QCD and Event Generators

Lecture 1 of 2

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Some slides of these Lectures are based on:

- talk by P. Skands at HCPSS, 2020
- Lectures by T. Sjöstrand at Lund, 2018

Contents:

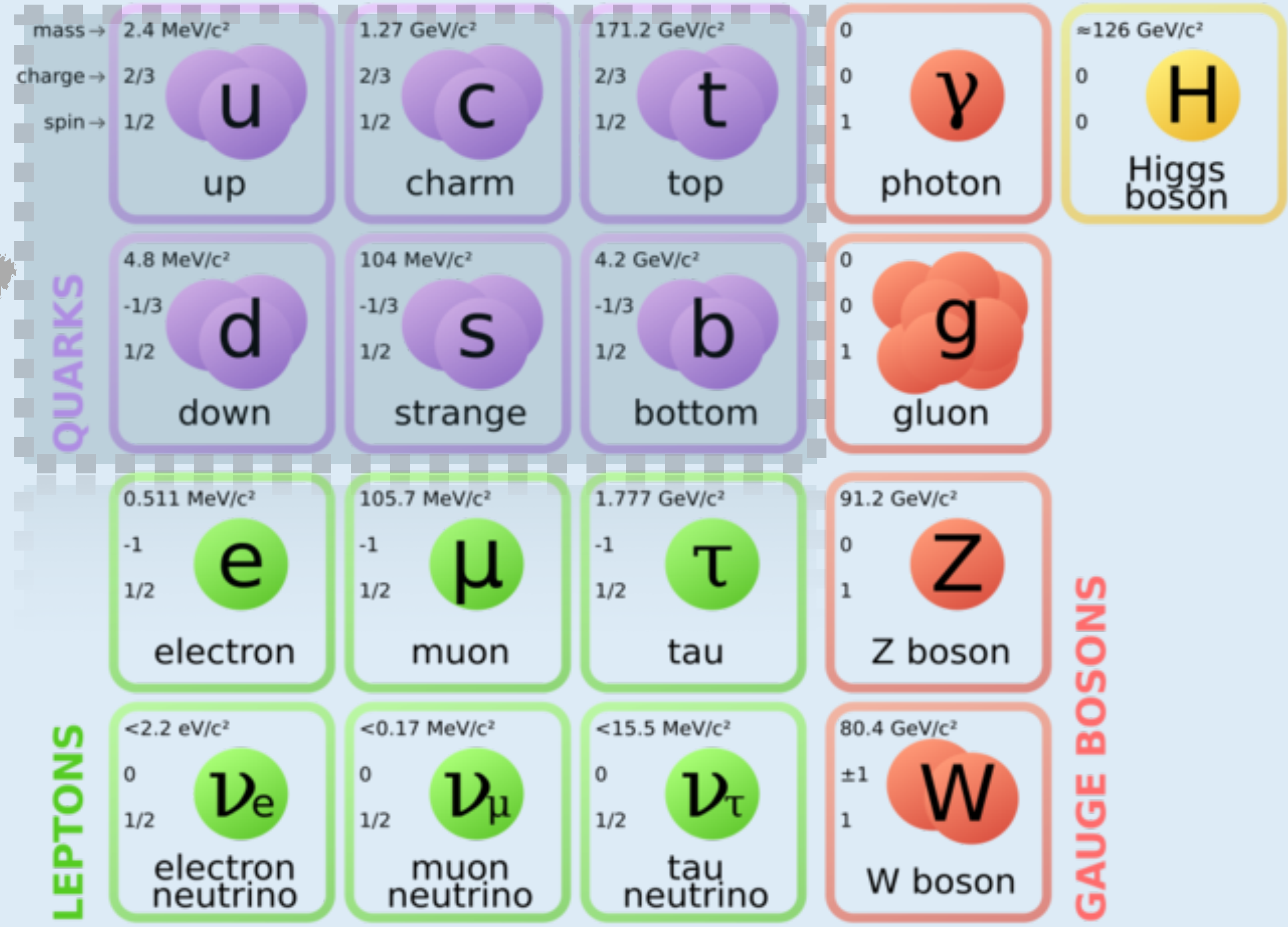
A. Fixed-order QCD.
B. All-orders QCD: Parton showers and Merging.
C. Hadronisation (tutorials?).

Some excellent references

- Peter Skands, *"Introduction to QCD"*, arXiv:1207.2389.
- Michelangelo Mangano, *"QCD and the physics of Hadronic collisions"*, CERN Yellow Rep.School Proc. 4 (2018) 27-62.
- MCnet review, "General-Purpose Event Generators", Phys. Rept. 504 (2011) 145.
- J. Campbell, J. Huston, F. Krauss, *"The Black Book of Quantum Chromodynamics: a Primer for the LHC era"*, Oxford University Press.
- G. Dissertori, I. Knowles, M. Schmelling, *"Quantum Chromodynamics: High Energy Experiments and Theory"*, Oxford Science Publications.
- R. K. Ellis, W. J. Stirling, B. R. Webber, *"QCD and Collider Physics"*, Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology.

Why QCD is important?

Quantum Chromodynamics or QCD is a quantum field theory which describes the strong interaction between quarks (constituents of the hadrons) and gluons.



Only these states feel the strong interaction

- Hadronic collisions involve protons in the initial state
- Even for electroweak physics, lepton and photon isolation depends on the QCD interaction (e.g. photons misidentified as QCD jets).
- Searching for new physics beyond the SM does not exclusively involve leptons.
- New physics searches involves lots of backgrounds of QCD nature.
- Higher order corrections are important to make the theory uncertainties under control. However, the higher we go in perturbation theory the more QCD is involved.
- Dark-matter annihilation leads to final-states particles whose spectra depend on QCD.
- The study of secondary cosmic rays depend on QCD and challenges existing models of fragmentation.
- QCD is based on SU(3) which is the richest gauge group we have so far: many studies are ongoing on unitarity properties, color structure, non-perturbative dynamics...etc.

First Hint for colour charge

The Δ^{++} baryon discovered in 1951 has bring the first hint for color

This configuration is symmetric while the overall fermionic wave function should be anti-symmetric

$$|\Delta^{++}\rangle \propto |u_\uparrow u_\uparrow u_\uparrow\rangle$$

Note that this is a fermion $(S = 3/2)$

Almost fourteen years after the discovery of Δ^{++}, this puzzle has been solved by the introduction of a new quantum number; the color charge. Each quark comes with three different colors (let's call them red, blue and green): $N_c = 3$.

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Three-Triplet Model with Double $SU(3)$ Symmetry*

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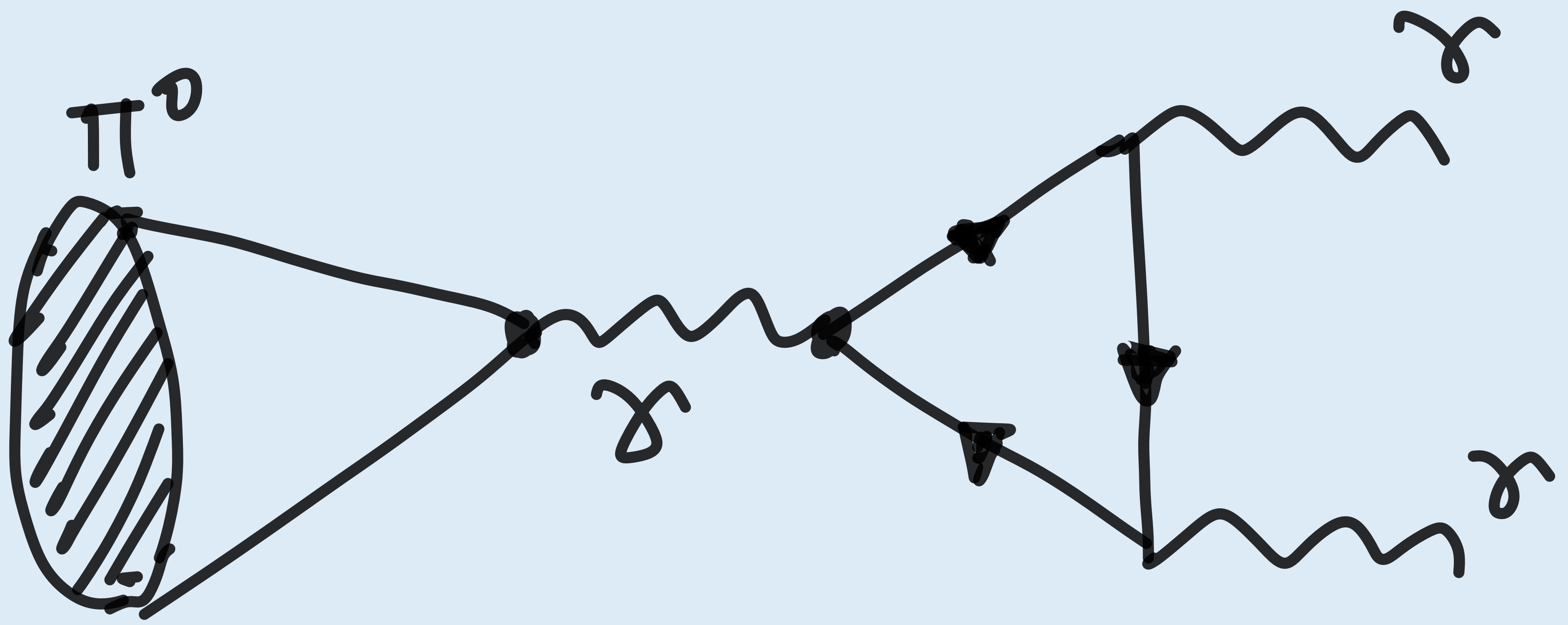
(Received 12 April 1965)

With a view to avoiding some of the kinematical and dynamical difficulties involved in the single-triplet quark model, a model for the low-lying baryons and mesons based on three triplets with integral charges is proposed, somewhat similar to the two-triplet model introduced earlier by one of us (Y. N.). It is shown that in a $U(3)$ scheme of triplets with integral charges, one is naturally led to three triplets located symmetrically about the origin of I_3-Y diagram under the constraint that the Nishijima–Gell-Mann relation remains intact. A double $SU(3)$ symmetry scheme is proposed in which the large mass splittings between different representations are ascribed to one of the $SU(3)$, while the other $SU(3)$ is the usual one for the mass splittings within a representation of the first $SU(3)$.

Further evidence for colour

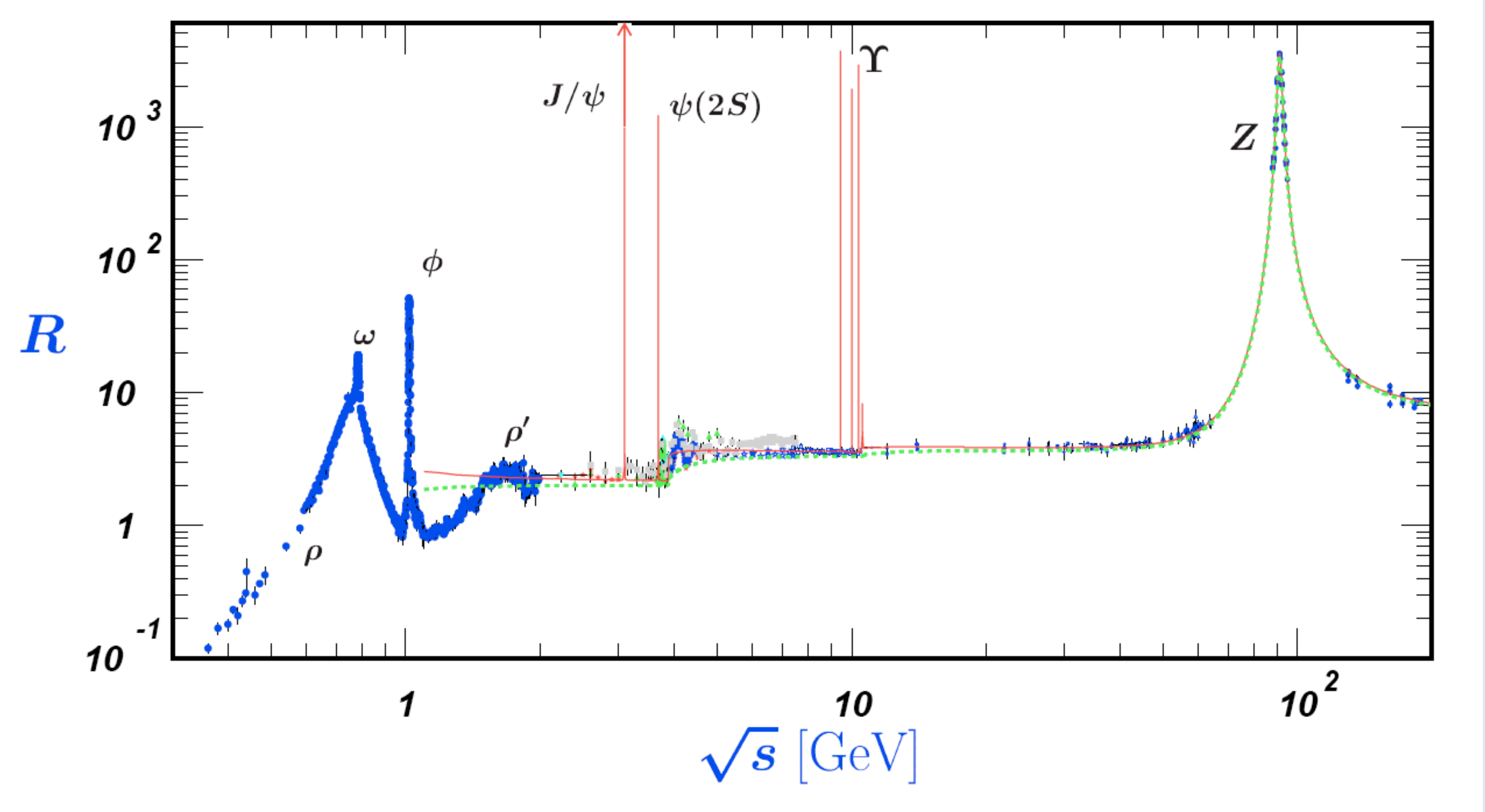


Measurement of the decay width of $\pi^0 \to \gamma\gamma$

$$\Gamma(\pi \to \gamma\gamma) = \frac{\alpha_e^2}{64\pi^3}\frac{m_\pi^3}{F_\pi^2}N_c^2\left[\left(\frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2\right]^2$$

