Cosmology

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LECTURE 1 & 2

HEP GRADUATE WORKSHOP

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April 03, 2021

Historical Bkg

Cosmology as a science started in early 20th century:

"Some highlights"

1915 : A. Einstein publishes his theory of general relativity

1917: A. Einstein introduces the cosmological constant into his equations to represent a static Universe

1922 : A. Friedmann showed that Einstein eqs allowed for non-static solutions → possibility an expanding Universe.

1925: E. Hubble showed that the MWY ≠ Universe; the Andromeda nebula is another galaxy (≈ 2.5 Mlys)

1929: E. Hubble showed that the Universe is expanding.

Distance Scales & Units

Earth ~ 40,000 km

Astronomical unit

Earth - Sun dist ~ 1.5×10^8 km : ≡ 1 AU

Size of Solar syst ~ 100 AU

The closest star to us ~ 4.2 lys ≈ 1.2 pc

(proxima centauri)

(≃ 3.26 lys)

The size of the MWy ~ 10^5 lys ~ 30 kpc

The closest Galaxy to us ~ 2.5 Mlys ~ 0.7 Mpc

The size cluster of galaxies ~ few × Mpc

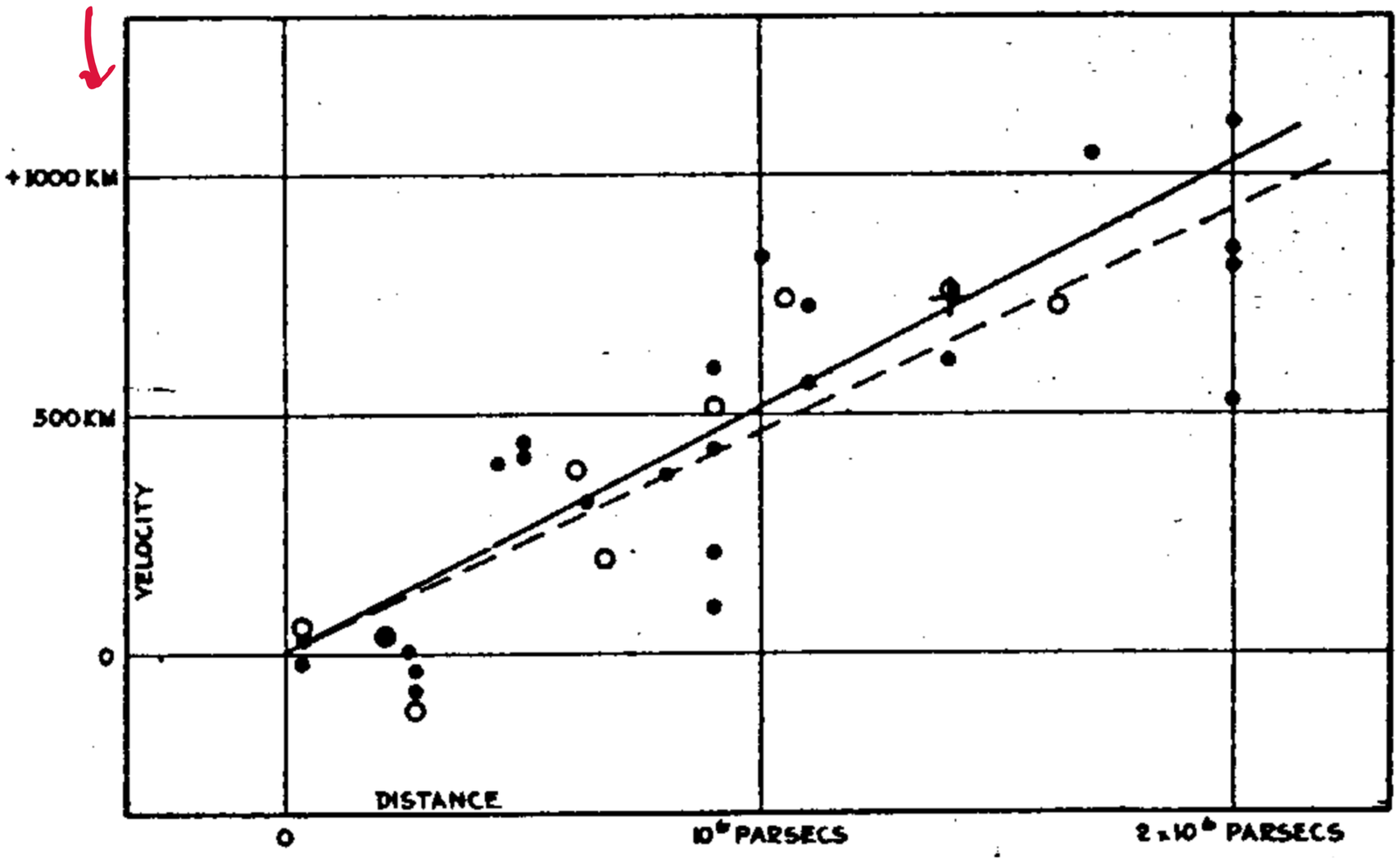
e.g. Virgo ~ 4.5 Mpc

The scale of the Universe ~ 3 Gpc

Observational fact

(1) Universe is expanding

(This is the velocity)



Doppler shift

Recall: Relativistic formula

$$z := \sqrt{\frac{1+\beta}{1-\beta}} - 1$$

from E. Hubble 1929, Proc. Nat. Acad. Sci, 15, 168

Note: on the vertical axis, the units are accidentally written as km rather than km/s.

Hubble measured the Doppler effect of light: $\frac{\Delta\lambda}{\lambda} =: z = \frac{v}{c}$ and distances d of cepheid (variable) stars. He found:

$$c\,z = v = H_0\, d \quad \leftarrow \quad \text{Hubble – Lemaitre Law}$$

IAU (2018) (1927)

Hubble measured $H_0 \simeq 500$ km/s/Mpc

sets a time scale for the age of the universe

$\Rightarrow \tau_{universe} \sim 2$ Gyrs !! ← However, in 1920's and 1930's it was known that there were significant geological evidence that $\tau_{Earth} \gg 2$ Gyrs !!

