Stefano Frixione

On colour flows

Based on: 2106.13471 (SF, B. Webber)

Taming the accuracy of event generators, CERN, 25/8/2021

- 2106.13471 shows how the use of colour flows leads to remarkably simple expressions for tree-level matrix elements and their soft limits

- By doing so in a language strictly related to that used by MCs, it hopefully paves the way to implementing subleading-colour effects in the latter

- Can also be employed for LL showers off QED$\times$QCD matrix elements

- 2106.13471 builds upon the results of 1106.0155, which dealt with the problem of the commutation of the FKS subtraction with the $1/N$ expansion

A flow is a set of ordered lists of parton labels

A flow is a set of ordered lists of parton labels. It stems from the representation of a scattering amplitude in terms of colour factors, after choosing a representation of the colour algebra. E.g. n gluons:

$$\begin{aligned}\mathcal{A}^{(n)}(a_1,\dots a_n) &= \sum_{\sigma\in P'_n} \mathrm{Tr}\left(\lambda^{a_{\sigma(1)}}\dots\lambda^{a_{\sigma(n)}}\right)\widehat{\mathcal{A}}^{(n)}\left(\sigma(1),\dots\sigma(n)\right) \\ &\equiv \sum_{\sigma\in P'_n} \Lambda\left(\{a_i\},\sigma\right)\widehat{\mathcal{A}}^{(n)}(\sigma)\end{aligned}$$

Colour configuration:

$$\{a_i\}_{i=1}^{n}\ , \qquad a_i \in \left\{1,\dots N^2-1\right\}$$

Colour flow (for gluons, there is a single list):

$$\left(\sigma(1),\dots\sigma(n)\right)\ , \qquad \sigma(1)=1$$

with σ an n-object non-cyclic permutation

Use orthonormal colour basis:

$$|a_1,\ldots a_n\rangle\,, \qquad \langle b_1,\ldots b_n|a_1,\ldots a_n\rangle = \prod_{i=1}^{n}\delta_{a_i b_i}$$

One can thus define amplitudes as vectors in colour space

$$|\mathcal{A}^{(n)}(a_1,\ldots a_n)\rangle = \mathcal{A}^{(n)}(a_1,\ldots a_n)|a_1,\ldots a_n\rangle$$

$$|\mathcal{A}^{(n)}(\sigma)\rangle = \sum_{\{a_i\}_{i=1}^{n}} \Lambda\left(\{a_i\},\sigma\right)\widehat{\mathcal{A}}^{(n)}\left(\sigma\right)|a_1,\ldots a_n\rangle$$

“Physical” amplitude:

$$|\mathcal{A}^{(n)}\rangle \quad = \quad \sum_{\{a_i\}_{i=1}^{n}} |\mathcal{A}^{(n)}(a_1,\ldots a_n)\rangle$$

$$|\mathcal{A}^{(n)}\rangle \quad = \quad \sum_{\sigma\in P'_n} |\mathcal{A}^{(n)}(\sigma)\rangle$$

The amplitude squared is what is relevant to cross sections

$$\mathcal{M}^{(n)} = \langle\mathcal{A}^{(n)}|\mathcal{A}^{(n)}\rangle$$

$$\mathcal{M}^{(n)} = \sum_{\sigma,\sigma'\in P'_n} \mathcal{M}^{(n)}(\sigma',\sigma)$$

$$\mathcal{M}^{(n)}(\sigma',\sigma) = \langle \mathcal{A}^{(n)}(\sigma') | \mathcal{A}^{(n)}(\sigma) \rangle$$

In terms of scalar quantities:

$$\mathcal{M}^{(n)}(\sigma',\sigma) = \widehat{\mathcal{A}}^{(n)}(\sigma')^\star \, C(\sigma',\sigma) \, \widehat{\mathcal{A}}^{(n)}(\sigma)$$

Colour-flow matrix (real and symmetric):

$$C(\sigma',\sigma) = \sum_{\{a_i\}_{i=1}^n} \Lambda\left(\{a_i\},\sigma'\right)^\star \Lambda\left(\{a_i\},\sigma\right)$$

- What matters is the *closed flow* (σ',σ), i.e. the combination of the flows on the lhs of the cut (*L-flow*, σ') and on the rhs of the cut (*R-flow*, σ)

At leading N, $\sigma' = \sigma \Longrightarrow$ closed flows and (R/L-)flows can be identified with each other (thus one doesn't bother to introduce the former)

For q quark lines, n gluons, with labelling:

$$\begin{array}{ccc} -2q \le i \le -q-1 & \longrightarrow & \text{antiquarks} \\ -q \le i \le -1 & \longrightarrow & \text{quarks} \\ 1 \le i \le n & \longrightarrow & \text{gluons} \end{array}$$

the flows are:

$$\gamma = \bigcup_{p=1}^{q} \gamma_p$$

$$\gamma_p = \Big(-p\,;\sigma(t_{p-1}+1),\ldots\sigma(t_p);\mu(-p-q)\Big)$$

- μ, σ: q- and n-object permutations (including cyclic ones)
- $t=\{t_0,\ldots t_q\}$, $0=t_0\le t_1\le\ldots t_{q-1}\le t_q=n$: a partition of the ordered set of the first n integers into q subsets of ordered integers

- γ_p is the colour line that connects quark $-p$ with antiquark $\mu(-p-q)$, emitting gluons $\sigma(t_{p-1}+1),\ldots\sigma(t_p)$

Colour loops

0. Write the L- and R-flow as two vertical sets of indices (the former to the left of the latter). Each open colour line that belongs to these flows is a separate subset of such sets. The relative order of these subsets is irrelevant.

1. Start from either a quark or a gluon that belongs to the R-flow; this is the first element of the loop.

2. Go down one place in the R-flow; a gluon or an antiquark is found.

3. Jump to the L-flow, landing on the element whose label is identical to that of the gluon or the antiquark of the previous step.