Measurement of the R-ratio in e^+e^- collisions

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_{i\in\text{quarks}} Q_i^2 N_c$$



The QCD Lagrangian

Let's write the Lagrangian for the free quark field

$$\mathscr{L} = i\bar{q}^i_\alpha \gamma^\mu_{\alpha\beta}\partial_\mu q^i_\beta - m_q \bar{q}^i_\alpha q^i_\alpha \qquad \text{with} \qquad q^i_\alpha = \begin{pmatrix} q_\alpha \\ q_\alpha \\ q_\alpha \end{pmatrix}$$

Under the following transformation under $SU(3)$

$$U(x) = e^{iT_a\theta_a(x)}$$

with T_a are the generators of the SU(3) Lie group $\implies [T_a, T_b] = if_{abc}T_c$, $\theta_a(x); a = 1,\ldots,8$ are real parameters.

f_{abc} are the structure constants

The Lagrangian $\mathscr{L}$ transforms as

$$\mathscr{L} \to i\bar{q}^i(x)\gamma^\mu\partial_\mu q^i(x) - m_q\bar{q}^i(x)q^i(x) + \bar{q}^i(x)\gamma^\mu U^{-1}(x)(\partial_\mu U(x))q^i(x)$$

$$\to \mathscr{L} + \bar{q}^i(x)\gamma^\mu U^{-1}(x)(\partial_\mu U(x))q^i(x)$$

not invariant under local SU(3) transformations!!!

The QCD Lagrangian

We introduce a spin-1 field $\mathscr{B}_\mu$ which can be represented by a 8×8 matrix (in colour space).
Suppose that $\mathscr{B}_\mu(x)$ transforms as

$$\mathscr{B}_\mu(x) \to U(x)\mathscr{B}_\mu(x)U^{-1}(x) + U(x)(\partial_\mu U^{-1}(x))$$

Now, the Lagrangian

$$\mathscr{L} = i\bar{q}^i(\partial_\mu + \mathscr{B}_\mu)q^i - m_q\bar{q}^i q^i,$$

transforms as

$$\begin{aligned}
\mathscr{L} \to\ & i\bar{q}^i(x)\gamma^\mu\partial_\mu q^i(x) - m_q\bar{q}^i(x)q^i(x) + i\bar{q}^i(x)\gamma^\mu U^{-1}(x)(\partial_\mu U(x))q^i(x) \\
& + i\bar{q}^i(x)\gamma^\mu\mathscr{B}_\mu(x)q^i(x) + i\bar{q}^i(x)\gamma^\mu(\partial_\mu U^{-1}(x))U(x)q^i(x) \\
& = \mathscr{L} + i\bar{q}^i(x)\gamma^\mu U^{-1}(x)(\partial_\mu U(x))q^i(x) + i\bar{q}^i(x)\gamma^\mu(\partial_\mu U^{-1}(x))U(x)q^i(x) \\
& = \mathscr{L} + i\bar{q}^i(x)\gamma^\mu\partial_\mu(U^{-1}(x)U(x))q^i(x) = \mathscr{L}
\end{aligned}$$

Invariant!

The QCD Lagrangian

We need fields which propagate in space-time $\Longrightarrow$ construct the kinetic energy term for $\mathscr{B}_\mu(x)$

$$\mathscr{L}_{\text{kinetic}} \equiv \frac{1}{4g_s^2}\text{Tr}(F_{\mu\nu}F^{\mu\nu}),$$

with

$$F_{\mu\nu} = \partial_\mu \mathscr{B}_\nu - \partial_\nu \mathscr{B}_\mu + [\mathscr{B}_\mu, \mathscr{B}_\nu]$$

It is easy to check that $F_{\mu\nu} \to U(x) F_{\mu\nu} U^{-1}(x)$

Remember that $\mathscr{B}_\mu(x) \in SU(3) \Longrightarrow$ can be expanded in terms of the generators (T_a) of $SU(3)$

$$(\mathscr{B}_\mu)_{ij} = -ig_s T^a_{ij} A^a_\mu$$

A^a_μ is the gauge field (there are 8 of them) and g_s is the coupling constant

$$\mathscr{L} = -\frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu} + i\bar{q}^i_\alpha(\gamma^\mu)_{\alpha\beta}(\delta_{ij}\partial_\mu - ig_s T^a_{ij}A^a_\mu)q^j_\beta - m_q \bar{q}^i_\alpha q^i_\alpha$$

and

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f_{abc} A^b_\mu A^c_\nu$$

We constructed the QCD Lagrangian
Let us study its implications!!!

Gell-Mann Matrices

The generators of SU(3) are defined as (traceless and Hermitian)

$$\mathrm{Tr}(T_a T_b) = g_{ab} = T_F \delta_{ab}$$

Cartan metric; $g_{ab} = -f_{amn} f_{bnm}$

IMPORTANT NOTE: If you change this convention, you have to change the definition of the coupling constant g_s, since $g_s T_a$ appears in the QCD Lagrangian (see previous slide).

We can choose a parameterization of T_a such that $T_F = 1/2$; i.e. define $T_a = \frac{1}{2}\lambda_a$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

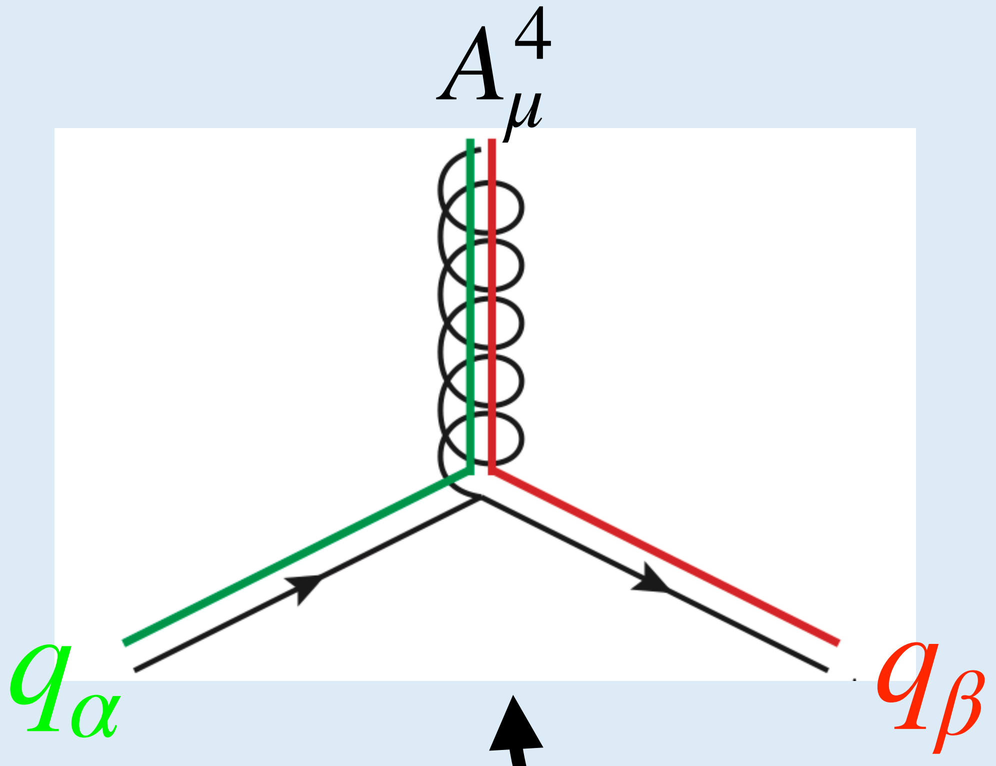
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Interactions in Colour Space

$$\boxed{\mathcal{L}_{q\bar{q}g} : \quad \bar{q}^i_\alpha (i\gamma^\mu)_{\alpha\beta} (D_\mu)_{ij} q^j_\beta}$$

The matrix-element calculations involve two independent parts: color space and Lorentz space. Let us focus on the color space first.

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - i g_s T^a_{ij} A^a_\mu$$



$$\begin{array}{cccc} \propto -\frac{i}{2} g_s & \bar{q}_\alpha & \lambda_4 & q_\beta \\ -\frac{i}{2} g_s & (1\ 0\ 0) & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

$$-i g_s T^4_{ij} \gamma^\mu_{\alpha\beta} A^4_\mu$$

Gluon (adjoint) colour index ∈ [1,8]

Gluon Lorentz-vector index ∈ [0,3]

Quark colour indices ∈ [1,3]

Fermion spinor indices ∈ [1,4]

Amplitudes Squared **summed over colors** → traces over products of T_a matrices
→ ***Color Factors*** (next slides)

Peter Skands (HCPSS 2020)

Interactions in Colour Space: Gluon self-interactions

$$\boxed{\mathscr{L}_{\text{gauge}} : -\frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu}}$$

$$\Longrightarrow$$

Gluon self-interactions:

$\mathscr{L}_{ggg}$ $\quad$ $\mathscr{L}_{gggg}$

$$G^a_{\mu\nu} = \underbrace{\partial_\mu A^a_\nu - \partial_\nu A^a_\mu}_{\text{Abelian}} + \underbrace{g_s f^{abc} A^b_\mu A^c_\nu}_{\text{non-Abelian}}$$

(Absent in QED)

$$-g_s f^{abc}(k_3 - k_2)^\rho g^{\mu\nu} \qquad -g_s^2 f^{yac} f^{ybd}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho})$$
$$-g_s f^{abc}(k_2 - k_1)^\mu g^{\nu\rho} \qquad -g_s^2 f^{yad} f^{ybc}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})$$
$$-g_s f^{abc}(k_1 - k_3)^\nu g^{\rho\nu} \qquad -g_s^2 f^{yab} f^{ycd}(g^{\mu\rho}g^{\nu\rho} - g^{\mu\sigma}g^{\nu\rho})$$

Note about f^{abc} :

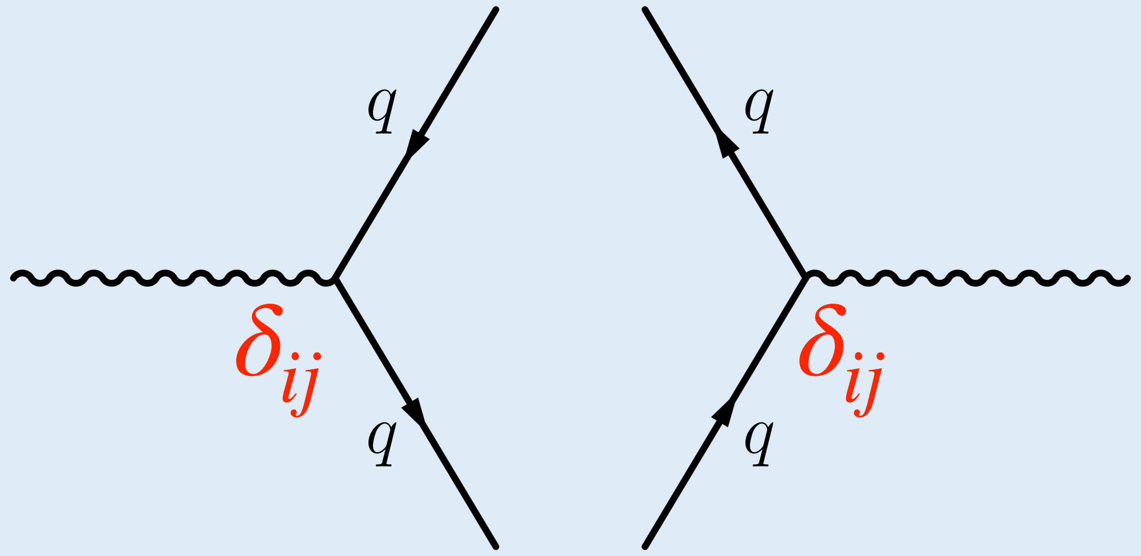
- f_{abc} are called the structure constants of the SU(3) group.
- They uniquely define its structure.
- They can provide a representation for SU(3) — called the adjoint representation —, if we define $(T_a)_{bc} = -if_{abc}$ we have $[T_a, T_d] = if_{ade}T_e$

$$\begin{aligned}
f_{123} &= 1 \\
f_{147} = f_{246} = f_{257} = f_{345} &= 1 \\
f_{156} = f_{367} &= -\frac{1}{2} \\
f_{458} = f_{678} &= \frac{\sqrt{3}}{2} \\
f_{abc} = -f_{bac} & \text{ for } a, b, c \in [1,8] \\
f_{aab} = f_{abb} = 0 & \text{ for } a, b \in [1,8] \\
f_{aaa} = 0 & \text{ for } a \in [1,8]
\end{aligned}$$

More about colour algebra

Z Decay: (aka color-singlet decays)