Recent measurement: $H_0 \simeq 70$ km/s/Mpc with precision $< 10\%$

Time scale: $H_0^{-1} = 13.97$ Gyr ; Hubble radius: $c\,H_0^{-1} = 4.29$ Mpc

(2) The Universe is homogeneous and isotropic at scales $\gg$ 100 Mpc

→ 3-D maps of large structure distribution

2dFGRS & SDSS

→ Cosmic microwave background (CMB)

WMAP, Planck $\qquad \frac{\delta T}{T} \sim 10^{-5}$

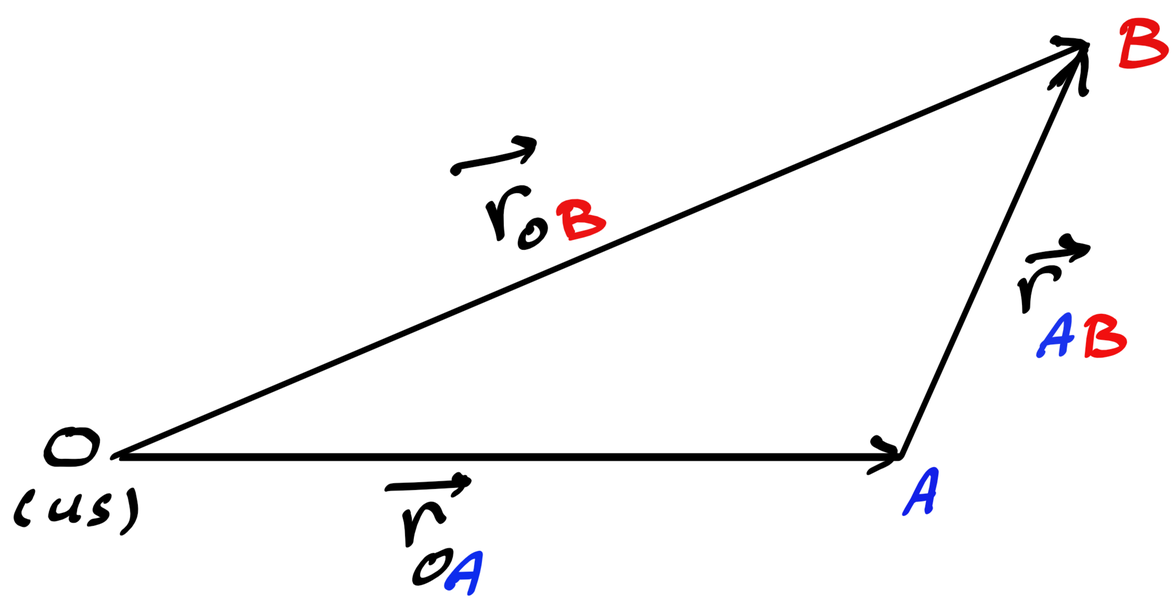
⇒ COSMOLOGICAL PRINCIPLE (CP)

" The Universe is homogenous and isotropic when averaged over very large scales (> 100 Mpc) "

Does Hubble - Lemaitre Law violate the CP ?

Someone may say " since according to HL law galaxies seem to move away from us, are we the center of the Universe ?

→ NO. To see this consider two arbitrary galaxies, say A and B, moving away from us (O) according to HL law.



$$\vec{r}_{AB} = \vec{r}_{oB} - \vec{r}_{oA}$$

$$\vec{v}_{oA} = H_o \vec{r}_{oA} \quad \text{and} \quad \vec{v}_{oB} = H_o \vec{r}_{oB}$$

Now, in the reference frame of the galaxy A, the velocity of the galaxy B reads:

$$\vec{v}_{AB} = \vec{v}_{oB} - \vec{v}_{oA} = H_o \vec{r}_{oB} - H_o \vec{r}_{oA} = H_o \vec{r}_{AB}$$

Same HL in the ref. frame of galaxy A

So, if galaxies are moving away from each other, then at very early time they were infinitesimally close.

→ Big Bang model of the Universe

CP $\Longrightarrow$ FLRW model of the Universe

(Friedmann - Lemaitre - Robertson - Walker)

Let's write the space-time line element in the form.

$$ds^2 = c^2 dt^2 - R^2(t)\, \gamma_{ij}\, dx^i dx^j$$

length scale

Dimensionless

comoving coordinates

i.e. the separation b/w two comoving observers @ rest can change only by the factor $R(t)$.

CP $\Longrightarrow$ MAXIMALLY SYMMETRIC SPACE

i.e. has the largest # of possible killing vectors,

$$\frac{d(d+1)}{2} \quad ; \quad d: \text{dimension of space}$$

(in our case $d=3$)

the space-time is NOT maximally symmetric

things looked different in the past and will look different in the future.

* One can show that a maximally symmetric space its Ricci tensor MUST satisfies:

$$R_{\alpha\beta} \propto g_{\alpha\beta} \quad \text{(Ricci scalar / d)}$$

* Maximally symmetric space is spherically symmetric

$$\rightarrow \quad \gamma_{ij}\, dx^i dx^j = f(\tilde{r})\, d\tilde{r}^2 + \tilde{r}^2 (d\theta^2 + S_\theta^2 d\varphi^2)$$

Dimensionless

Now, compute one of the component of ${}^{(3)}R_{ij}$ and set it, then use the fact it is a maximally symmetric space, i.e

$$ {}^{(3)}R_{ij} = 2k\,\gamma_{ij} $$

constant ($\equiv$ Ricci scalar/6)

(just to simplify the calculation)

Detail:

$$ R_{\tilde{r}\tilde{r}} = \frac{1}{r}\,\frac{f'(\tilde{r})}{f(\tilde{r})} = 2k\,f(\tilde{r}) $$

$$ \Rightarrow f(r) = \frac{1}{1-k\tilde{r}^2} $$

(Plz: don't confuse it with Ricci scalar)

$$ \Rightarrow ds^2 = c^2dt^2 - R^2(t)\left[\frac{d\tilde{r}^2}{1-k\tilde{r}^2} + \tilde{r}^2\,d\Omega^2\right] $$

$$ \underbrace{(d\theta^2 + s_\theta^2\,d\varphi^2)}_{\text{"Wrongly" called solid angle}} $$

<u>Note</u>:

Rescalling $k \to \lambda k$, $\tilde{r} \to \frac{1}{\sqrt{\lambda}}\tilde{r}$, $R(t) \to \sqrt{\lambda}\,R(t)$ leaves ds^2 unchanged.

$\Rightarrow$ We can normalize k to the values $+1, 0, -1$.

$$ \Rightarrow \boxed{ds^2 = c^2dt^2 - R^2(t)\left[\frac{d\tilde{r}^2}{1-k\tilde{r}^2} + \tilde{r}^2\,d\Omega^2\right]} $$

recall that $\tilde{r}$ is dimensionless

Another parametrization of ds^2

Define: $$d\chi = \frac{d\tilde{r}}{\sqrt{1-k\tilde{r}^2}}$$

$$\rightarrow \int_0^{\chi} d\chi' = \int_0^{\tilde{r}} \frac{d\tilde{r}'}{\sqrt{1-k\tilde{r}'}}$$

$$\rightarrow \tilde{r} = S_k(\chi) = \begin{cases} \sin\chi & ; \quad k=+1 \\ \chi & ; \quad k=0 \\ \sinh\chi & ; \quad k=-1 \end{cases} \qquad (*)$$

$$\rightarrow \boxed{ds^2 = c^2dt^2 - R^2(t)\left[d\chi^2 + S_k^2(\chi)\, d\Omega^2\right]}$$