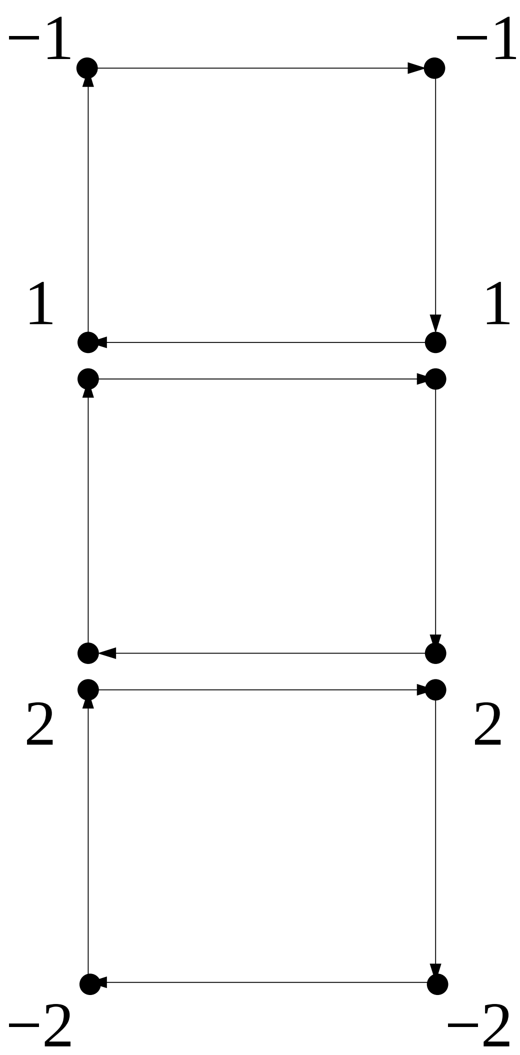
4. Go up one place in the L-flow; a gluon or a quark is found.

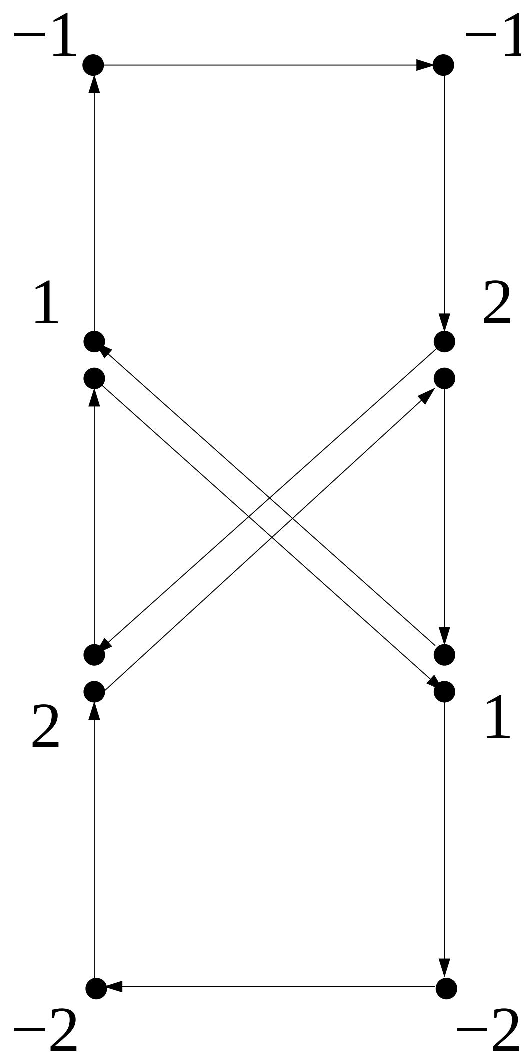
5. Jump to the R-flow, landing on the element whose label is identical to that of the gluon or the quark of the previous step.

6. If the landing element coincides with the first element of the loop then go to step 7, otherwise iterate steps 2–5.

7. A loop is the ordered set composed of the starting element and of all of the landing elements subsequently found, except for the last one (because it coincides with the starting element).

Consider $0 \to q\bar{q}gg \Longrightarrow \gamma_1 = (-1; 1, 2; -2)$, $\gamma_2 = (-1; 2, 1; -2)$





$\mathcal{L}(\gamma_1, \gamma_1) = \Big\{ \big(-1, 1 \big), \big(1, 2 \big), \big(2, -2 \big) \Big\}$ $\mathcal{L}(\gamma_1, \gamma_2) = \Big\{ \big(-1, 2, 1, -2, 2, 1 \big) \Big\}$

Colour loops

- Are a property of closed flows
- Easy to teach a computer to determine them
- For any closed flow (γ', γ), we denote their set by $\mathcal{L}(\gamma', \gamma)$

■ Closed flows generalise "ordinary" flows beyond leading N;
colour loops generalise "ordinary" colour connections

Colour partners of k: all the particles that belong to the loop(s) to which k belongs $\Longrightarrow$ up to $m-1$ for quarks, $2(m-1)$ for gluons, $m = 2q + n$

A remarkable fact in the gluon-only case. Colour loops are in one-to-one correspondence with:

- the alternating cycles of the cycle graph of a permutation that is constructed (trivially) with the two permutations that form the given closed flow
- the ordinary cycles of a permutation that is constructed (non-trivially) with the two permutations above

There is a growing mathematics literature on these things, since they are important for genome-rearrangement studies

See appendix A of 2106.13471 for other peculiarities of gluon-only flows

Tree-level matrix elements

Quark only:

$$\mathcal{M}^{(2q;0)} = \sum_{\gamma',\gamma\in\mathcal{F}_{2q;0}} \widehat{\mathcal{A}}^{(2q;0)}(\gamma')^\star\, \widehat{\mathcal{A}}^{(2q;0)}(\gamma)\, N^{-\rho(\gamma')-\rho(\gamma)}\, N^{\left|\mathcal{L}(\gamma',\gamma)\right|}$$

with (Mangano&Parke):

$$\rho(\gamma) = \min\left\{q-1, \sum_{p=1}^{q} \delta\Big(-p-q, \mu(-p-q)\Big)\right\}$$

- Colour algebra replaced by loop counting

  $|\mathcal{L}(\gamma',\gamma)|$ is the number of loops in the closed flow (γ',γ)

- The only non-trivial ingredients are the dual amplitudes
- Hierarchy controlled by the number of colour loops

Tree-level matrix elements

Quark-gluon:

$$\mathcal{M}^{(2q;n)} = \sum_{k=0}^{n} \sum_{s_k \in S_k^{(n)}} \frac{(-1)^k}{2^n} N^{-k} \times \sum_{\bar{\gamma},\bar{\gamma}' \in \mathcal{F}_{2q;n-k}^{(s_k)}} \bar{\mathcal{A}}^{(2q;n)}(\bar{\gamma}')^{\star} \, \bar{\mathcal{A}}^{(2q;n)}(\bar{\gamma}) \, N^{-\rho(\bar{\gamma})-\rho(\bar{\gamma}')} N^{\left|\mathcal{L}(\bar{\gamma}',\bar{\gamma})\right|}$$

- Striking similarity with the quark-only case
- With the additional complication of the sums over the number of $U(1)$ gluons and their identities, embedded in the *secondary flows*
- The amplitudes $\bar{\mathcal{A}}^{(2q;n)}(\bar{\gamma})$ are linear combinations of the original dual amplitudes (insertion in $\bar{\gamma}$ of the missing $U(1)$ gluons in all possible manners)