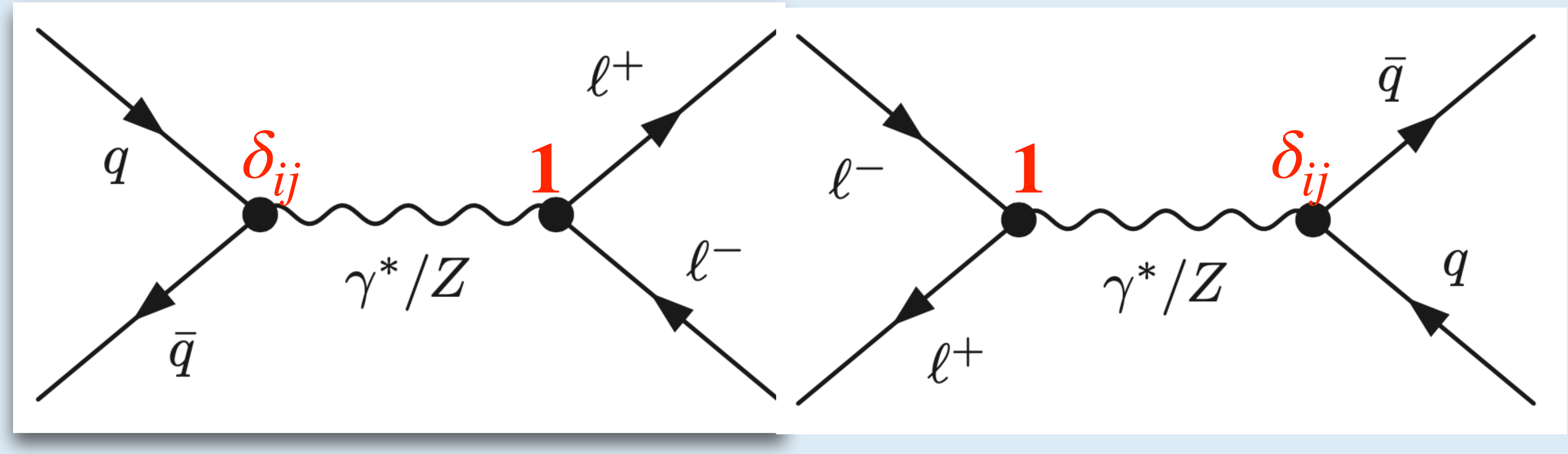
$$\sum_{\text{colors}} |\mathcal{M}|^2 \equiv$$



$$\boxed{\propto \delta_{ij}\delta^*_{ji} = \mathrm{Tr}(\delta_{ij}) = N_c}$$

Drell-Yan ($q\bar{q} \to Z^*$)

$$\frac{1}{N_c^2}\sum_{\text{colors}} |\mathcal{M}|^2 \equiv$$

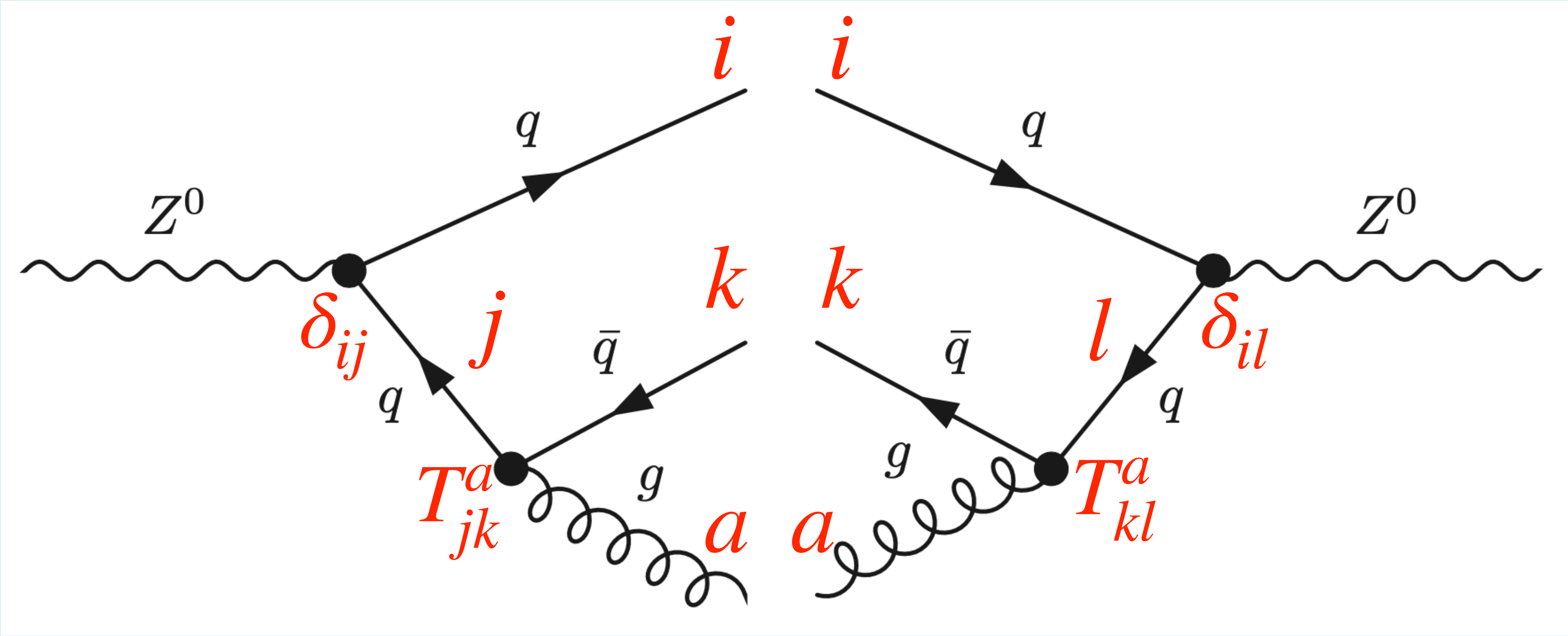


$$\boxed{\propto \delta_{ij}\delta^*_{ji}\frac{1}{N_c^2} = \frac{\mathrm{Tr}(\delta_{ij})}{N_c^2} = \frac{1}{N_c}}$$

More about colour algebra

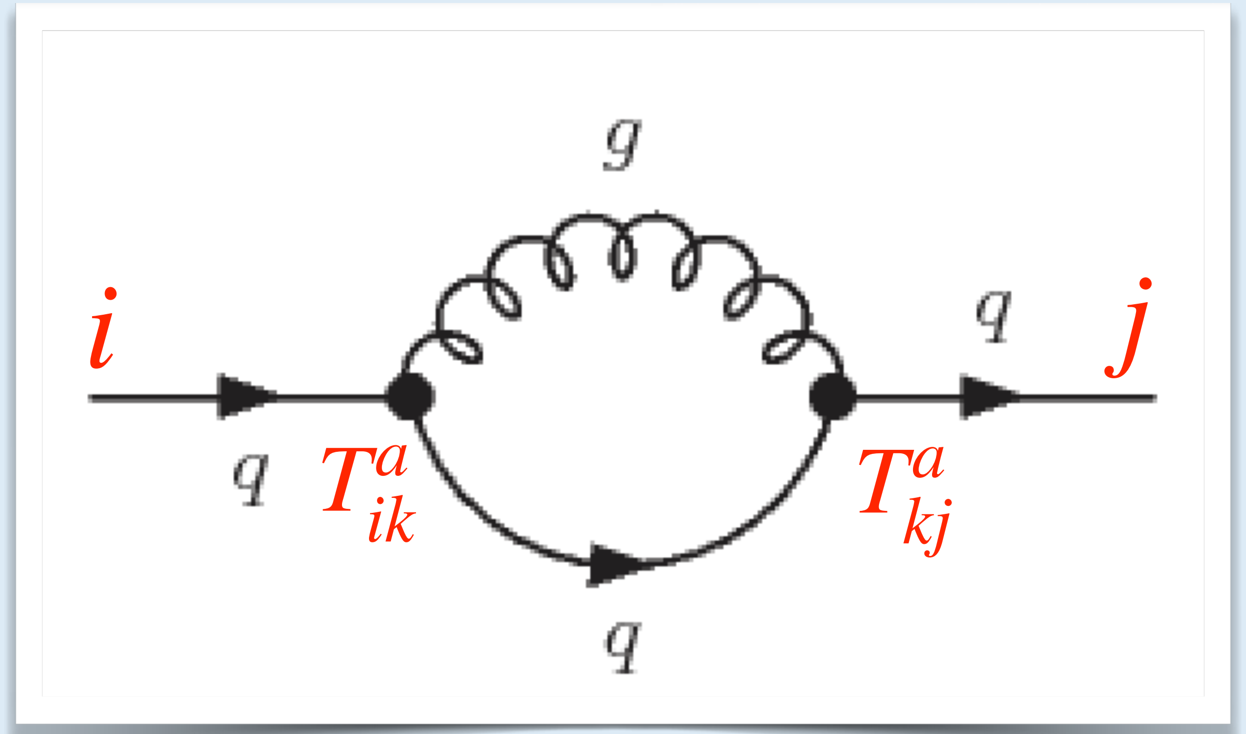
Z → 3 jets:

$$\sum_{\text{colors}} |\mathscr{M}|^2 \equiv$$



$$\propto \delta_{ij}\delta^*_{li}T^a_{jk}T^a_{kl} = T^a_{ik}T^a_{ki} = (T^aT^a)_{ii}$$
$$= T_F \mathrm{Tr}(\delta^{aa}) = (N_c^2 - 1)T_F$$

Quark self-energy



$$\propto \sum_a (T_aT_a)_{ij} = C_F\delta_{ij} \implies T_F\delta_{aa} = T_F(N_c^2 - 1)$$

$$\implies C_F = T_F\frac{N_c^2 - 1}{N_c}$$

Break: An important relation

$$T^a_{ij} T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$$

PROOF

Let $\mathbb{M}$ be an arbitrary Hermitian $N_c \times N_c$ matrix. It can be expanded as:

$$\mathbb{M} \equiv \alpha_0 \mathbb{I}_{N_c} + \alpha_a T^a; \;\; \alpha_0, \alpha_a \in \mathbb{R}$$

The coefficients (α_0 and α_a) can be estimated from the traces over $\mathbb{M}$ and $T^a\mathbb{M}$. We have:

$$\alpha_0 = \frac{1}{N_c}\mathrm{Tr}(\mathbb{M}) \text{ and } \alpha_a = \frac{1}{T_F}\mathrm{Tr}(\mathbb{M}T^a) \Longrightarrow \mathbb{M} = \frac{1}{N_c}\mathrm{Tr}(\mathbb{M})\mathbb{I}_{N_c} + \frac{1}{T_F}\mathrm{Tr}(T^a\mathbb{M})T^a$$

Now, let us take the (i,j) element of the matrix $\mathbb{M}$

$$\mathbb{M}_{ij} = \frac{1}{N_c}\mathbb{M}_{kk}\delta_{ij} + \frac{1}{T_F}(T^a\mathbb{M})_{kk}T^a_{ij} = \frac{1}{N_c}\mathbb{M}_{kk}\delta_{ij} + \frac{1}{T_F}T^a_{kl}\mathbb{M}_{lk}T^a_{ij}$$

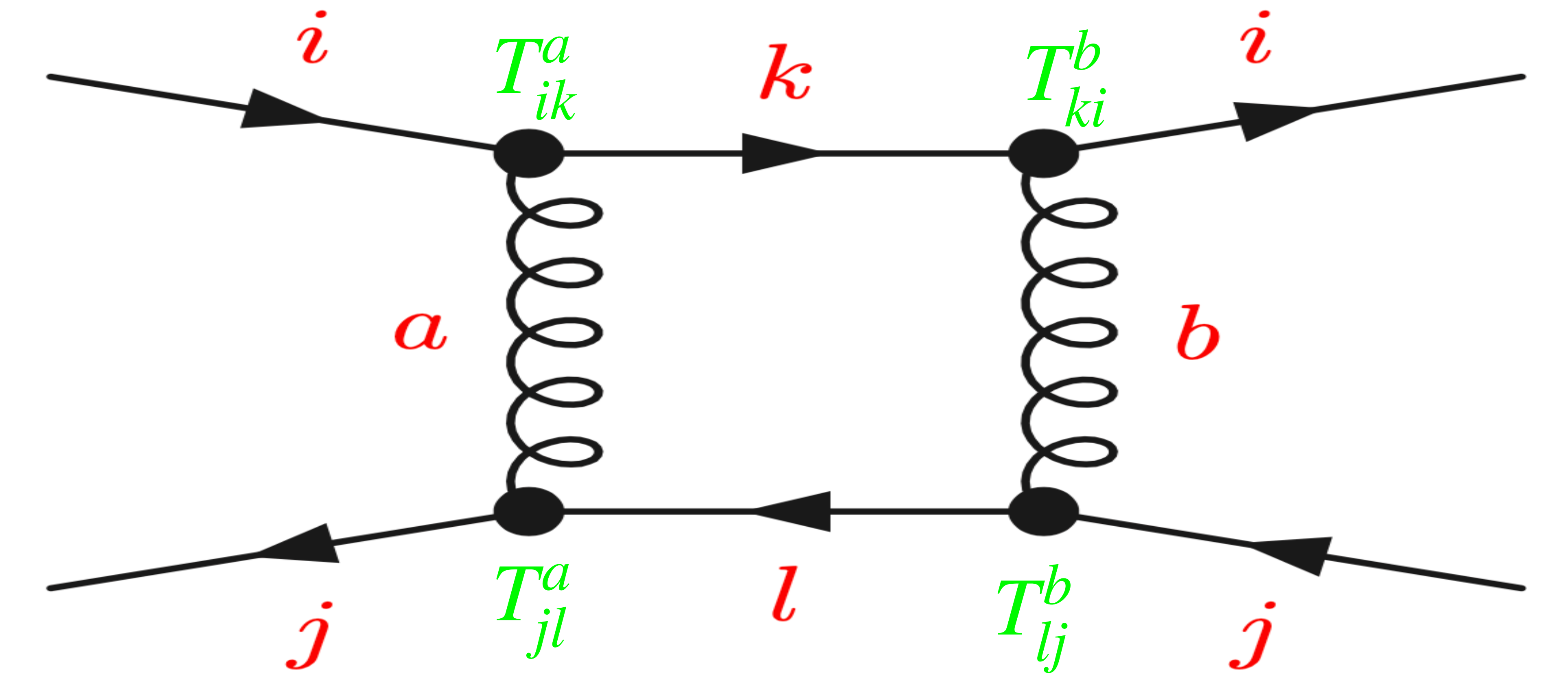
Or $\mathbb{M}_{ij} = \mathbb{M}_{lk}\underbrace{\left(\frac{1}{N_c}\delta_{kl}\delta_{ij} + \frac{1}{T_F}T^a_{kl}T^a_{ij}\right)}_{\delta_{ik}\delta_{jk}} \Longrightarrow T^a_{ij}T^a_{kl} = T_F\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$

$$\mathrm{Tr}(\mathbb{M}) = \alpha_0 \underbrace{\mathrm{Tr}(\mathbb{I}_{N_c})}_{=N_c} + \alpha_a \underbrace{\mathrm{Tr}(T^a)}_{=0} = \alpha_0 N_c \Longrightarrow \alpha_0 = \frac{1}{N_c}\mathrm{Tr}(\mathbb{M})$$

$$\mathrm{Tr}(\mathbb{M}T^a) = \alpha_0\mathrm{Tr}(T^a) + \alpha_b\mathrm{Tr}(T^aT^b) = \alpha_b T_F\delta_{ab} \Longrightarrow \alpha_a = \frac{1}{T_F}\mathrm{Tr}(\mathbb{M}T^a)$$