Define

$$a(t) := \frac{R(t)}{R(t_0)} \Rightarrow a(t_0) = 1$$

scale factor

Now

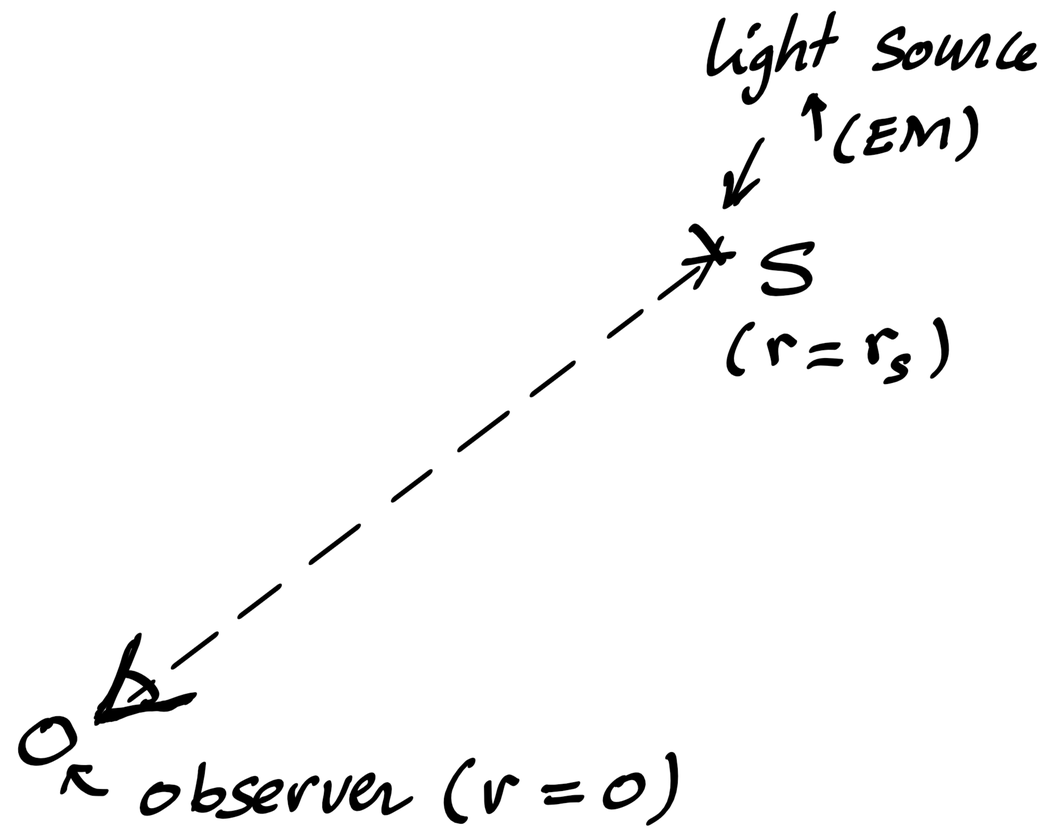
Normalization

$$r := R(t_0)\,\chi$$

dimension of length

$$\boxed{ds^2 = c^2dt^2 - a^2(t)\left[dr^2 + R_0^2\, S_k^2(r/R_0)\, d\Omega^2\right]}$$

Redshifts / Distances



* Redshift $\longleftrightarrow$ Expansion

EM travelling $\Rightarrow$ $ds^2 = 0$

Consider a radial direction, $(d\theta = d\varphi = 0)$

(I'll set $c = 1$)

$$\Rightarrow c\,dt = -a(t)\frac{dr}{\sqrt{1-kr^2}}$$

observer $r = 0$ $\rightarrow$ r decreasing from S to O.

Source emit a pulse of light @ $t = t_e$

observer receive it @ $t = t_o$

$$\int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_0^{r_s} \frac{dr}{\sqrt{1-kr^2}}$$

independent of time (comoving coord's)

The next pulse: $\begin{cases} \text{emitted by the source @ } t = t_e + \delta t_e \\ \text{received by the observer @ } t = t_o + \delta t_o \end{cases}$

$$\Rightarrow \int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_{t_e+\delta t_e}^{t_o+\delta t_o} \frac{dt}{a(t)} \Longrightarrow \int_{t_e}^{t_e+\delta t_e} \frac{dt}{a(t)} = \int_{t_o}^{t_o+\delta t_o} \frac{dt}{a(t)}$$

$$\Rightarrow \frac{\delta t_e}{a(t_e)} = \frac{\delta t_o}{a(t_o)} \xrightarrow{\delta t = \lambda} \frac{\lambda_{obs}}{\lambda_{em}} = \frac{a_o}{a(t_e)} := 1 + z$$

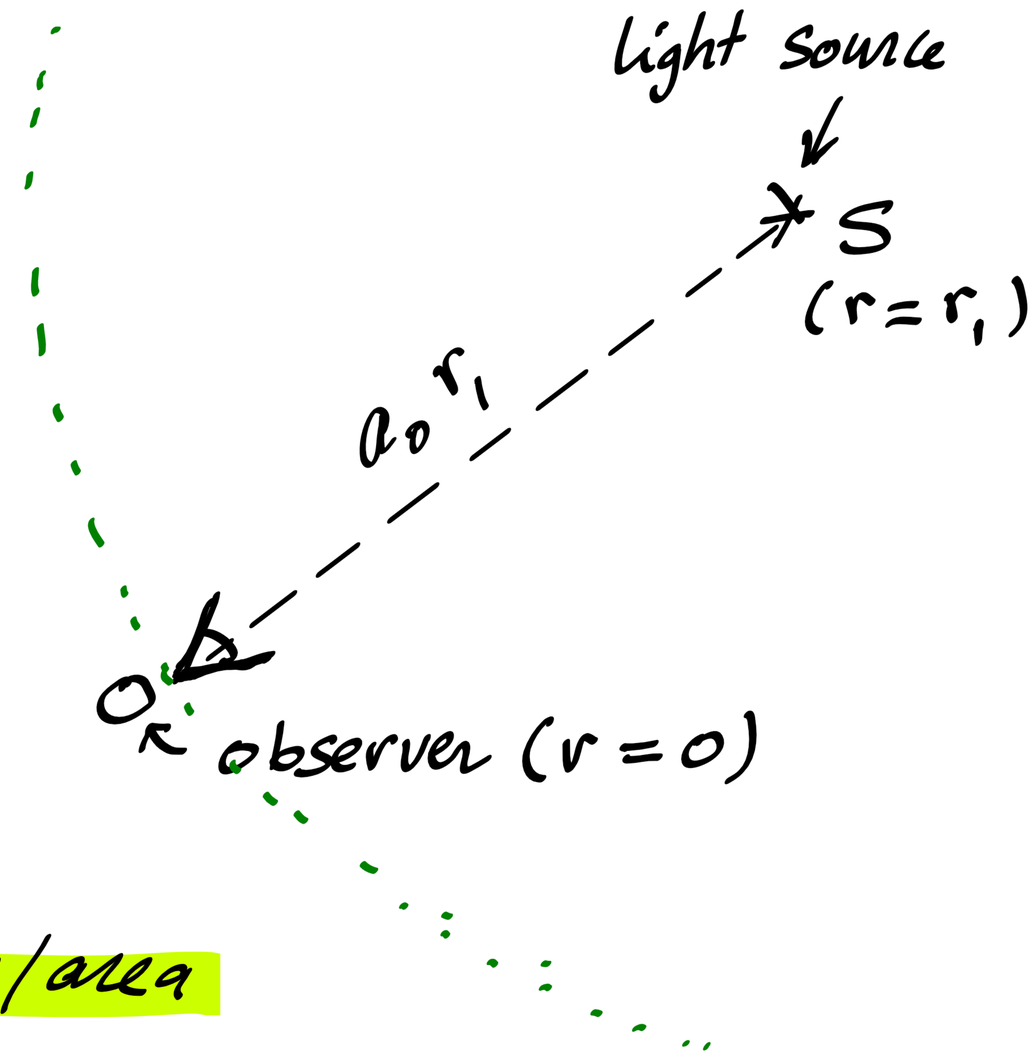
Thus: $\begin{cases} \text{if } \lambda_{obs} > \lambda_{em} \Rightarrow a_o > a(t_e) \Rightarrow \text{Expansion} \\ \text{if } \lambda_{obs} < \lambda_{em} \Rightarrow a_o < a(t_e) \Rightarrow \text{Contraction} \end{cases}$