Secondary flows

- Constructed as the (primary) flows: insert onto the q colour lines the gluon labels in all possible manners
- At variance with the case of the primary flows, this must be done for all possible subsets of the set of gluon labels (including both itself and the empty set)
- Stem from the repeated application of the Fierz identity. Gluons are replaced by fictitious $Q\bar{Q}$ pairs (first term), or eliminated (second term $\Longrightarrow$ $U(1)$ gluons). In the former case, they are denoted by the same gluon label they originate from, with a bar on top

Secondary flows

Consider again $0 \to q\bar{q}gg$

- **Primary flows**

$$\gamma_1 = (-1;1,2;-2)\,, \qquad \gamma_2 = (-1;2,1;-2)$$

- **Secondary flows**

$$\begin{aligned}
\bar{\gamma}_{0,1} &= \big(-1;\bar{1},\bar{2};-2\big)\\
\bar{\gamma}_{0,2} &= \big(-1;\bar{2},\bar{1};-2\big)\\
\bar{\gamma}_{1,1} &= \big(-1;\bar{2};-2\big)\\
\bar{\gamma}_{1,2} &= \big(-1;\bar{1};-2\big)\\
\bar{\gamma}_{2,1} &= \big(-1;-2\big)
\end{aligned}$$

Tree-level matrix elements

Gluon only:

$$\mathcal{M}^{(n)} = \frac{1}{2^n} \sum_{\bar{\sigma},\bar{\sigma}' \in P'_n} \widehat{\mathcal{A}}^{(n)}(\bar{\sigma}')^{\star}\, \widehat{\mathcal{A}}^{(n)}(\bar{\sigma})\, N^{\left|\mathcal{L}(\bar{\sigma}',\bar{\sigma})\right|},$$

- Derived from the quark-gluon case
- Significant simplification: all $\bar{\mathcal{A}}^{(n)}$ amplitudes vanish owing to the dual Ward identities, except those whose secondary flows feature all gluons

  (i.e. there are no $U(1)$ gluons left)
- $U(1)$-gluon decoupling is a well-known result, which *does not happen* if one works at fixed flows

Soft limits

Quark only (i.e. a single soft gluon):

$$\mathcal{M}^{(2q;1)}_{\text{SOFT}} = 2g_S^2\, C_F \sum_{\gamma',\gamma\in\mathcal{F}_{2q;0}} \widehat{\mathcal{A}}^{(2q;0)}(\gamma')^\star\, \widehat{\mathcal{A}}^{(2q;0)}(\gamma)\, N^{-\rho(\gamma)-\rho(\gamma')} N^{\left|\mathcal{L}(\gamma',\gamma)\right|}$$
$$\times \sum_{\ell\in\mathcal{L}(\gamma',\gamma)} \sum_{k,l\in\ell}^{k<l} (-1)^{\delta_\ell(k,l)} \left[k,l\right]$$

- Colour algebra replaced by loop counting

  $|\mathcal{L}(\gamma',\gamma)|$ is the number of loops in the closed flow (γ',γ)

- Radiation pattern entirely determined by the structure of the loops, in particular by the *distance* $(\delta_\ell(k,l))$ within a loop between the two radiators that identify an eikonal

  $\delta_\ell(k,l)$ is simply the number of partons between k and l

Soft limits

Quark-gluon:

$$\begin{aligned}
\mathcal{M}^{(2q;n+1)}_{\text{SOFT}} \quad &= \quad 2g_S^2\, C_F \sum_{p=0}^{n} \sum_{s_p \in S_p^{(n)}} \frac{(-1)^p}{2^n}\, N^{-p} \\
&\times \sum_{\bar\gamma,\bar\gamma' \in \mathcal{F}^{(s_p)}_{2q;n-p}} \bar{\mathcal{A}}^{(2q;n)}(\bar\gamma')^\star\, \bar{\mathcal{A}}^{(2q;n)}(\bar\gamma)\, N^{-\rho(\bar\gamma)-\rho(\bar\gamma')}\, N^{\left|\mathcal{L}(\bar\gamma',\bar\gamma)\right|} \\
&\times \sum_{\ell \in \mathcal{L}(\bar\gamma',\bar\gamma)} \sum_{k,l\in\ell}^{k<l} (-1)^{\delta_\ell(k,l)} \left[k, l\right]
\end{aligned}$$

- As at the tree level, this is identical to the result for the quark-only case, bar for the sum over $U(1)$ gluons

Soft limits

Gluon only:

$$\mathcal{M}_{\text{SOFT}}^{(n+1)} = \frac{2g_S^2\, C_F}{2^n} \sum_{\bar{\sigma},\bar{\sigma}' \in P_n'} N^{\left|\mathcal{L}(\bar{\sigma}',\bar{\sigma})\right|}\, \widehat{\mathcal{A}}^{(n)}(\bar{\sigma}')^{\star}\, \widehat{\mathcal{A}}^{(n)}(\bar{\sigma})$$

$$\times \sum_{\ell \in \mathcal{L}(\bar{\sigma}',\bar{\sigma})} \sum_{k,l \in \ell}^{0<k<l} (-1)^{\delta_\ell(k,l)} \left[k, l\right],$$

- As at the tree level, significant simplifications induced by the dual Ward identities

Monte Carlos

- Reproduce the soft-gluon emission pattern of an m-parton process by associating single partons/dipoles with eikonal terms in the soft limit of the $(m+1)$-parton process
- The associations are equivalent to finding colour connections, i.e. the colour partner(s) of any given parton
- Leading N: quark-antiquark, quark-g anticolour, antiquark-g colour, gg colour-anticolour pairings, as dictated by colour loops

  (At leading N colour loops are two-object sets)

- Proceed in the same way beyond leading N, by exploiting the properties of generic colour loops

  (colour-colour and anticolour-anticolour connections become possible)

Example: angular-ordered showers

Angular separation(s) of colour partners define intensity of radiation (~ Sudakov exponent pre-factor) in the relevant regions of emission

$$\Delta_k(\xi_k) = \exp\left[-C(\mathcal{I}_k)\frac{\alpha_S}{2\pi}\int_{\xi_0}^{\xi_k}\frac{d\xi}{\xi}\int P(z)\,dz \right]$$

Leading N (l sums over colour partners of k):

$$[\Delta_k(\xi_k)]^{p_{\rm LC}(\xi_k)} \, , \qquad p_{\rm LC}(\xi_k) = \frac{\sum_l \Theta(\xi_{kl}-\xi_k)}{\sum_l 1}$$