Return to colour algebra

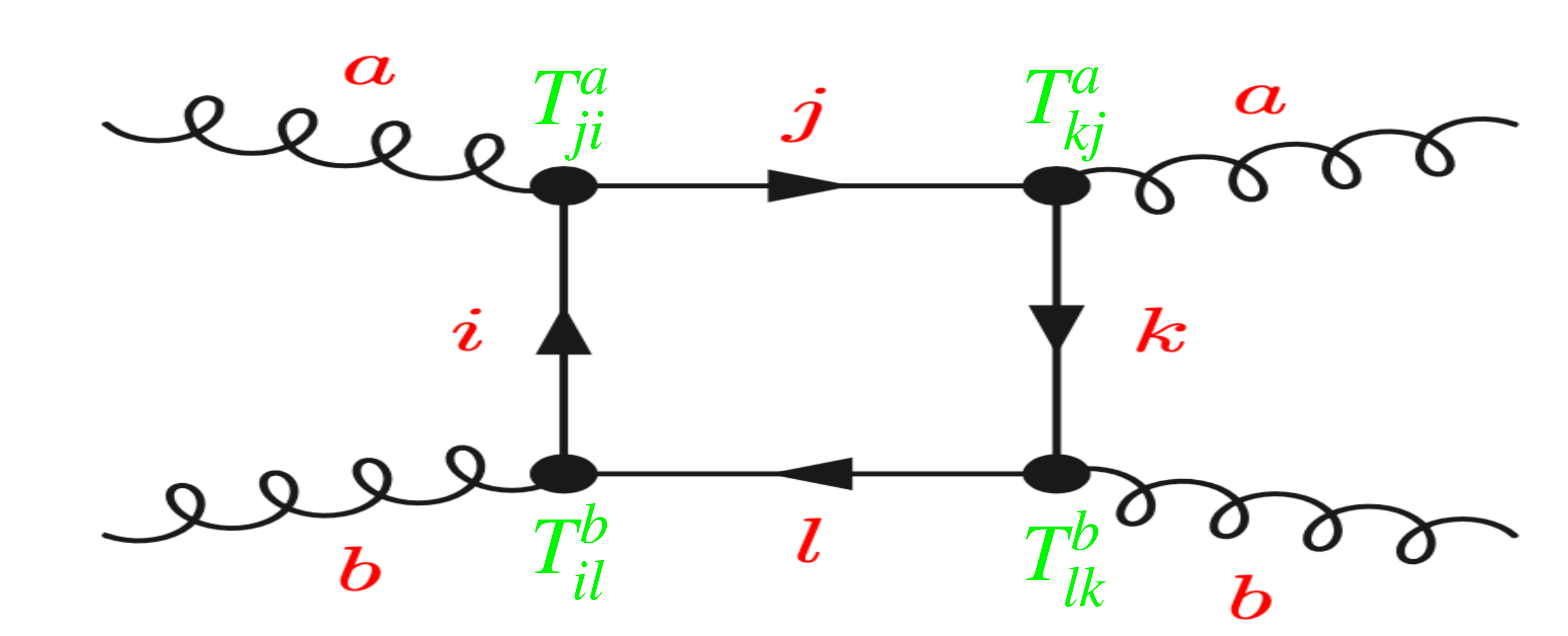
$q\bar{q} \to q\bar{q}$ at the one-loop order:



$$\begin{aligned}
&\propto T^a_{ik} T^b_{ki} T^a_{jl} T^b_{lj} = \mathrm{Tr}(T^a T^b)\mathrm{Tr}(T^a T^b) \\
&= T_F^2 \delta_{ab}\delta_{ab} = T_F^2 (N_c^2 - 1) \\
&= T_F C_A C_F
\end{aligned}$$

After averaging over the initial colors, we get $\frac{1}{N_c^2} C_A C_F T_F = \frac{T_F C_F}{C_A}$

$gg \to gg$ at the one-loop order:

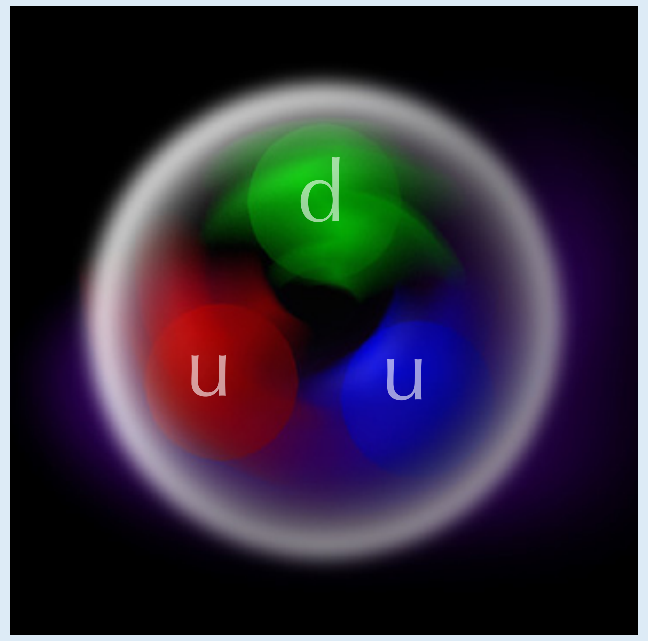


$$\begin{aligned}
&\propto T^a_{ji} T^b_{il} T^b_{lk} T^a_{kj} = (T^a_{ji} T^a_{kj})(T^b_{il} T^b_{lk}) \\
&= T_F^2 \left(\delta_{jj}\delta_{ik} - \frac{1}{N_c}\delta_{ji}\delta_{kj}\right)\left(\delta_{ik}\delta_{ll} - \frac{1}{N_c}\delta_{il}\delta_{lk}\right) \\
&= T_F^2 \left(N_c - \frac{1}{N_c}\right)^2 \delta_{ik}\delta_{ki} = T_F^2 N_c \left(\frac{N_c^2 - 1}{N_c}\right)^2 = C_F^2 C_A
\end{aligned}$$

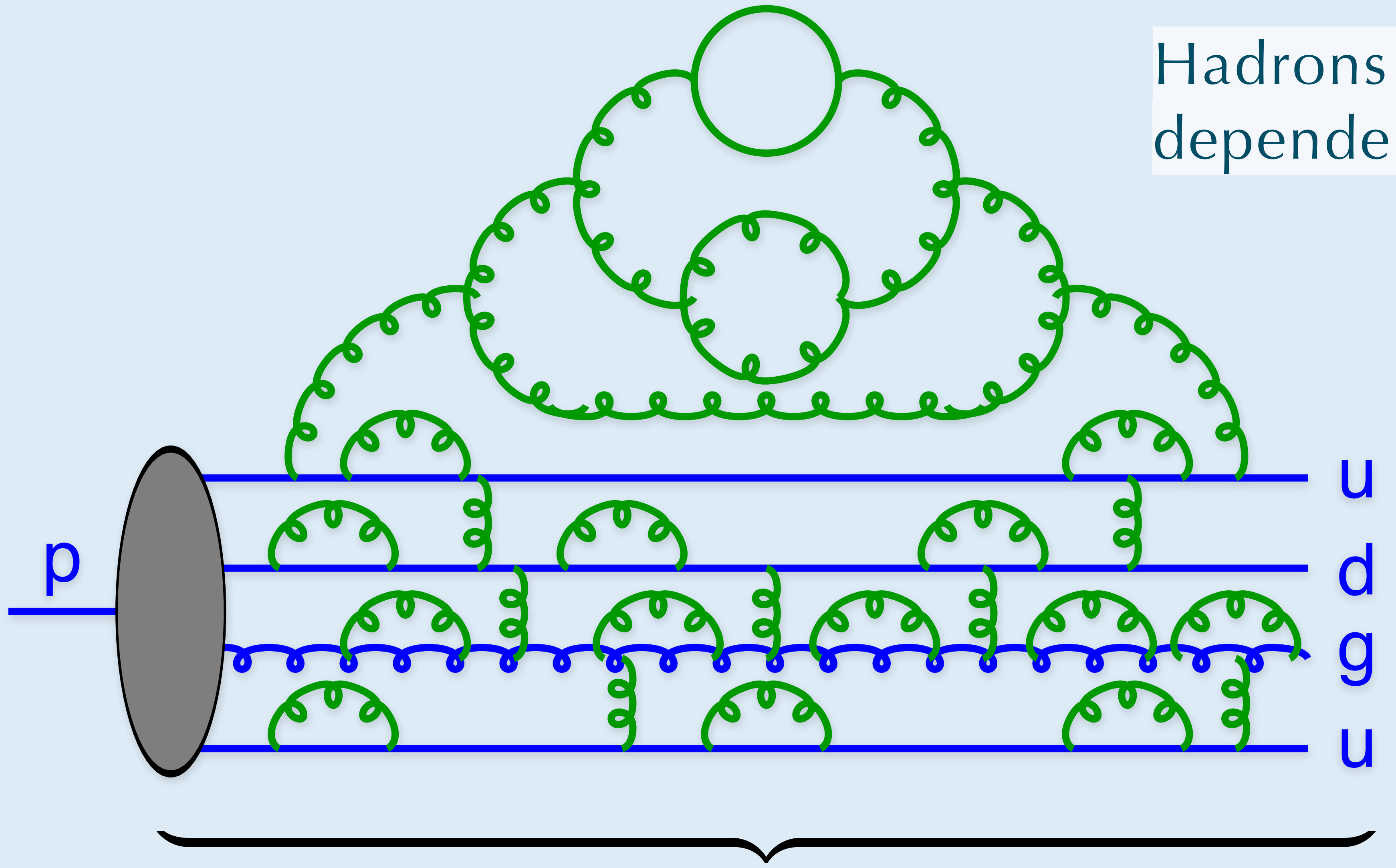
Can we calculate LHC processes now?

What are we really colliding?

Take a look at the quantum level



Hadrons are composite, with time-dependent structure



(illustration by T. Sjöstrand)

Describe this mess **statistically** ➜ **parton distribution functions (PDFs)**

PDFs: $f_i(x, Q_F^2)$ $\quad i \in [g, u, d, s, c, (b), (t), (\gamma), (\ell)]$

Probability to find parton of flavour i with momentum fraction $x = p_i/p_{\mathrm{hadron}}$, as function of "resolution scale" Q_F ~ virtuality / inverse lifetime of fluctuation

Peter Skands (HCPSS 2020)

Why PDFs work 1: heuristic explanation

Lifetime of typical fluctuation $\approx r_p/c$ (=time it takes light to cross a proton)

$\approx 10^{-23}$ seconds; Corresponds to a frequency of ~ 500 billion THz

To the LHC, that's slow! (reaches "shutter speeds" thousands of times faster)

Planck-Einstein: $E = h\nu \implies \nu_{\text{LHC}} = 13\ \text{TeV}/h = 3.14$ million billion THz

$\implies$ Protons look "frozen" at moment of collision

But they have a lot more than just two "u" quarks and a "d" inside

Difficult/impossible to calculate, so use statistics to parametrise the structure: **parton distribution functions** (PDFs)

Every so often I will pick a gluon, every so often a quark (antiquark)

Measured at previous colliders (+ increasingly also at LHC)

Expressed as functions of energy fractions, x, and resolution scale, Q^2

+ obey known scaling laws df / dQ^2 : "**DGLAP equations**".