* Luminosity distance

EM source with intrinsic luminosity $\mathcal{L}$

(a.k.a. absolute luminosity)

Energy / Time



The observer measures flux $(\mathcal{F})$

Energy / time / area

$$\mathcal{F} = \underbrace{\mathcal{L}\left(\frac{a_1}{a_0}\right)^2}\ \underbrace{\frac{1}{\text{Area}\,(S^2)}}$$

of radius $a_0 r_1$

$$= 4\pi R_0^2 S_k^2(r_1/R_0)$$

$$\begin{cases} \left(\frac{a_1}{a_0}\right) \leftarrow \text{photon energy gets redshifted} \\ \left(\frac{a_1}{a_0}\right) \leftarrow \text{time b/w arrival of photon is stretched due to frequency redshift} \end{cases}$$

$$= 4\pi \begin{cases} r_1^2 & ; k = 0 \\ R_0^2 \sin^2(r_1/R_0) & ; k = +1 \\ R_0^2 \sinh^2(r_1/R_0) & ; k = -1 \end{cases}$$

Using the fact that $\frac{a_0}{a_1} := (1+z)$, we can write

$= \frac{1}{a_1}$ (since $a_0 = 1$)

$$\rightarrow \quad \mathcal{F} = \frac{L}{4\pi d_L^2(z)}$$

where $d_L(z) = (1+z) R_0 S_k(\chi_1)$

$(\chi_1 = r_1/R_0)$

For flat space $(k=0)$: $d_L(z) = (1+z)\, r_1$

Let's Taylor the scale factor $a(t)$ about the present time t_0:

$$a(t) = \underset{\substack{\uparrow \\ (a(t_0)=1)}}{1} + H_0(t-t_0) - \frac{1}{2}\underset{\uparrow}{q_0} H_0^2 (t-t_0)^2 + \dots$$

Here $q_0 := -\frac{\ddot{a}\,a}{\dot{a}^2} = -\frac{\ddot{a}}{aH^2}$: deceleration parameter

light $\Rightarrow$ $$\chi_1 = c\int_{t_1}^{t_0} \frac{dt}{a(t)} = c\int_{t_1}^{t_0} \left[1 - H_0(t-t_0) + \dots\right] dt$$

$$= c(t_0 - t_1)\left[1 + \frac{1}{2}H_0(t_0-t_1) + \dots\right] \qquad (*)$$

Recall : $(1+z) = \frac{1}{a(t_1)}$

$$\Rightarrow \quad z = \frac{1}{a_1} - 1 \simeq H_0(t_0-t_1) + \frac{1}{2}(2+q_0)H_0^2(t_0-t_1)^2 + \dots$$

Now, we invert it to solve for (t_0-t_1), and obtain

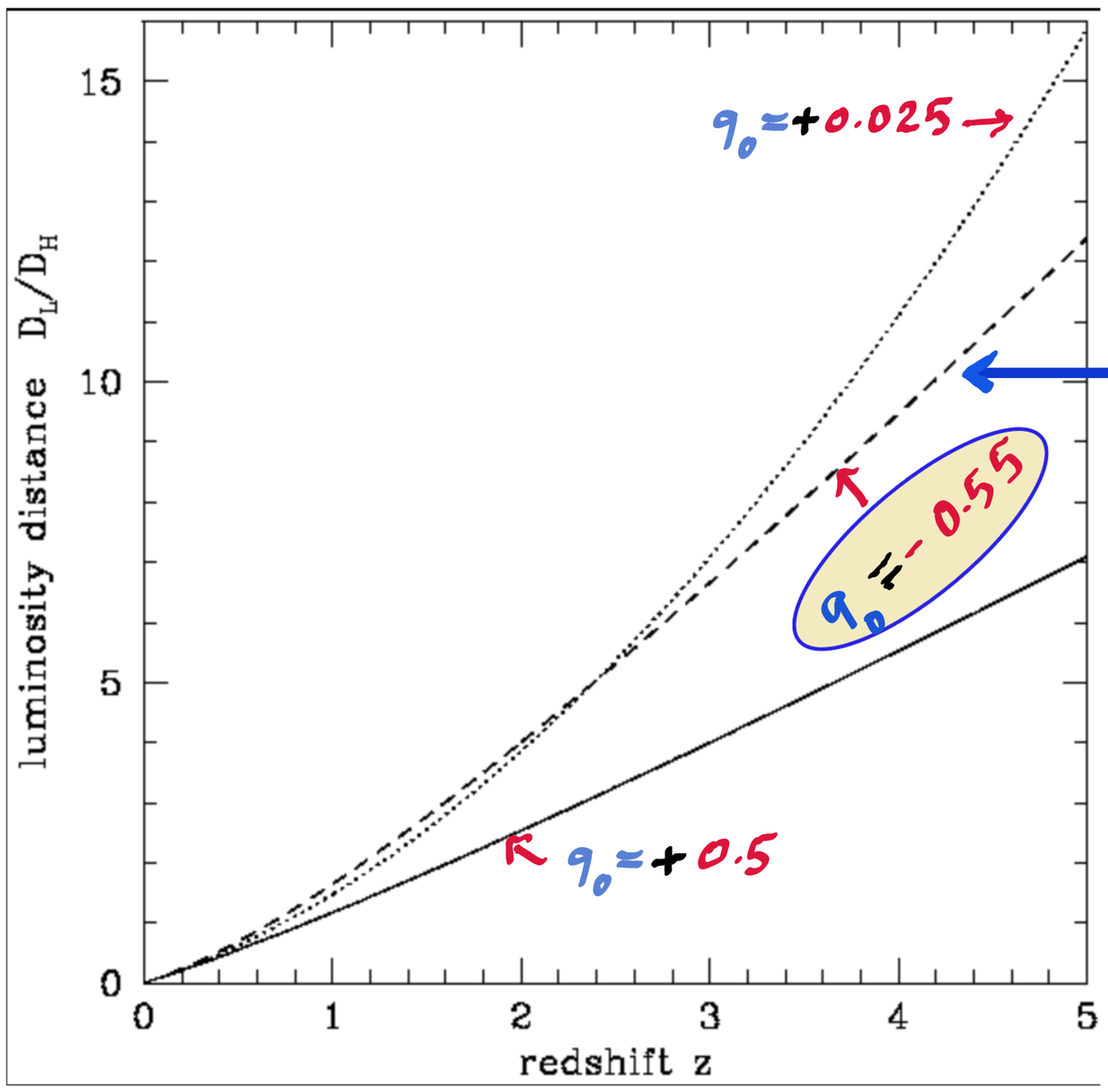
$$H_0(t_0-t_1) = z - \frac{1}{2}(2+q_0)z^2 + \dots \qquad (z<1)$$