Beyond leading N (l sums over colour partners of k):

$$[\Delta_k(\xi_k)]^{p_{(\bar\gamma',\bar\gamma)}(\xi_k)} \, , \qquad p_{(\bar\gamma',\bar\gamma)}(\xi_k) = \frac{\sum_{\ell\in\mathcal{L}(\bar\gamma',\bar\gamma)}\sum_l (-1)^{\delta_\ell(k,l)}\Theta(\xi_{kl}-\xi_k)}{\sum_{\ell\in\mathcal{L}(\bar\gamma',\bar\gamma)}\sum_l (-1)^{\delta_\ell(k,l)}}$$

- Multiple regions of radiation, negative contributions ("countershowers"), $U(1)$ gluons do not radiate, colour reconnections among showers

An alternative: dipole graphs

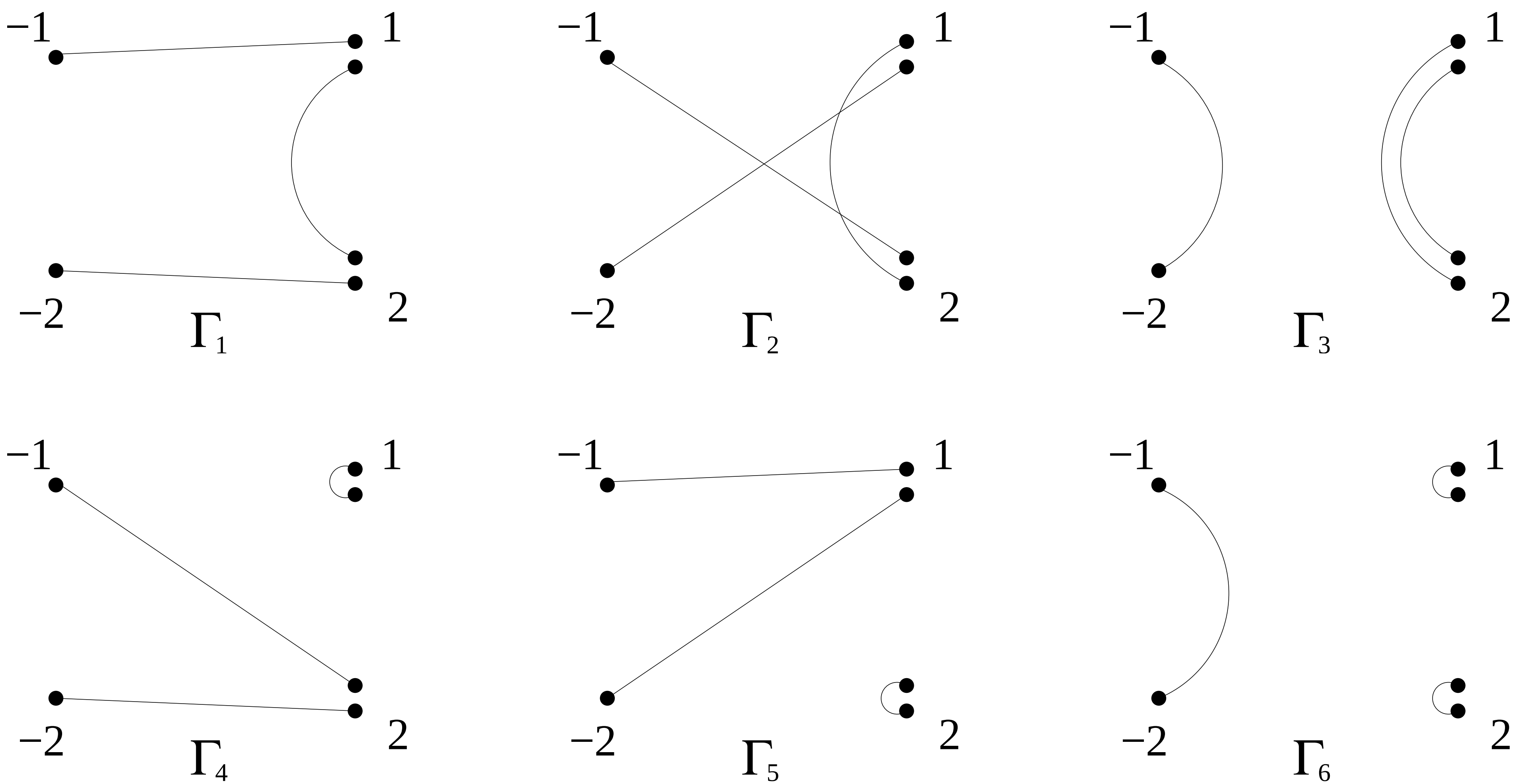
Dipole graphs are the generalisation of standard colour connections; thus, any (anti)colour-(anti)colour combination must be possible

$$\begin{aligned} q_s &= \Big\{ -2q, -2q+1, \ldots -1 \Big\}, \qquad g_s = \Big\{ 1, 2, \ldots n \Big\} \\ \Gamma &= \Big\{ \big(\Gamma_{1,1}, \Gamma_{1,2} \big), \ldots \big(\Gamma_{q+n,1}, \Gamma_{q+n,2} \big) \, \Big| \, \Gamma_{\alpha,\beta} \in q_s \cup g_s \, , \ \forall \alpha \ \ \Gamma_{\alpha,1} \neq \Gamma_{\alpha,2} \, , \\ & \qquad\qquad \forall \, k \in q_s \ \ \exists! \ (\alpha, \beta) \ \ \text{s.t.} \ \ k = \Gamma_{\alpha,\beta} \ \ \text{and} \\ & \qquad\qquad \forall \, k \in g_s \ \ \exists! \ (\alpha_1, \beta_1) \, , (\alpha_2, \beta_2) \ \ \text{s.t.} \ \ k = \Gamma_{\alpha_1,\beta_1} = \Gamma_{\alpha_2,\beta_2} \Big\} \end{aligned}$$

That is: a dipole graph Γ is a set of pairs of non-identical parton labels; each quark and antiquark (gluon) label will appear in it exactly once (twice)

- Each pair $(\Gamma_{\alpha,1}, \Gamma_{\alpha,2})$ is a dipole; its two ends are colour-connected
- There are $n+q$ dipoles in a dipole graph
- Self-connected gluons, that correspond to self-eikonals and arise from gluons which appear twice in the same colour loop, are not part of dipole graphs

Consider $0 \to q\bar{q}gg$: there are six $(= 3 + 2 \times 1 + 1)$ dipole graphs