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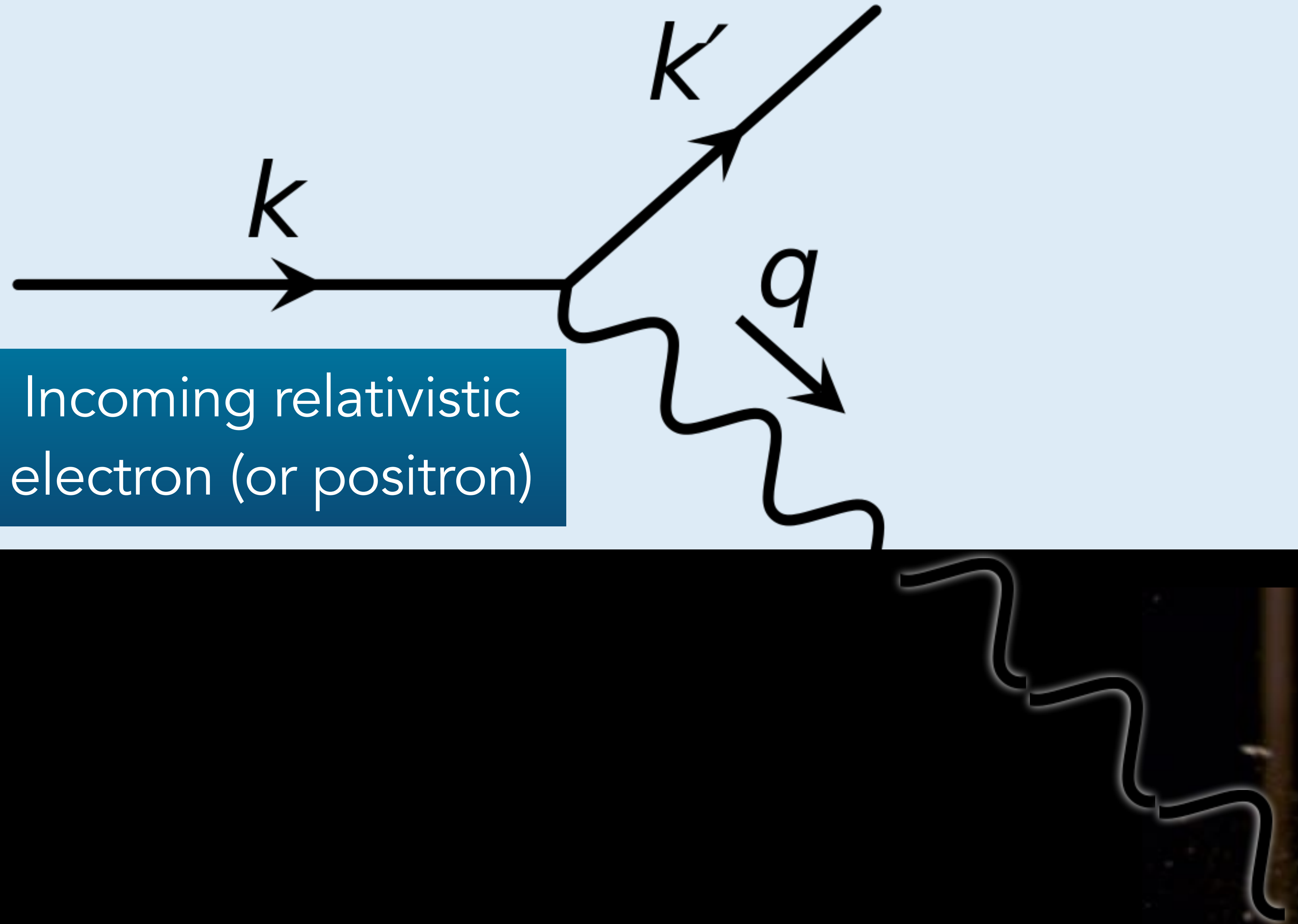
Deep Inelastic Scattering (DIS)



Incoming relativistic electron (or positron)

Hard (i.e. high-energy) photon
$q^2 = (k - k')^2 < 0$ (spacelike)
$\Longrightarrow$ often use $Q^2 = -q^2 > 0$ instead

Leptonic part : **clean**

"Deep" = invariant mass of final hadronic system $\gg M_{\text{proton}}$

Hadronic part : **messy**

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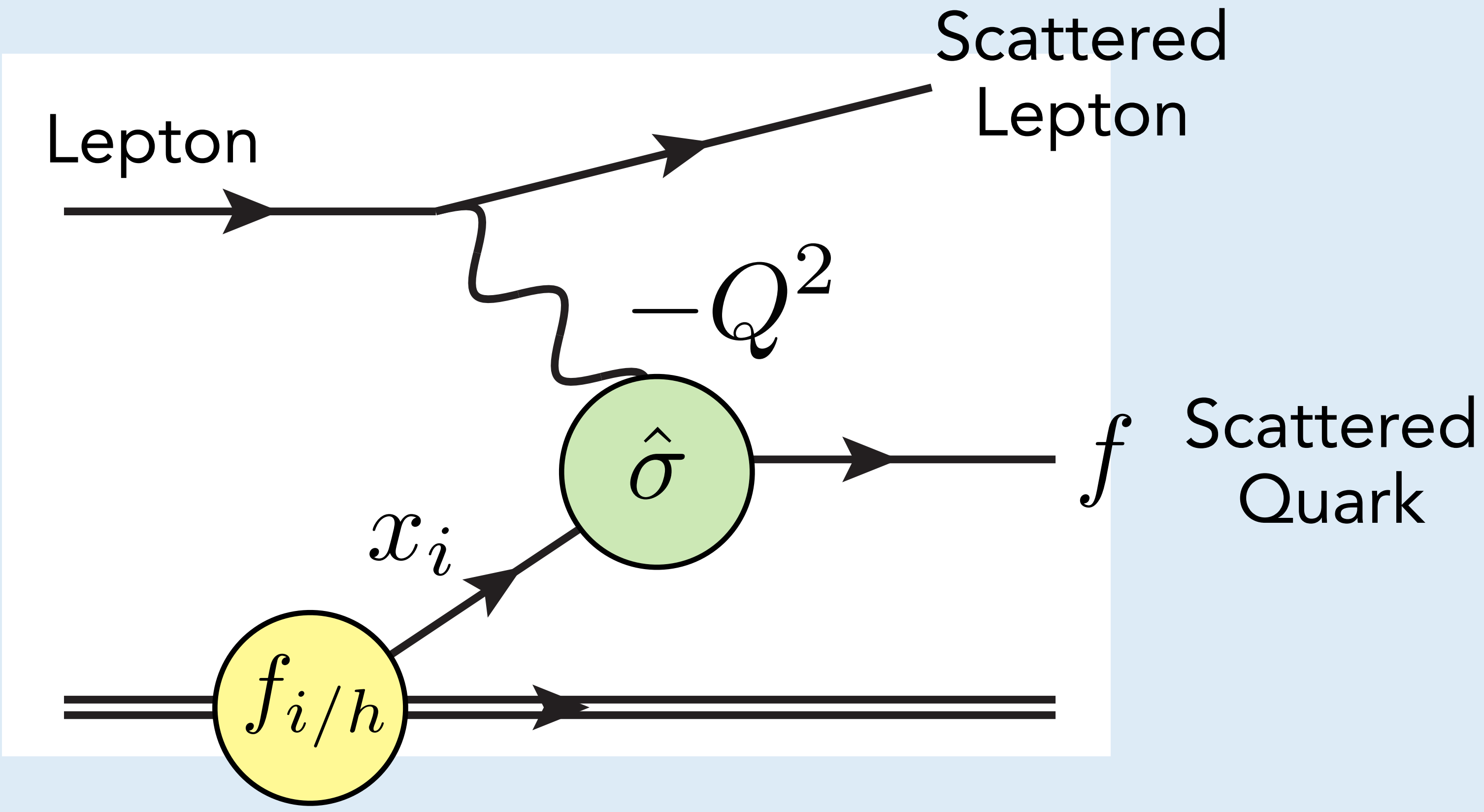
Why PDFs work 2: factorisation in DIS

Deep Inelastic Scattering (DIS)



We **assume*** that an analogous factorisation works for pp

*caveats are beyond the scope of this course

→ The cross section can be written in **factorised** form :

$$\sigma^{\ell h} = \sum_i \sum_f \int dx_i \int d\Phi_f \, f_{i/h}(x_i, Q_F^2) \, \frac{d\hat{\sigma}^{\ell i \to f}(x_i, \Phi_f, Q_F^2)}{dx_i \, d\Phi_f}$$

Sum over initial (i) and final (f) parton flavors

Φ_f = Final-state phase space

$f_{i/h}$ = PDFs
Assumption: $Q^2 = Q_F^2$

Differential partonic **Hard-scattering** Matrix Element(s) at Q^2

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Factorisation $\Longrightarrow$ we can still calculate!

pQCD = perturbative QCD

PDFs: connect incoming hadrons with the high-scale process

Fragmentation Functions: connect high-scale process with final-state hadrons

Both combine **non-perturbative input** + all-orders (perturbative) bremsstrahlung **resummations**

$$\frac{\mathrm{d}\sigma}{\mathrm{d}X} = \sum_{a,b} \sum_{f} \int_{\hat{X}_f} f_a(x_a, Q_i^2) f_b(x_b, Q_i^2) \frac{\mathrm{d}\hat{\sigma}_{ab\to f}(x_a, x_b, f, Q_i^2, Q_f^2)}{\mathrm{d}\hat{X}_f} D(\hat{X}_f \to X, Q_i^2, Q_f^2)$$

PDFs: needed to compute inclusive cross sections

Hard Process
Fixed-Order QFT

FFs: needed to compute (semi-)exclusive cross sections

In MCs → **initial-state radiation** + non-perturbative hadron (beam-remnant) structure + multi-parton interactions

Matching & Merging

In MCs: **resonance decays + final-state radiation +** hadronisation + hadron decays (+ final-state interactions?)

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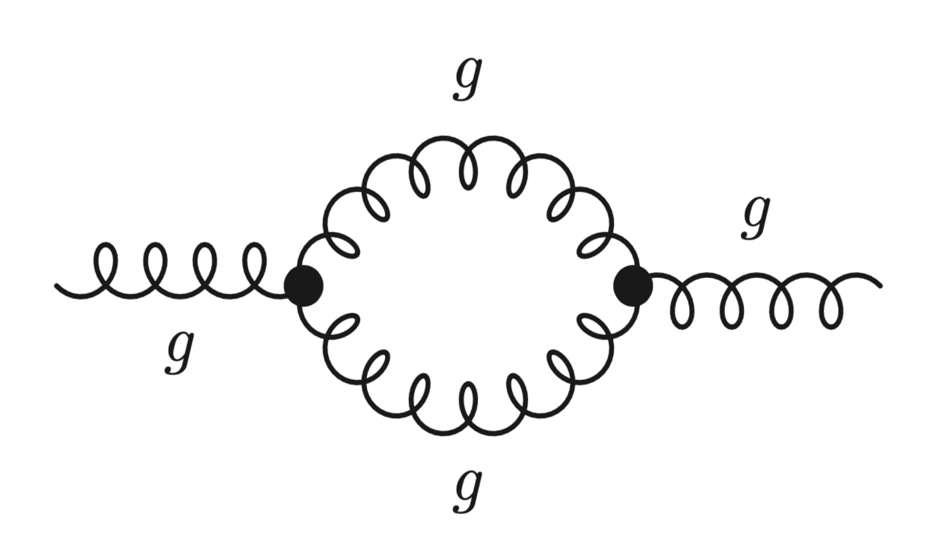
Beyond trees: Infinities

The QCD Lagrangian is no-linear in the fields $\Longrightarrow$ Physical observables can only be computed approximatively.

$$\mathcal{O} = \sum_k g_s^k \mathcal{O}_k$$

Example

$k = 2 \Longrightarrow$



$$\Longrightarrow \mathcal{I} \approx \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^4} \approx \int \frac{dq}{q} = \lim_{q \to +\infty} \log(q)$$

Clearly divergent

Divergences (here called UV) mean that we cannot make predictions!

An ad-hoc solution is to cut-off the integral at some scale Λ_c
$\Longrightarrow$ our theory cannot say anything above Λ_c

$$\Longrightarrow \mathcal{I} \propto \log(\Lambda_c)$$

Prescription leads to gauge dependent quantities + add some arbitrariness to the theory predictions (dependence on an unknown parameter Λ_c)

$\Longrightarrow$ Define a better prescription; aka *renormalisation*

Renormalization or infinities are not so scary

We say that the fields and parameter are just bare at a given order

$\Longrightarrow$ At the quantum level, the fields and parameters are defined as

$$p_i^0 \to Z_{p_i} p_i \text{ and } \mathscr{F}_{i,0} \to Z_{\mathscr{F}}^{1/2} \mathscr{F}_i$$

+ suitable regularization scheme

In a nutshell

- Choose a set of independent parameters in the theory. In QCD, we have only one parameter g_s (if we ignore quark masses).
- Split the bar parameters (fields) into renormalised parameters (fields) and counter-terms.
- Find renormalisation conditions to fix the counter-terms.
- The final result should be free of UV infinities.