$$(*) \Rightarrow \frac{H_0\chi_1}{c} = z - \frac{1}{2}(1+q_0)z^2 + \dots$$

$$\Rightarrow \quad d_L(z) = \frac{c}{H_0}\left[z + \frac{1}{2}(1-q_0)z^2 + \dots\right]$$

$d_L(z)$ v.s z $\Rightarrow$ H_0 and $q_0 = \frac{1}{2}\Omega_M^{(0)} - \Omega_\Lambda^{(0)}$

(can be extracted for $z \sim 0$)



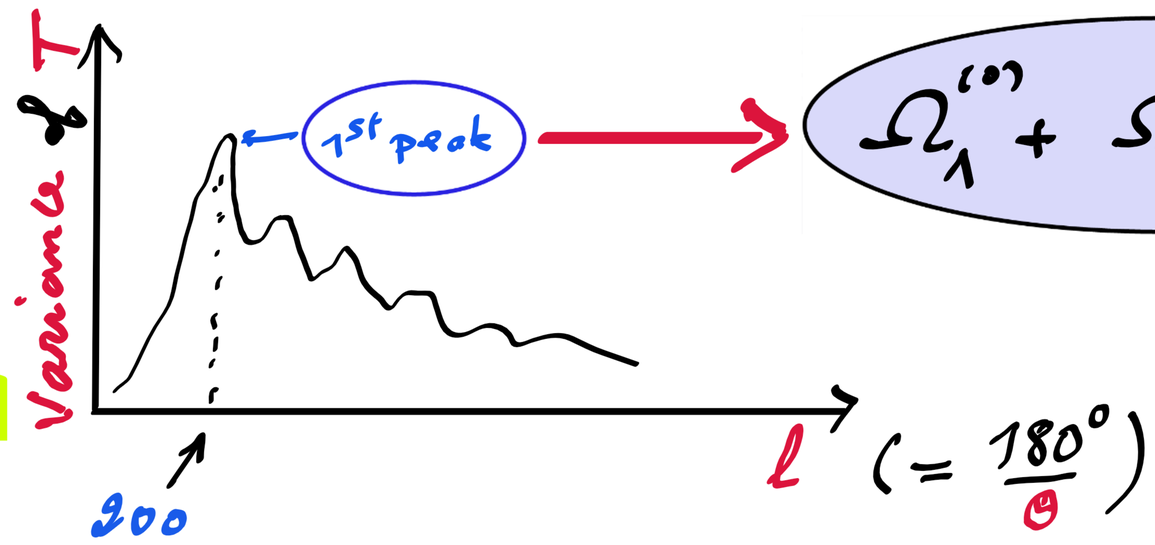
presented in terms of $\log(D_L/\mathrm{Mpc})$

SNIa Data

$$\frac{1}{2}\Omega_M^{(0)} - \Omega_\Lambda^{(0)} \simeq -0.55$$

CMB

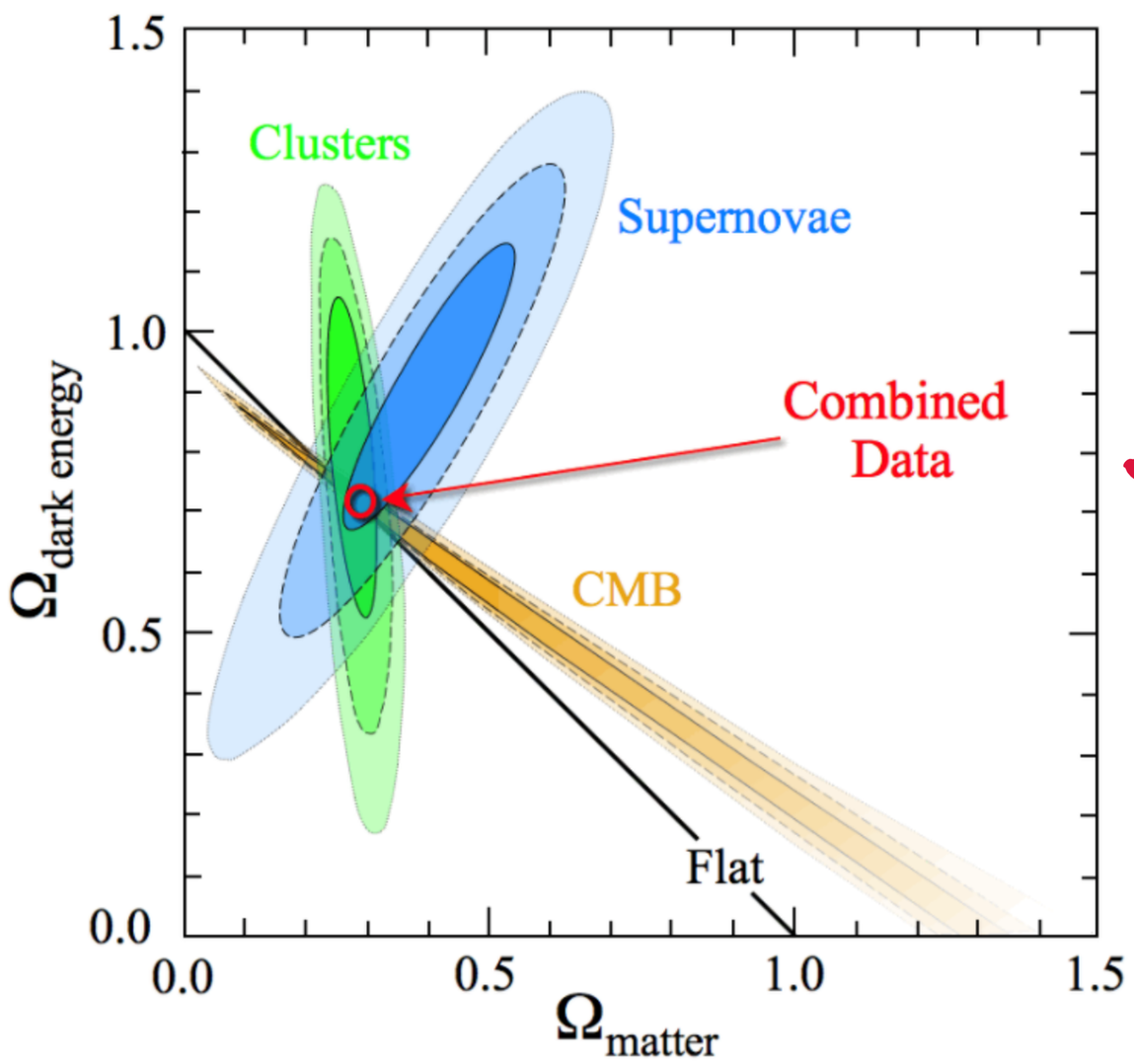
think of it $\sim \left(\frac{\Delta T}{T}\right)^2$



$$\Omega_\Lambda^{(0)} + \Omega_M^{(0)} \simeq 1$$

SNIa & CMB

$$\Omega_\Lambda^{(0)} \simeq 70\% \ \& \ \Omega_M^{(0)} \simeq 30\%$$



The Standard Model of Cosmology

Energy Content of the Universe

Visible matter : $\Omega_{vis}^{(0)} \sim 0.3\,\%$

(stars, galaxies, ...)