Γ_1 and Γ_2 are the only leading-N connections. Self-connected gluons are depicted in spite of not belonging to dipole graphs, to show their identities

An alternative: dipole graphs

Dipole graphs are the generalisation of standard colour connections; thus, any (anti)colour-(anti)colour combination must be possible

- For any given closed-flow radiation pattern, there exists at least one representation of it in terms of dipole-graph radiation patterns

$$
\begin{aligned}
\mathcal{E}(\bar{\gamma}',\bar{\gamma}) &= \sum_{\ell\in\mathcal{L}(\bar{\gamma}',\bar{\gamma})} \sum_{k,l\in\ell}^{k<l} (-1)^{\delta_\ell(k,l)} \Big[k,l\Big] \\
&\equiv \sum_a c^+(\Gamma_a)\,\mathcal{D}(\Gamma_a) - \sum_b c^-(\Gamma_b)\,\mathcal{D}(\Gamma_b) \\
\mathcal{D}(\Gamma) &= \sum_{\alpha=1}^{q+n} \Big[\Gamma_{\alpha,1},\Gamma_{\alpha,2}\Big]
\end{aligned}
$$

- The representation may not be unique (in particular, it could be positive-definite)
- Even if $(\bar{\gamma}',\bar{\gamma})$ does not feature any $U(1)$ gluons, the corresponding dipole graphs may contain self-connected gluons

How many dipole graphs?

$q \diagdown n$	0	1	2	3	4	5	6
0	1	0	1	1	6	22	130
1	1	1	3	10	46	252	1642
2	3	6	21	93	510	3306	24762
3	15	45	195	1050	6750	50280	425490
4	105	420	2205	13965	103110	867510	8183490

These sequences admit the generating function:

$$\mathcal{F}(x,y) = \sum_{q,n} |\mathcal{G}_{2q;n}| \frac{x^q}{q!} \frac{y^n}{n!} = \frac{\mathrm{e}^{-y/2+y^2/4}}{\sqrt{1-2x-y}}$$

- Bear in mind the (possible) redundancy: this helps design strategies with different efficiencies
- Except for the $q = 0$ case, the sequences above were not known to OEIS

A possible MC implementation: subevents

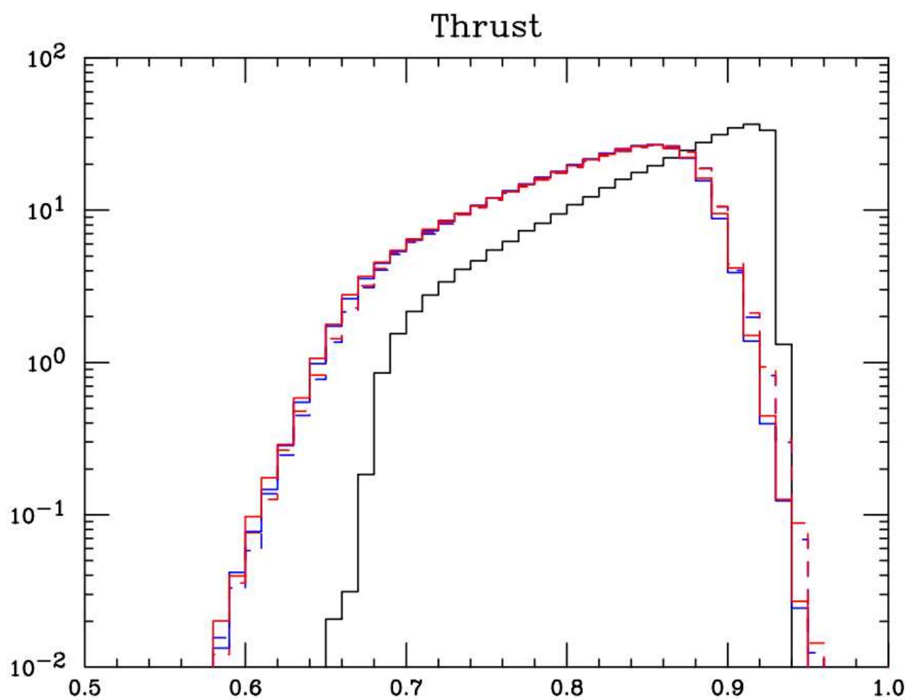
- Express the closed-flow radiation patterns that appear in the soft limit in terms of their equivalent dipole-graph radiation patterns
- Shower (in the usual way) a given Born configuration as many times as there are dipole graphs, using the colour connections associated with such graphs *(subevents)*
- Each subevent has a weight equal to the factor that multiplies the radiation pattern of the corresponding dipole graph in the expression of the soft limit
- Fill histograms with the kinematics of each subevent and its associated weight (or, alternatively, go through an unweighting)

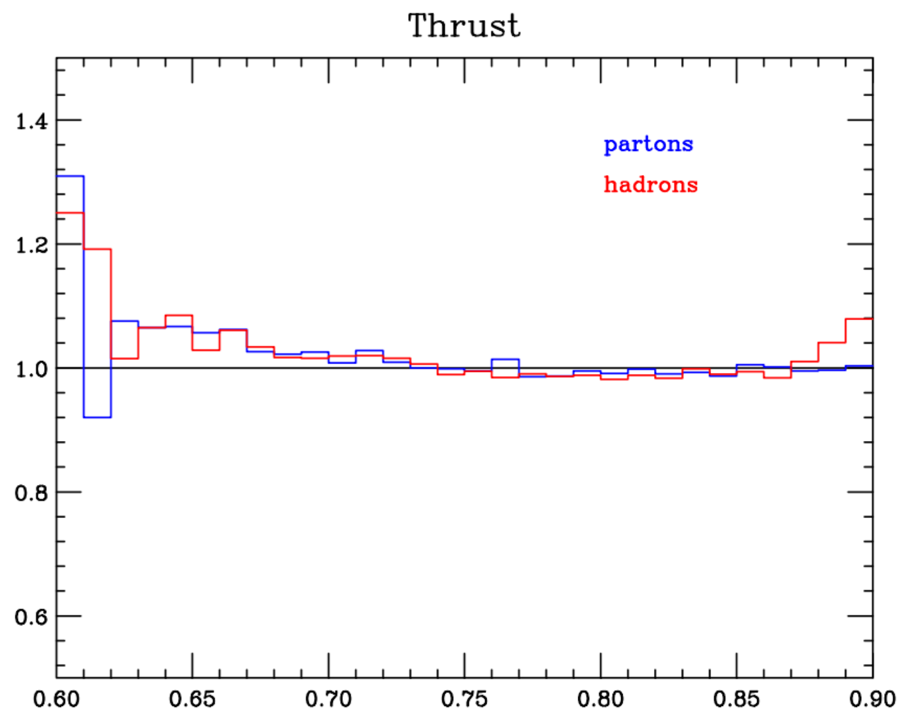
Very preliminary MC results

- With subevents only
- With HW6
- Off $e^+e^- \to jjj$ and $e^+e^- \to jjjj$ matrix elements
- Ratio plots obtained with larger statistics (i.e. not the same runs)

