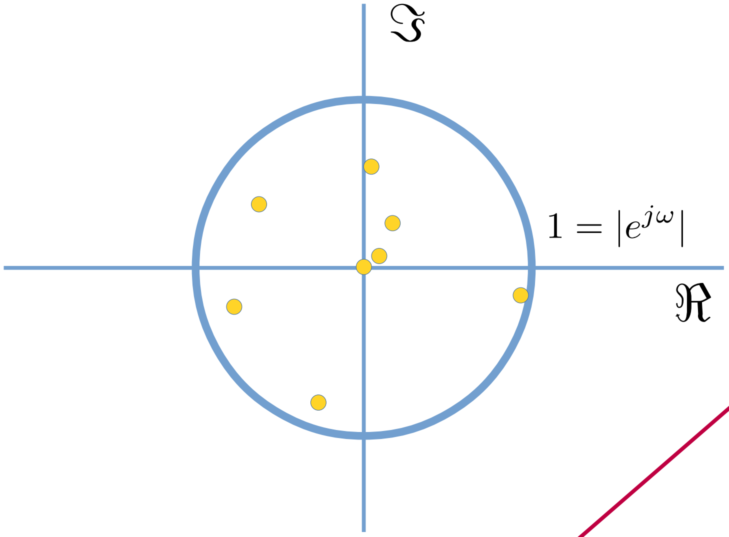
- 75/128 and 78.5/128 frequency sinusoids with 1 and ¾ amplitude added. Signal length: 256 samples.
- Hanning window + FFT.
- CTA calculated for the 256(+50) point signal. ~6x resolution.
- Calculating an only two point CTA would have been sufficient in this example.



CTA computation and use

- Cost: fast convolution needs 2xFFT and 1xIFFT, when implemented the usual way.
- CTA can avoid long FFTs by shortening the input or simplifying binning.
- CTA can zoom in on parts of the spectrum at relatively low computational cost.
- It is particularly suited to systems where the signal processing involves a convolution with a fixed impulse response.
- Typical use case: CCD sensors.

Chirp z-Transform (CZT)



- Going off the unit circle using a spiral.
- Getting closer to points in the z-plane.
- Amplifying buried poles.
- One, simple generalisation compared to CTA:

$$g[n] = x[n]e^{-j\omega_0 n}W^{n^2/2} \rightarrow g[n] = x[n]A^{-n}W^{n^2/2}$$

- "Radial gain" added to "tangential binning".
- Traditional inverse via IFFT, but that is slow.
- Direct, fast inverse exists since 2019.
- CZT-ICZT became a similar pair to FFT-IFFT (N log N effort).

Same as CTA: slide 8.

Implementations in Python

- A summary: https://gist.github.com/endolith/2783807
- Scipy: https://github.com/scipy/scipy/issues/4288
- The new inverse computation is implemented: https://pypi.org/project/czt/ Needs testing at the moment.
- Own code used for examples.

Conclusions

- To overcome the limitations of uniform sampling and the unit circle, the discrete Fourier transform can be conveniently generalised using the z-transform and z-plane.
- CTA adds flexibility to the DFT by allowing a high resolution subsampling of an arbitrary arch on the unit circle. It still maintains N log N performance.
- CZT, the generalised CTA, evaluates the z-transform along a spiral contour of the z-plane. Performance is N log N.
- The recently invented N log N performance ICZT completes the CZT-ICZT pair similar to the FFT-IFFT pair of transforms.

Outlook for BLonD

- CTA / CZT are efficient tools for enhanced spectral analysis. → Mode analysis.
- Because the convolution is calculated via FFT, frequency domain multiplication can be implemented for some relevant samples only that can be spaced flexibly. → Potential gain for induced voltage.
- Next step: check performance with the example from Markus for fast induced voltage.

References

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