Baryonic matter : $\Omega_b^{(0)} \sim 4\,\%$

photons : $\Omega_\gamma^{(0)} \sim 5 \times 10^{-5}$

Neutrino : $\Omega_\nu^{(0)} \sim 10^{-5}$

Dark matter : $\Omega_{DM}^{(0)} \sim 27\,\%$

Dark energy : $\Omega_{DE}^{(0)} \sim 68\,\%$

Dynamics of the FLRW Universe

Einstein Eqns:

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\equiv\, G_{\mu\nu}} = \frac{8\pi G}{3} T_{\mu\nu}$$

Ricci scalar

Ricci tensor

Einstein tensor

stress-energy-momentum tensor.

can be derived from Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G} \int R \sqrt{-g}\, d^4x$$

Note that under gen coord. transf.:

$x^\mu \to x'^\mu$

$g \to J^{-2}$

$d^4x \to |J| d^4x$

$\left(J_{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\nu} : \text{Jacobian of the transf} \right)$

Exercise:

show that in the FLRW universe, we've

$$G_{00} = \frac{3k}{a^2} + \frac{3\dot{a}}{a^2}$$

$$G_{i0} = 0$$

$$G_{ii} = -\frac{(2a\ddot{a} + \dot{a}^2 + k)}{1 - kr^2}$$

$$T_{\mu\nu} = ?$$

$$CP \Longrightarrow \begin{cases} T_{00} \text{ is a fct of time only} \\ T_{0i} = 0 \\ T_{ij} \propto g_{ij} \end{cases}$$

energy density

$$\Longrightarrow \quad T_{\mu\nu} = \begin{pmatrix} \rho(t) & \\ & P(t)\, g_{ij} \end{pmatrix} \qquad (c=1)$$

pressure $\longrightarrow T_{ii} = P \dfrac{a^2}{1-kr^2}$

Einstein Eqns read: (I am setting $c=1$)

$$G_{00}: \qquad \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \qquad (1)$$

$$G_{11}: \qquad 2\left(\frac{\ddot a}{a}\right) + \left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = -8\pi G P \qquad (2)$$

These Eqns were derived in 1922 by A. Friedmann

(confirmed independently by G. Lemaître in 1927) $\longrightarrow$ (7 years before Hubble's discovery of the expansion of the universe.)

$$(2)-(1) \Longrightarrow \quad \frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3P) \qquad (3)$$

Exercise:

Show that the equ: $\nabla_\mu T^\mu{}_0 = 0$ yields

$$\dot{\rho} + 3H(\rho + P) = 0. \qquad (*)$$

Equation $(*)$ can be rewritten as:

$$\underbrace{d(\rho a^3)}_{} = -p\, d(a^3)$$

$$dE = -P\, dV \longleftarrow 1^{st} \text{ Law of thermodynamics (Conservation of energy)}$$

We have 3 unkown functions $\{\rho(t), P(t), a(t)\}$ and 2 equations (Eqns (1) and (3)) $\rightarrow$ Need Eq of State, i.e.

$$P = P(\rho) \qquad \text{(E.o.S)}$$

We'll consider $P(t) = \omega\, \rho(t)$ (4)

ω: constant

As we'll see all type of matter and energy that you're familiar with follow the E.o.S in (4).

Plugging (4) into $(*)$, we get

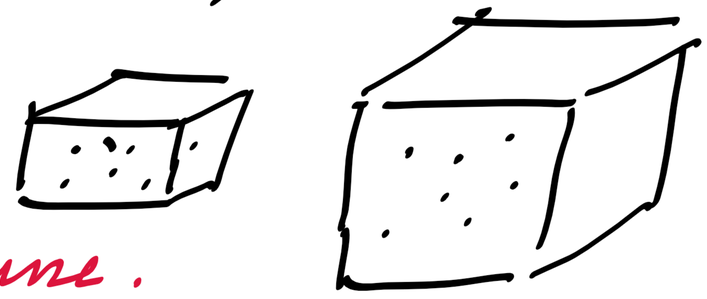
$$\rho(t) \propto a(t)^{-3(1+\omega)}$$

Examples

(1) Non-relativistic ptcles: Jargon → "Matter"

$$\omega \simeq 0 \Longrightarrow \rho \propto a^{-3}$$

expected: decrease as the inverse of volume.



Why $\omega \simeq 0$?

$$P_{(i)} = \frac{m v_i}{\Delta t} \frac{N}{A}$$ ← # of ptcles

pressure along direction x_i:

$$= \frac{m v_i}{\cancel{\Delta t}} \frac{n}{\cancel{A}} v_i \cancel{\Delta t} \cancel{A}$$ ← density

$$P_{(i)} = m n v_i^2 \underset{(av.)}{=} \frac{1}{3} m n \langle v^2 \rangle$$

$$= \frac{1}{3} \langle \vec{p} \cdot \vec{v} \rangle n$$

momentum

$$\rho \simeq \frac{N m c^2}{V} = n m c^2$$ ← energy @ rest

$$\Rightarrow P/\rho \sim \left(\frac{v}{c}\right)^2 \approx 0$$

(2) Relativistic ptcles: Jargon: "radiation"

$$\omega = \frac{1}{3} \Longrightarrow \rho \propto a^{-4}$$

a^{-3}: volume expansion

a^{-1}: wavelength streatching

Why?