All plots by Bryan Webber

$e^+e^- \to jjj$

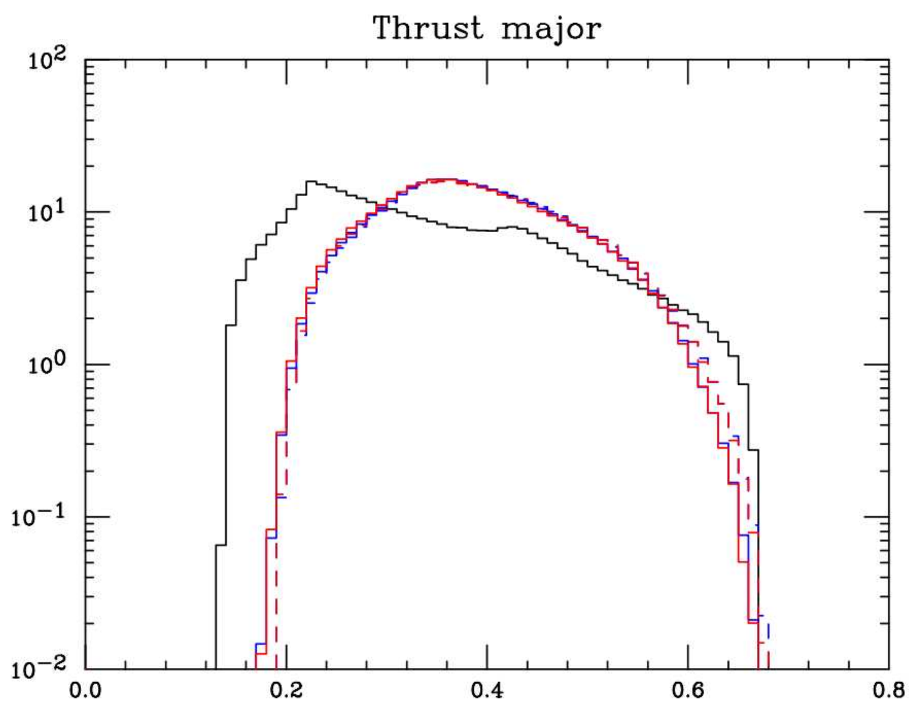


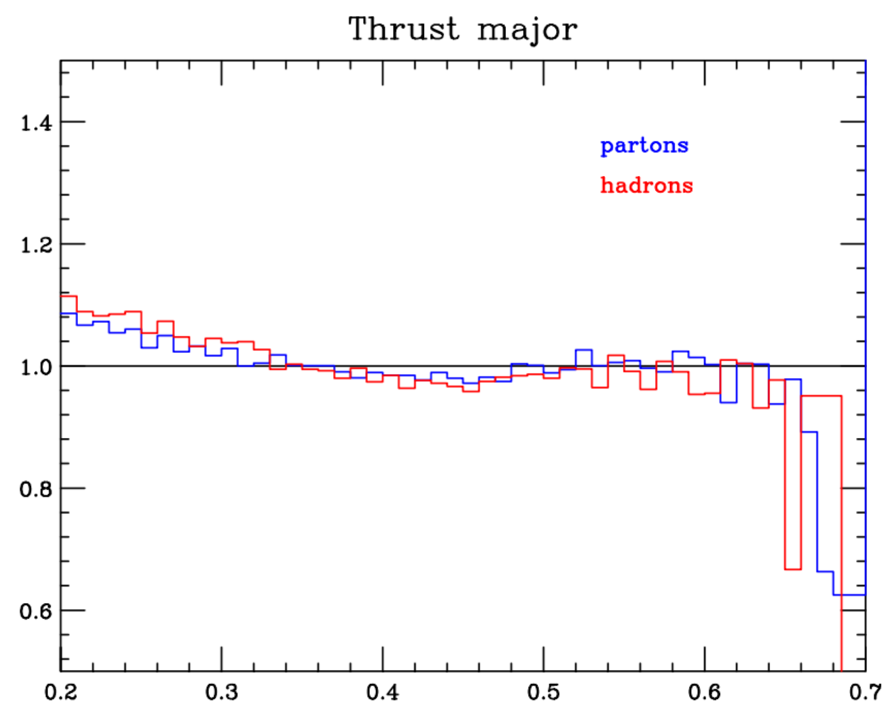


- Born: black; LC: blue; FC: red
- Parton level: dashed; hadron level: solid

- FC/LC

$e^+e^- \to jjj$





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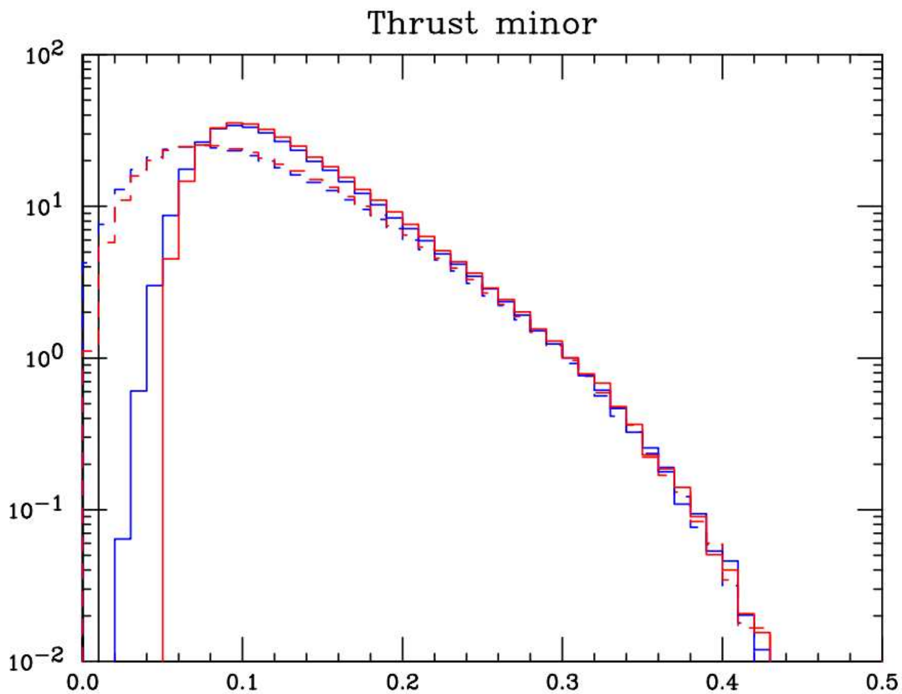