NOTE: There are further divergences for momenta $q \to 0$ (these are IR divergences and should be treated separately)

Application: the strong coupling constant

Let's return to the Lagrangian of $q\bar{q}g$ interaction (at the one-loop order)

$$\mathscr{L}_{qqg} = \mu^{\frac{4-D}{2}} g_s Z_1 \bar{q}_i \gamma^\mu T^a_{ij} q_j A^a_\mu = \mu^{\frac{4-D}{2}} g_s \frac{Z_1}{Z_2\sqrt{Z_3}} \bar{q}_i^{(0)} \gamma^\mu T^a_{ij} q_j^{(0)} A_\mu^{a,(0)} = \underbrace{g_{s,0} \bar{q}_i^{(0)} \gamma^\mu T^a_{ij} q_j^{(0)} A_\mu^{a,(0)}}$$

to preserve the natural dimension of $\mathscr{L}$ in D space-time dimensions. μ is called the renormalisation scale

$$q_i^{(0)} \to \sqrt{Z_1} q_i$$
$$A_\mu^{a,(0)} \to \sqrt{Z_3} A^a_\mu$$

Identical to the Lagrangian in slide (10)

$$\Longrightarrow \quad g_{s,0} = \frac{Z_1}{Z_2\sqrt{Z_3}} \mu^{\frac{4-D}{2}} g_s$$

$g_{s,0}$ is independent of μ

$\Longrightarrow$ differentiating with respect to μ gives

$$\beta(g_s) = -\frac{\epsilon}{2} g_s + \frac{\epsilon}{2} g_s^2 \frac{\partial}{\partial g_s}\left(\frac{Z_1}{Z_2\sqrt{Z_3}}\right)$$

$\epsilon = 4 - D$

Application: the strong coupling constant

We need to truncate at the one-loop order; we define $\boxed{Z_i^k = 1 + k\delta_i + \mathcal{O}(\delta_i^2)}$

$\Longrightarrow$ δ_i can be computed from explicit evaluation of one-loop integrals (see e.g. Peskin & Schroeder)

$$\boxed{\begin{aligned}
\delta_1 &= \frac{1}{\epsilon}\left(\frac{g_s}{4\pi}\right)^2\left[-2C_F - 2C_A + 2(1-\xi) + \frac{1}{2}(1-\xi)C_A\right] \\
\delta_2 &= \frac{1}{\epsilon}\left(\frac{g_s}{4\pi}\right)^2\left[-2C_F + 2(1-\xi)C_F\right] \\
\delta_3 &= \frac{1}{\epsilon}\left(\frac{g_s}{4\pi}\right)^2\left[\frac{10}{3}C_A - \frac{8}{3}n_f T_F + (1-\xi)C_A\right]
\end{aligned}}$$

$$\Longrightarrow \beta(g_s) = -\frac{\epsilon}{2}g_s - \frac{g_s^3}{16\pi^2}\left[\frac{11}{3} - C_F n_f T_F\right] = -\frac{g_s^3}{16\pi^2}\beta_0$$

$$\boxed{\beta_0 = \frac{11}{3}C_A - C_F n_f T_F}$$

> 0 unless $n_f > 17$

If we define the strong coupling constant as:

$$\alpha_s = \frac{g_s^2}{4\pi}$$

$$\Longrightarrow \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha_s = -\frac{\alpha_s^2}{2\pi}\beta_0$$

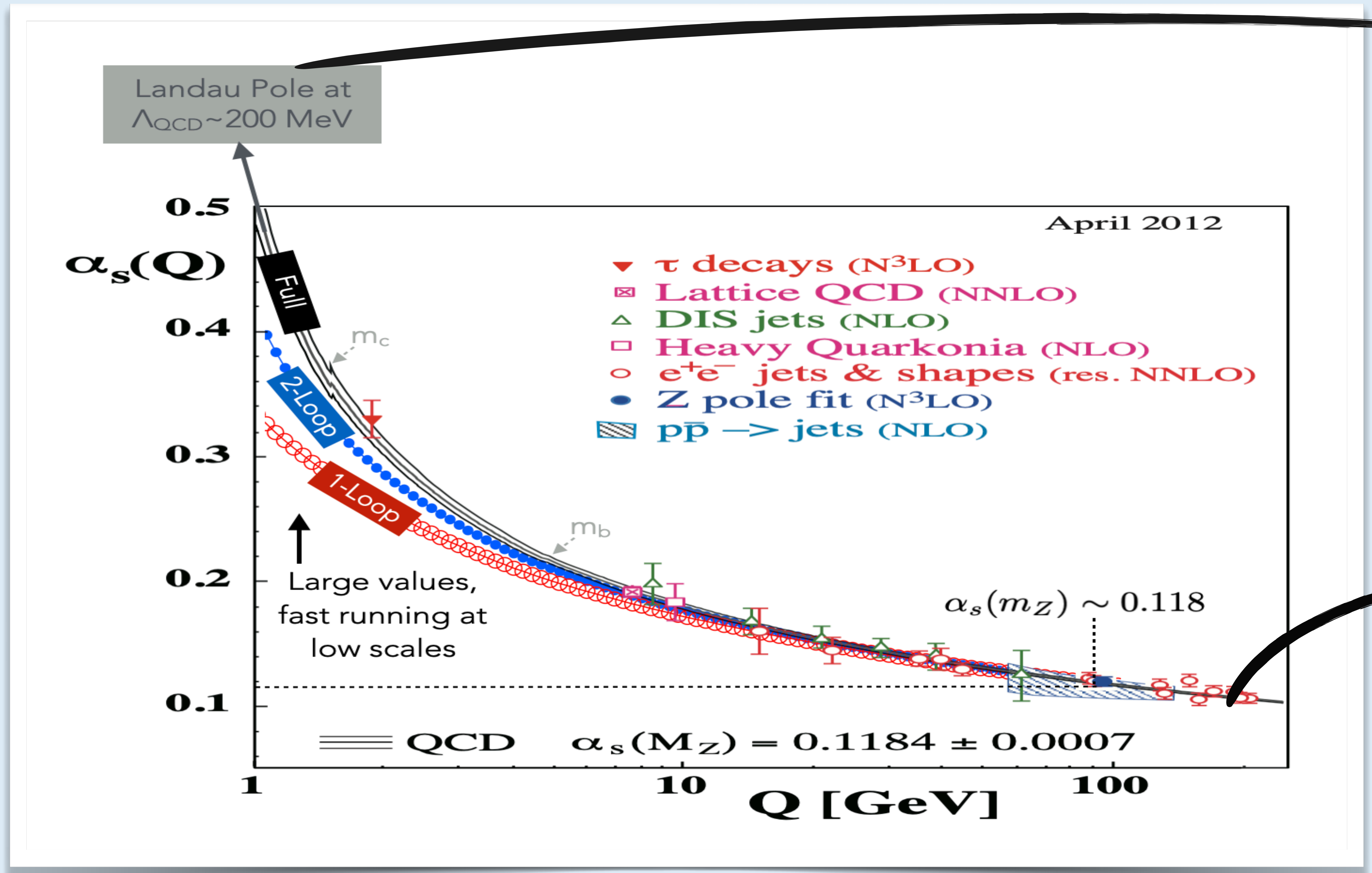
which can be solved to give

$$\alpha_s(\mu) = \frac{2\pi}{\beta_0}\frac{1}{\log\left(\frac{\mu}{\Lambda_{\mathrm{QCD}}}\right)}$$

The strong coupling constant

The strong coupling is the **main parameter** of perturbative QCD calculations. It controls:

- The size of **QCD cross sections** (& QCD **partial widths** for decays).
- The overall amount of **QCD radiation** (extra jets + **recoil effects** + jet substructure).
- Sizeable **QCD "K Factors"** to essentially all processes at LHC, and **ditto uncertainties**.



Confinement (IR slavery?)	in the infrared
Asymptotic freedom	in the ultraviolet

Nobel prize citation (taken from G. Salam's talk)

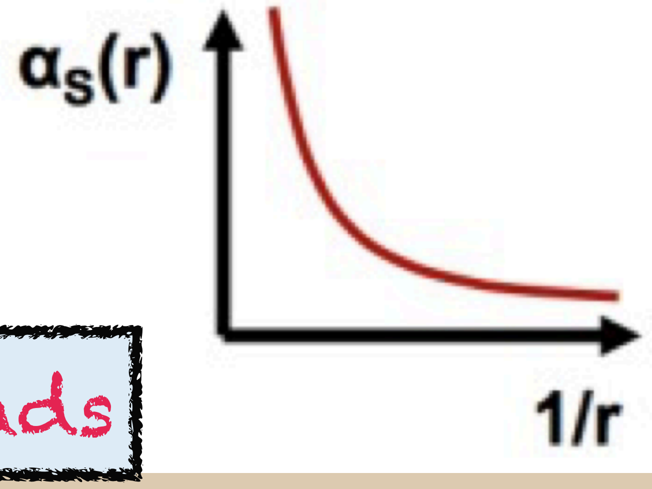
"What this year's Laureates discovered was something that, at first sight, seemed completely contradictory. The interpretation of their mathematical result was that the closer the quarks are to each other, the *weaker* is the 'colour charge'. When the quarks are really close to
*1 each other, the ~~force~~ charge is so weak that they behave almost as free particles. This phenomenon is called 'asymptotic freedom'. The converse is true when the quarks move apart:
*2 the ~~force~~ potential becomes stronger when the distance increases."

Nobelprize.org
The Official Web Site of the Nobel Prize

The Nobel Prize in Physics 2004
David J. Gross, H. David Politzer, Frank Wilczek

David J. Gross H. David Politzer Frank Wilczek

The Nobel Prize in Physics 2004 was awarded jointly to David J. Gross, H. David Politzer and Frank Wilczek *"for the discovery of asymptotic freedom in the theory of the strong interaction"*.

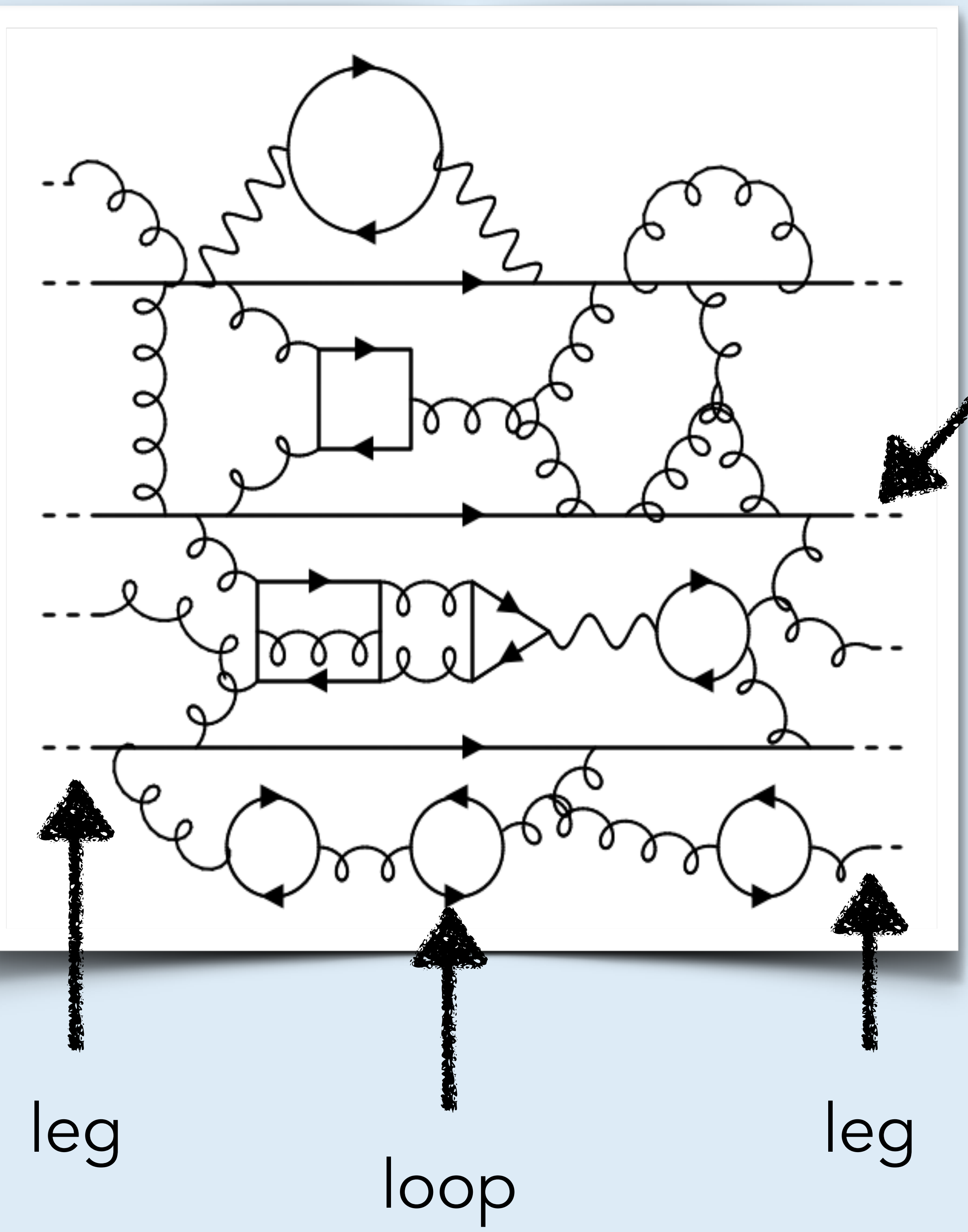


*1 The force still goes to ∞ as $r \to 0$ (Coulomb potential), just less slowly

*2 The potential grows linearly as $r \to \infty$, so the force actually becomes constant (even this is only true in "quenched" QCD. In real QCD, the force eventually vanishes for $r >> 1$ fm)

Annotation by Peter Skands

Cross sections at Fixed order



Now want to compute the distribution of some observable: $\mathcal{O}$

In $\underbrace{\text{“inclusive X production”}}_{\text{X + anything}}$ (suppressing PDF factors)

Fixed Order
(All Orders)

$$\left.\frac{d\sigma}{d\mathcal{O}}\right|_{\mathrm{ME}} = \sum_{k=0} \int \underbrace{\mathrm{d}\Phi_{X+k}}_{\text{Phase Space}} \left| \sum_{\ell=0} M_{X+k}^{(\ell)} \right|^2 \delta\left(\mathcal{O} - \mathcal{O}(\{p\}_{X+k})\right)$$

Sum over identical amplitudes, then square

Momentum configuration

Cross Section differentially in O

Sum over “anything” ≈ legs

Matrix Elements for X+k at (ℓ) loops

Evaluate observable → differential in $\mathcal{O}$

Truncate at $k = 0, \ell = 0$ → **Born Level = First Term**
Lowest order at which X happens