$$P_R = \frac{1}{3} \langle \vec{p} \cdot \vec{v} \rangle \times n_\gamma$$

$$\simeq \frac{1}{3} (\varepsilon_\gamma / c) \times c \times n_\gamma = \frac{1}{3} \varepsilon_\gamma n$$

$$\rho_R = \frac{N \varepsilon_\gamma}{V} = n \varepsilon_\gamma$$

$$\Rightarrow P_R / \rho_R \simeq 1/3$$

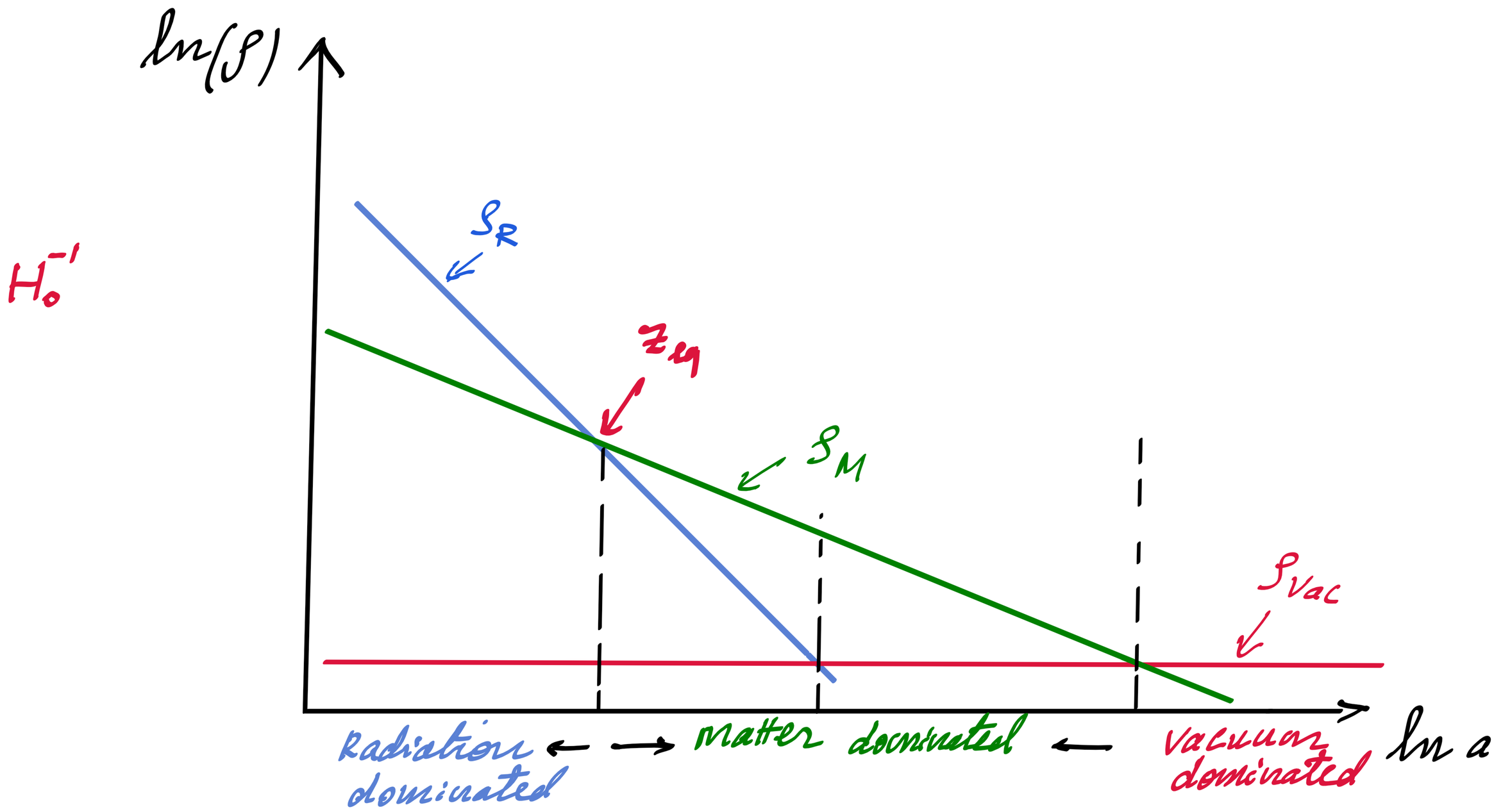
(3) Vacuum energy Jargon "cosmological const."

$$P = -\rho \Longrightarrow \omega = -1$$

Einstein added the term $\Lambda g_{\mu\nu}$ to the RHS of his equation

$$\Rightarrow \rho_{vac} = \frac{\Lambda}{8\pi G} \quad \text{and} \quad P_{vac} = -\frac{\Lambda}{8\pi G}$$

Note that $\Lambda g_{\mu\nu}$ does not vanish in the limit of flat spacetime.



z_{eq}: redshift @ which MAT energy density = RAD energy density

$$\Downarrow$$

$$\Omega_R^{(0)} (1+z_{eq})^4 = \Omega_M^{(0)} (1+z_{eq})^3$$

$$\left(\Omega_{CMB}^{(0)} + \Omega_\nu^{(0)}\right)$$

$\longrightarrow$ $z_{eq} \simeq 3400$ $\rightarrow$ $t_{eq} \simeq 50{,}000$ yrs

(indep of H_0) (b/c ratio)

($H_0 = 70$ km/s/Mpc)

Exercise:

Show that the epoch of equality of Vacum and matter energy densities occur at $z \simeq 0.55$

Age of the observable Universe

Define:

$$\Omega_X^{(0)} := \frac{\rho_X^{(0)}}{\rho_c^{(0)}}$$

$\leftarrow$ energy density of a component X

$\leftarrow$ critical energy density

$$\rho_c^{(0)} := \frac{3 H_0^2}{8\pi G} = 8.4\times 10^{-30}\ g/cm^3 \simeq 5\ m_{proton}/m^3$$

(1):

$$\left(\frac{\dot a}{a}\right)^2 = H_0^2\left[\Omega_M^{(0)}\left(\frac{a}{a_0}\right)^{-3} + \Omega_R^{(0)}\left(\frac{a}{a_0}\right)^{-4} + \Omega_\Lambda + (1-\Omega_{tot}^{(0)})\left(\frac{a}{a_0}\right)^{-2}\right]$$

or, in term of redshift as:

$$H^2(z) = H_0^2\underbrace{\left[\Omega_M^{(0)}(1+z)^3 + \Omega_R^{(0)}(1+z)^4 - (1-\Omega_{tot})(1+z)^2 + \Omega_\Lambda\right]}_{E(z)}$$

Recall: $(1+z) = \frac{1}{a}$ (a_0)

$$\Rightarrow \frac{dz}{dt} = -\frac{1}{a^2}\frac{da}{dt}$$

$$H(z) = \frac{1}{a}\frac{da}{dt} = -(1+z)^{-1}\frac{dz}{dt}$$

$$\Rightarrow \boxed{t_0 = \int_0^{t_0} dt = \frac{1}{H_0}\int_0^\infty \frac{dz}{(1+z)\sqrt{E(z)}}}$$

e.g. 1: $\underbrace{\Omega_{tot}^{(0)} = \Omega_M^{(0)} = 1}_{\text{called Einstein-deSitter Universe.}} \Rightarrow t_0^{(EdS)} = \frac{2}{3}H_0^{-1} \simeq 9.45\ Gys$

e.g. 2: $\Omega_M + \Omega_\Lambda = 1 \rightarrow t_0^{(\Lambda)} = \frac{2}{3}H_0^{-1}\overbrace{\frac{1}{\sqrt{\Omega_\Lambda^{(0)}}}\ln\left[\frac{1+\sqrt{\Omega_\Lambda^{(0)}}}{\sqrt{\Omega_\Lambda^{(0)}}}\right]}^{\geq 1}$

for $\Omega_\Lambda^{(0)} \simeq 0.7 \rightarrow t_0 \approx 0.96\ H_0^{-1} \simeq 13.7\ Gyrs$