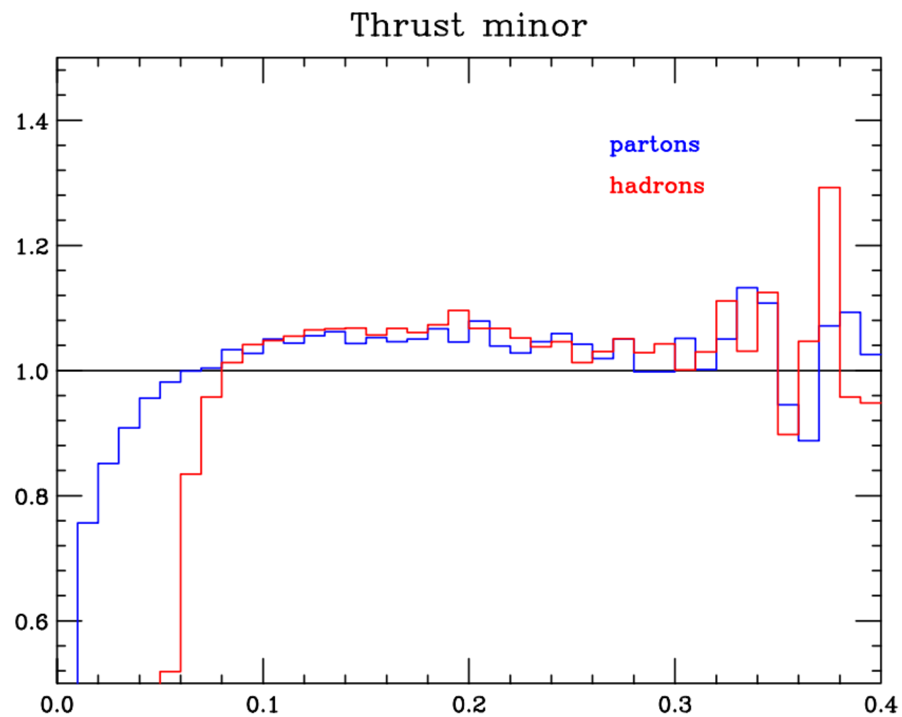

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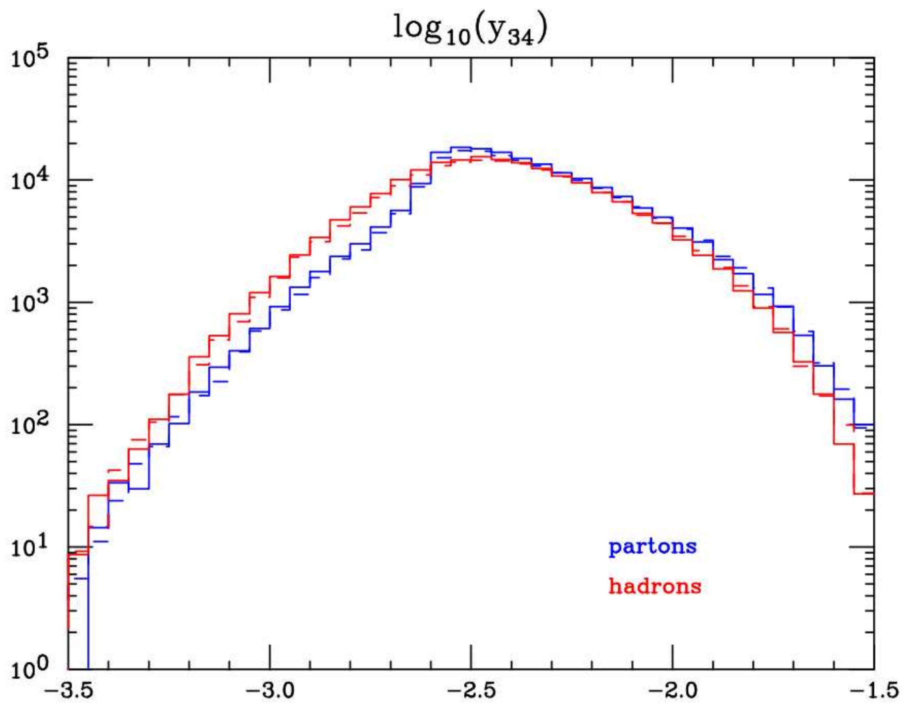
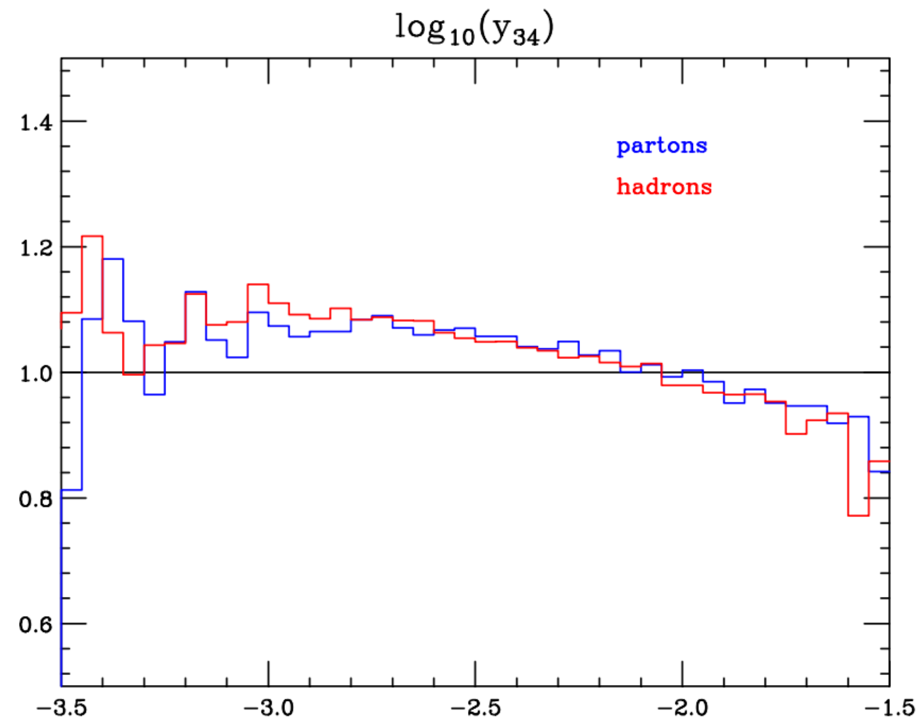
$e^+e^- \to jjjj$