Loops and Legs

ℓ (loops)					
2	$\sigma_0^{(2)}$	$\sigma_1^{(2)}$	$\dots$		
1	$\sigma_0^{(1)}$	$\sigma_1^{(1)}$	$\sigma_2^{(1)}$	$\dots$	
0	$\sigma_0^{(0)}$	$\sigma_1^{(0)}$	$\sigma_2^{(0)}$	$\sigma_3^{(0)}$	$\dots$
	0	1	2	3	$\dots$

k (legs)

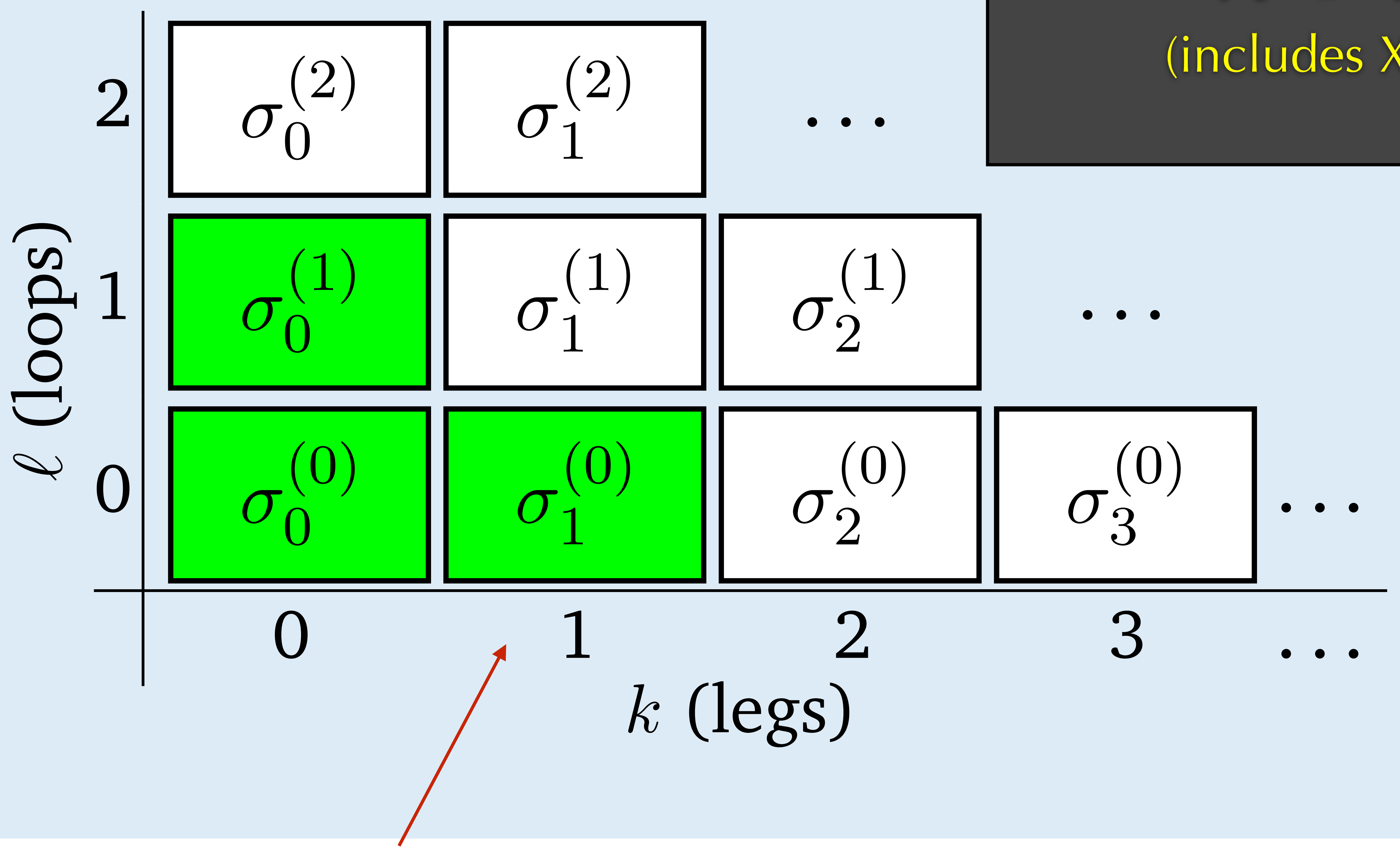
Born

Max Born

(1882-1970)
Nobel Prize 1954

$$k = 0,\ \ell = 0$$

Loops and Legs

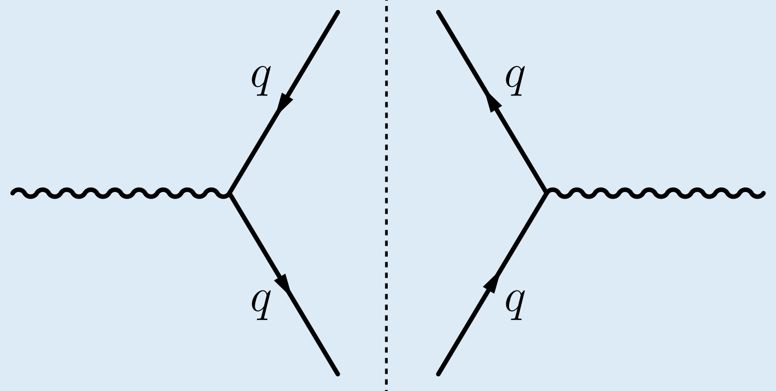


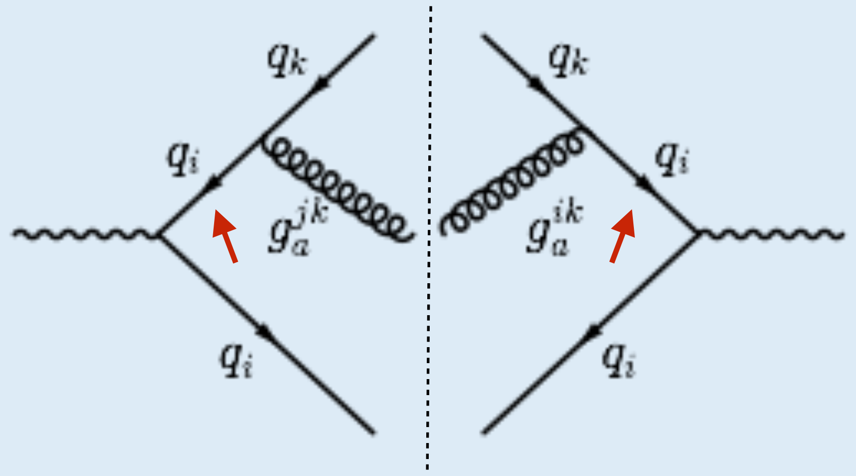
X @ NLO
(includes X+1 @ LO)

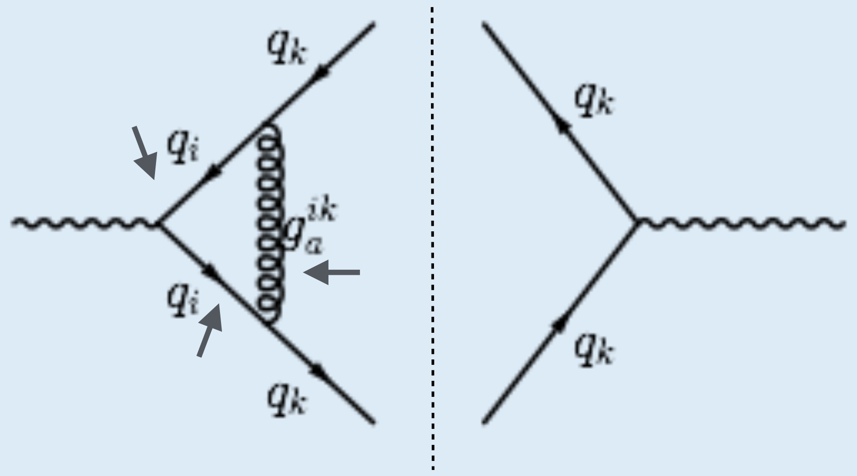
Note: (X+1)-jet observables will of course only be correct to LO

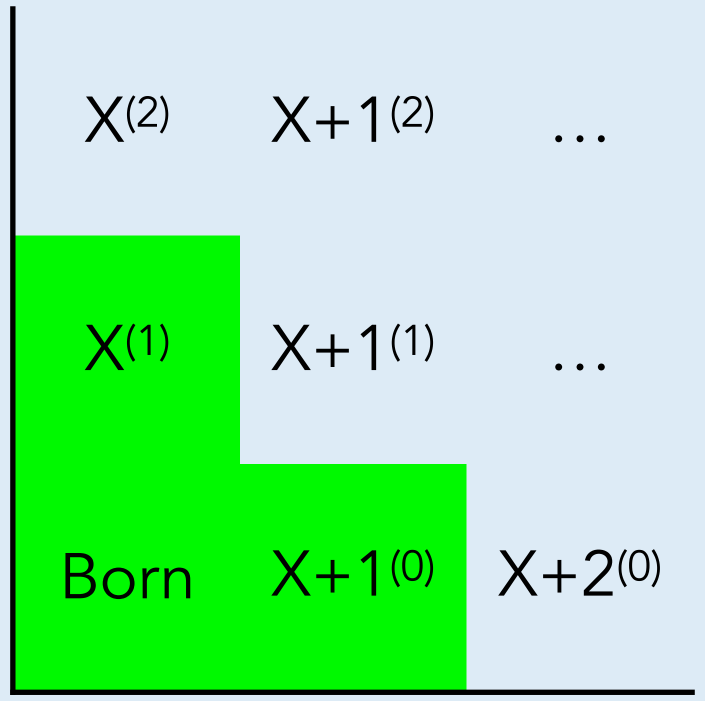
Cross sections at NLO: a closer look

(note: not the 1-loop diagram squared)









$$\sigma_{\rm X}^{\rm NLO} = \int |M_X^{(0)}|^2 + \int |M_{X+1}^{(0)}|^2 + \int 2\mathrm{Re}[M_X^{(1)} M_X^{(0)*}]$$

IR singularities

(from poles of propagators going on shell)

*for so-called IRC safe observables; more later

$$= \sigma_{\rm Born} + \mathrm{Finite}\left\{\int |M_{X+1}^{(0)}|^2\right\} + \mathrm{Finite}\left\{\int 2\mathrm{Re}[M_X^{(1)} M_X^{(0)*}]\right\}$$

$$\boxed{\sigma_{\rm NLO}(e^+e^- \to q\bar{q}) = \sigma_{\rm LO}(e^+e^- \to q\bar{q}) \left(1 + \frac{\alpha_s(E_{\rm CM})}{\pi} + \mathcal{O}(\alpha_s^2)\right)}$$

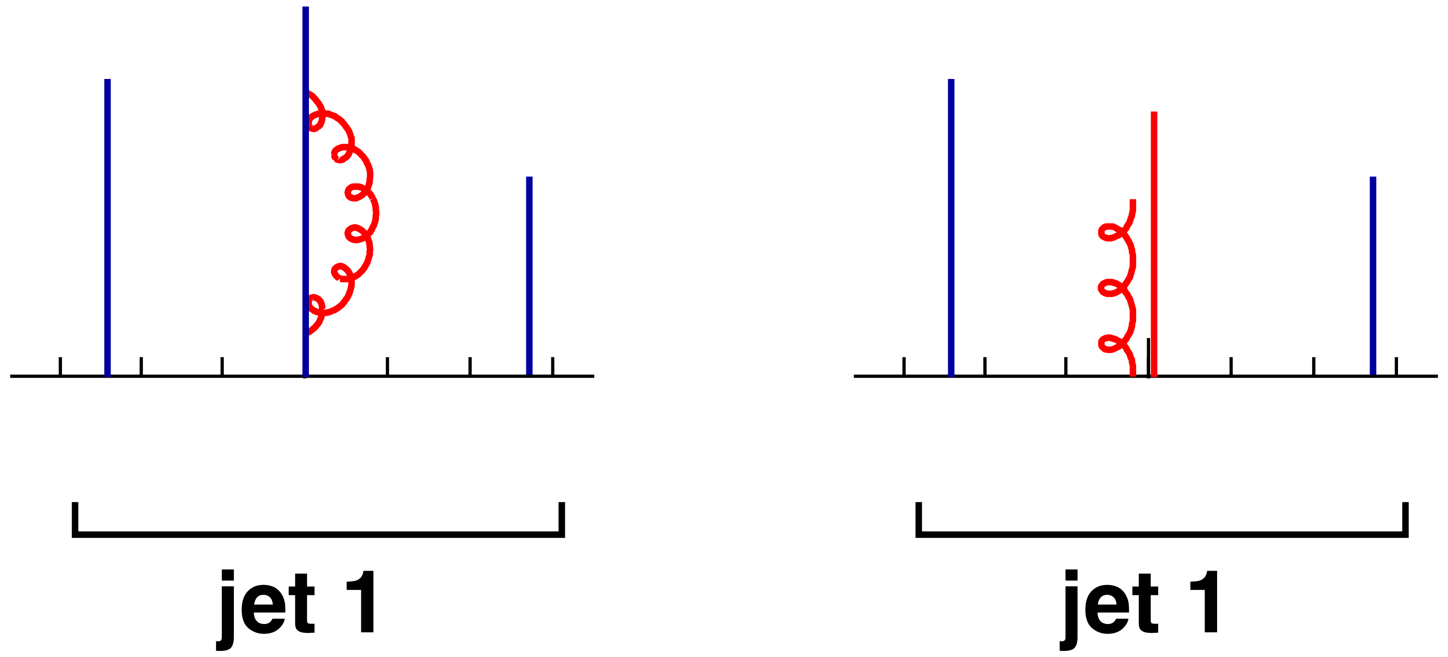
Note on Observables

(example by G. Salam)