Thermal History of the Universe

* The Early Universe was very hot
* There were thermal equilibrium (T)
* Depending on T, it was popullated with different type of "elementary" ptcles. e.g: MeV $< T <$ 100 MeV $\rightarrow \{ e^{\pm}, \gamma, \nu \}$

  ($\&$ Baryons $\approx 10^{-10} \times \gamma$'s)

  (D: ν_L and ν_R; M: $\nu = \nu^c$)

Define:

- $f(\vec{x}, \vec{p}; T)$: # ptcles per unit Volume of space & momentum

  "phase-space"

Homogeneity $\longrightarrow$ $f(\vec{x}, \vec{p}; T) = f(\vec{p}; T)$: indep of the position

Isotropy $\longrightarrow$ $f(\vec{p}; T) = f(p; T)$: depends only on $|\vec{p}|$

$p = (|\vec{p}|)$

Thus, CP $\Longrightarrow$ $f(\vec{x}, \vec{p}; T) = f(p; T)$

- g : # of internal degrees of freedom of the ptcle in the gas.

  (such as spin, color, or any other QM #'s)

e.g 1) Massive ptcle with spin S $\Rightarrow$ $g = 2S + 1$

2) Massless gauge boson (e.g. photon) (or graviton, $s=2$) $\Rightarrow$ $g = 2$

only transverse polarizations

- particle # density : $n = \frac{g}{(2\pi)^3} \int f(p,t)\, d^3\vec{p}$

  particle energy density : $\rho = \frac{g}{(2\pi)^3} \int E(p)\, f(p,t)\, d^3\vec{p}$

  particles pressure : $P = \frac{g}{(2\pi)^3} \int \frac{p^2}{3E(p)}\, f(p,t)\, d^3\vec{p}$

  (recall: pressure $= \frac{1}{3} \vec{p}\cdot\vec{v}\, \frac{N}{V}$)

- If ptcle species "i" is in thermal equilibrium then

  (kinetic equilibrium $i + j \to i + j$)
  i.e. elastic scattering

$$f_{(i)}(p,t) = \frac{1}{\exp\left[\frac{E_i - \mu_i}{T(t)}\right] \pm 1}$$

$+$: FD
$-$: BE

(I am working in USU) $k_B = c = \hbar = 1$

μ_i: chemical potential of the species "i".

If the species are also in chemical equilibrium, e.g.
$i + j \to k + l$, (ptcle species is not conserved),
then: $\mu_i + \mu_j = \mu_k + \mu_l$

ONLY the total # of species is conserved

* γ's are emitted and absorbed in arbitrary #'s

  e.g. Bremsstrahlung
  ect....

  $\Rightarrow \mu_\gamma = 0$

* ptcle + anti-ptcle $\xrightarrow[\text{(can)}]{}$ γ's

  $\Rightarrow \mu_{\text{anti-ptcle}} = -\mu_{\text{ptcle}}$

When Thermal equilibrium is achieved?

it's fast enough compared to the expansion rate of the Universe:

$$\Gamma_{(i)}^{(int)} \geq H \qquad (*)$$

A ptcle species for which (*) does not hold is said to have frozen out or decoupled.

(for the breaking of chemical equilibrium) (for the breaking of kinetic equilibrium)

Exercise

show that for relativistic ptcles (i.e. $T \gg m$):

1) Consider a gas of different species of ptcles. Show that if $T \gg$ the masses of the ptcles (i.e. they're relativistic), then

$$\rho_R = \frac{\pi^2}{30} g_* T^4 \quad \text{and} \quad P_R = \frac{1}{3}\rho_R$$

(i.e. $\omega = 1/3$)

where g_* is the effective # of massless (or relativistic) degrees of freedom, given by

$$g_* = \sum_{\text{bosons}} g_b + \frac{7}{8}\sum_{\text{fermions}} g_f$$

2) List all the ptcles that are in thermal equilibrium @ $T \ll$ MeV, and calculate the corresponding g_*.

3) What is g_* @ $T > 100$ GeV.

Consider a period in the early Universe when it was RAD.-dominated:

$$\Rightarrow \rho = \rho_R \propto T^4$$

But we saw that for radiation $\rho_R \propto a^{-4}$

$$\Rightarrow \quad T \propto a^{-1}$$

Recall: $H^2 \underset{(k=0)}{=} \frac{8\pi G}{3}\rho \propto T^4 \Rightarrow H(T) \propto T^2$

What about $\Gamma_{(i)}^{(int)}$?

$$n \langle \sigma . v \rangle$$

cross-section

relative velocity

density of the ptcles that ptcle "i" is colliding with

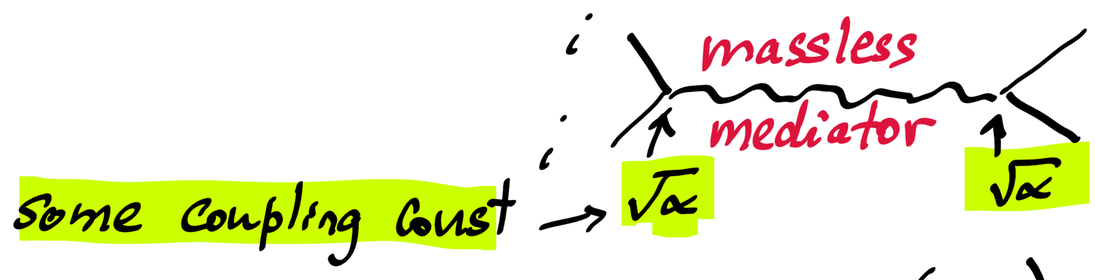
Just from dimension analysis we can write

$$\sigma(E) \sim E^2 |\mathcal{M}|^2 = \sigma(T)$$

(factors ~ 1) (which I don't care for them here)

energy of the process $\sim T$ for relativistic ptcles "i"

Consider interaction mediated by massless bosons (e.g. photon)



$$\Rightarrow \sigma(T) \sim \frac{\alpha^2}{T^2} \Rightarrow \Gamma \sim \overset{n}{T^3} \times \frac{\alpha^2}{T^2} \times \overset{(v \sim c)}{1}$$

$$\rightarrow \Gamma \sim \alpha^2 T \longrightarrow$$ Equilibrium holds for

$$T \lesssim \alpha^2 M_{Pl} \sim \left(\frac{\alpha}{10^{-2}}\right)^2 10^{15}\ \text{GeV}$$

Exercise

Show that for interactions mediated by massive boson X with $M_X \geq T$, thermal equilibrium holds @ $T \geq \left(\frac{M_X}{100\,\text{GeV}}\right)^{4/3}$ MeV

(assume the coupling consts ~ 1)