Not all observables can be computed perturbatively:

Collinear Safe

Virtual and Real go into **same bins!**



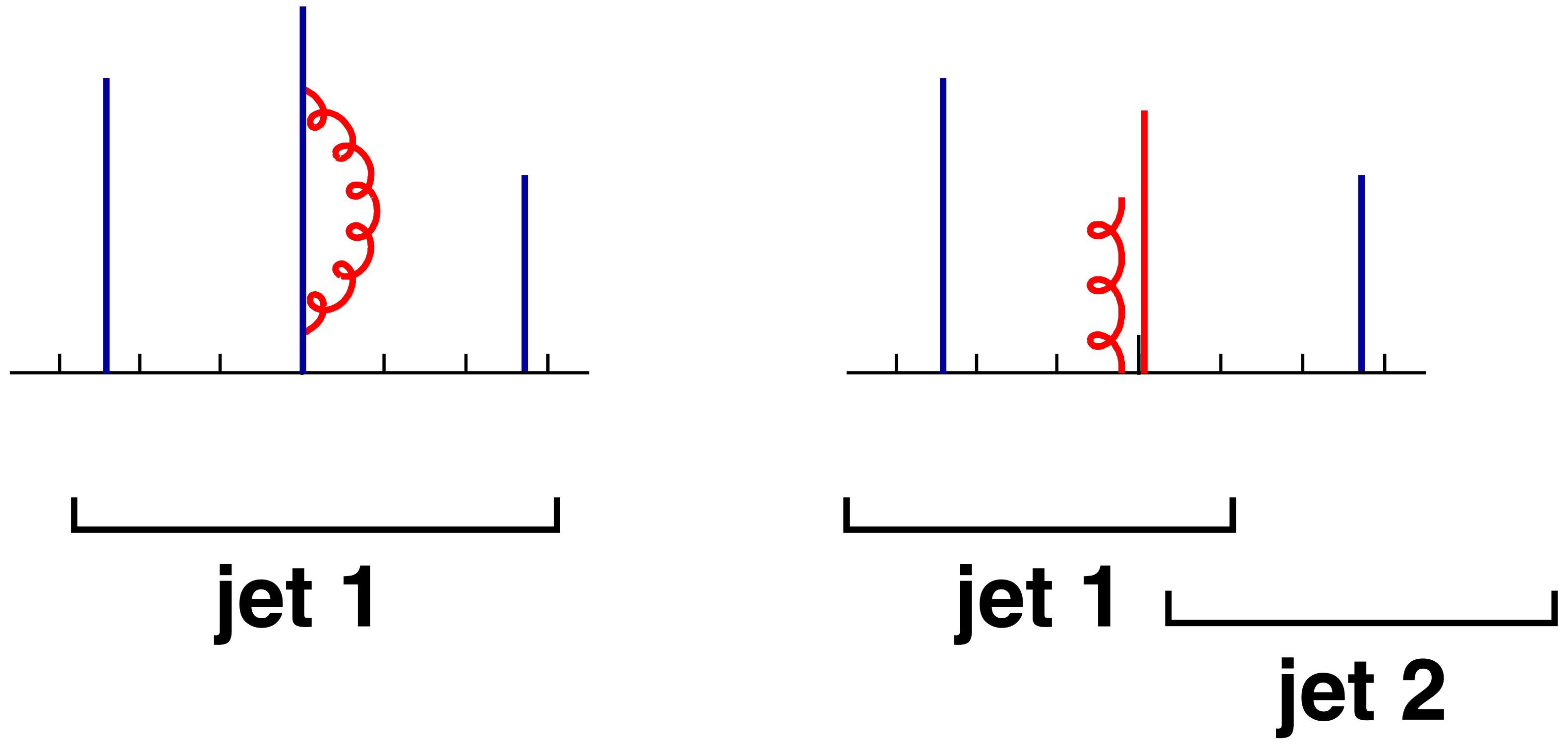
$\alpha_s^n \times (-\infty)$ $\alpha_s^n \times (+\infty)$

Infinities cancel

(KLN: 'degenerate states')

Collinear Unsafe

Virtual and Real go into **different bins!**



$\alpha_s^n \times (-\infty)$ $\alpha_s^n \times (+\infty)$

Infinities do not cancel

Invalidates perturbation theory

Perturbatively Calculable $\Longleftrightarrow$ "Infrared and Collinear Safe"

SOFT radiation:

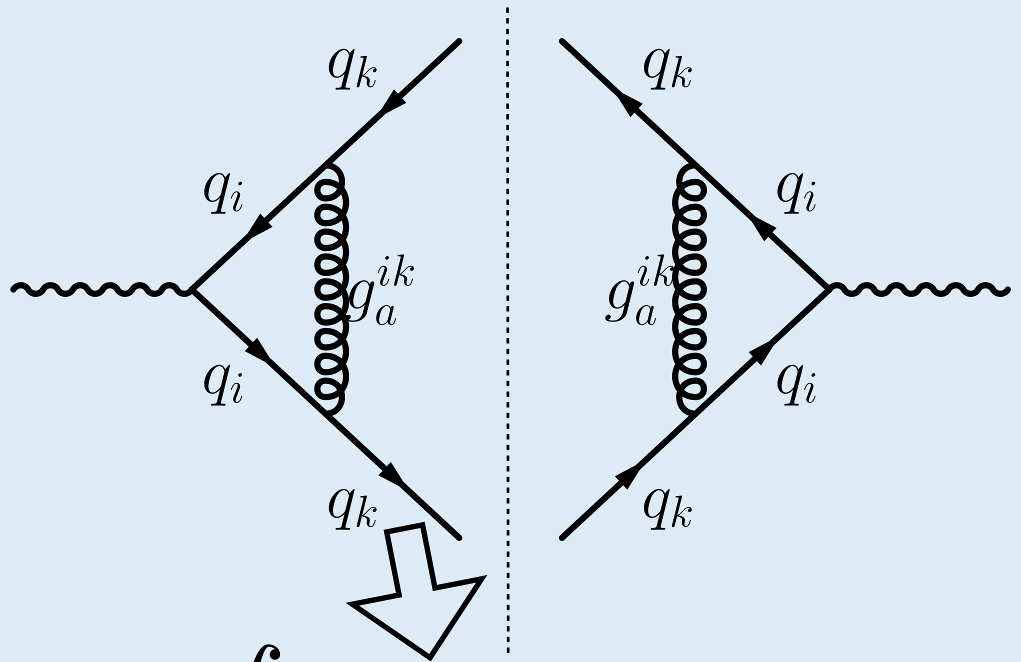
Adding any number of infinitely *soft* particles (zero-energy)
should not change the value of the observable

COLLINEAR radiation:

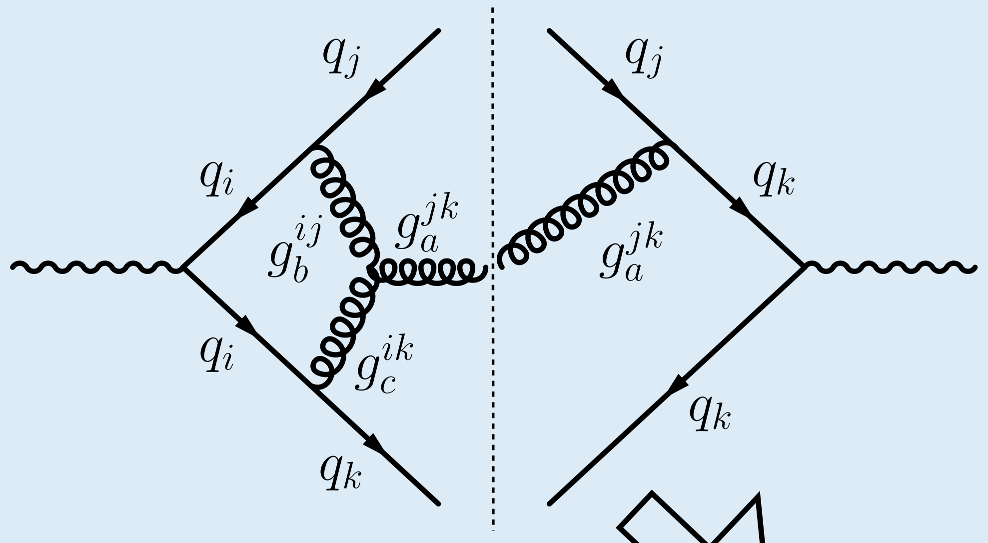
Splitting an existing particle up into two *comoving* ones
(conserving the total momentum and energy)
should not change the value of the observable

Structure of NNLO calculation

1-Loop × 1-Loop



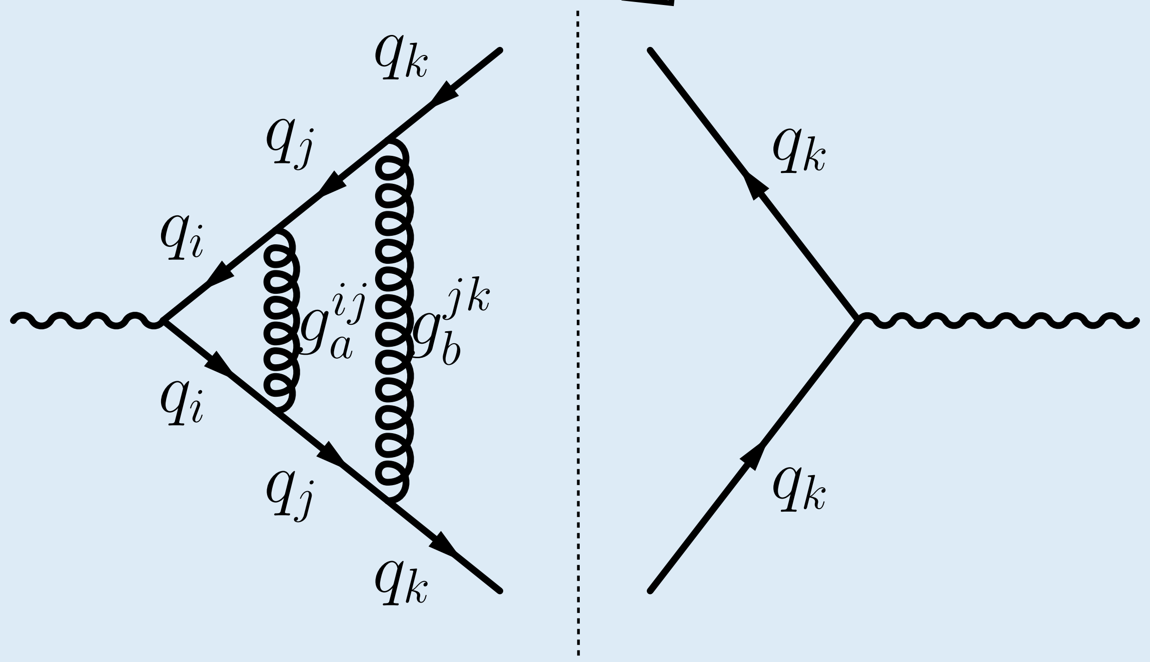
1-Loop × Real for (X+1)



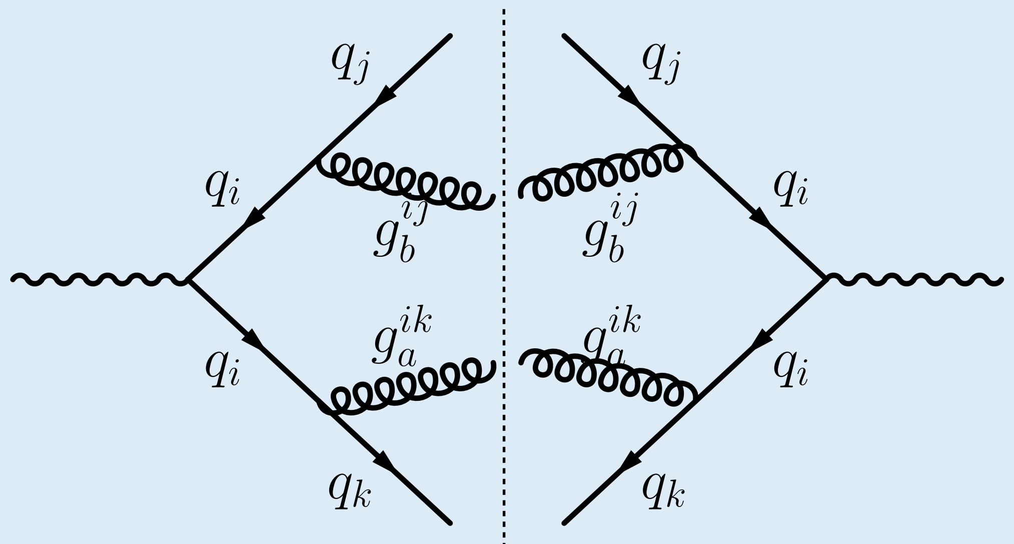
$X^{(2)}$	$X+1^{(2)}$	...
$X^{(1)}$	$X+1^{(1)}$	...
Born	$X+1^{(0)}$	$X+2^{(0)}$

$$\sigma_{\rm X}^{\rm NNLO} = \sigma_X^{\rm NLO} + \int \left(|M_X^{(1)}|^2 + 2{\rm Re}[M_X^{(2)} M_X^{(0)*}] \right) + \int 2{\rm Re}[M_{X+1}^{(1)} M_{X+1}^{(0)*}] + \int |M_{X+2}^{(0)}|^2$$

Everything we had at NLO



Two-Loop × Born Interference



Real × Real for (X+2)