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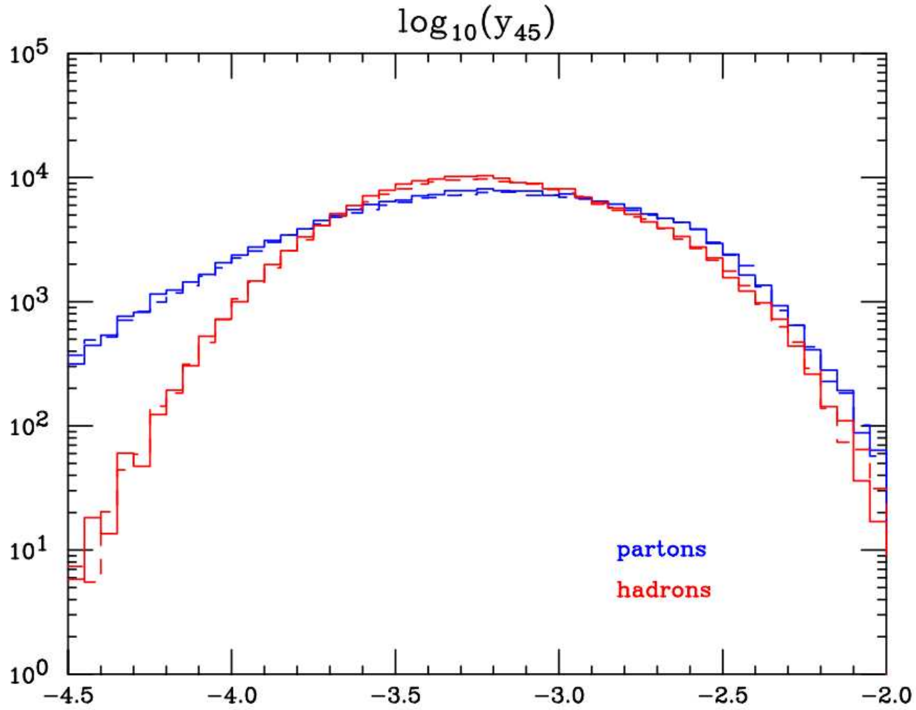

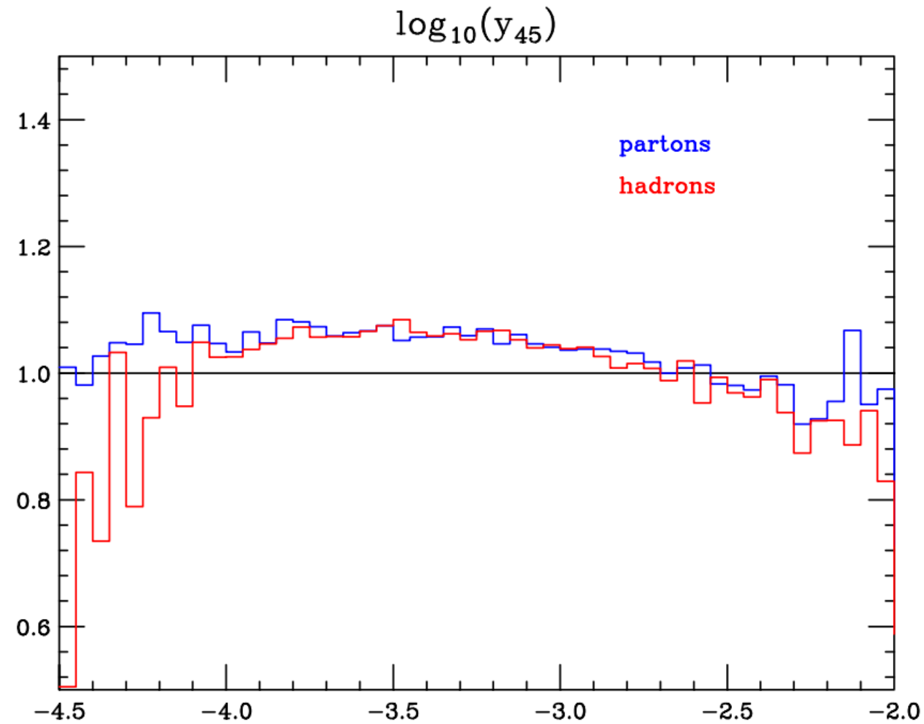

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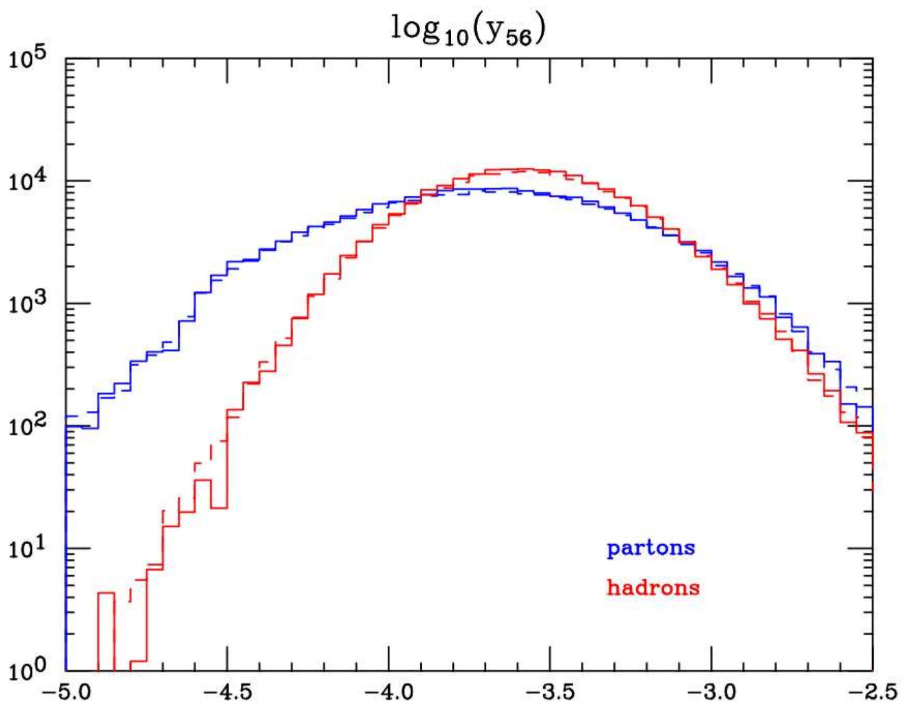

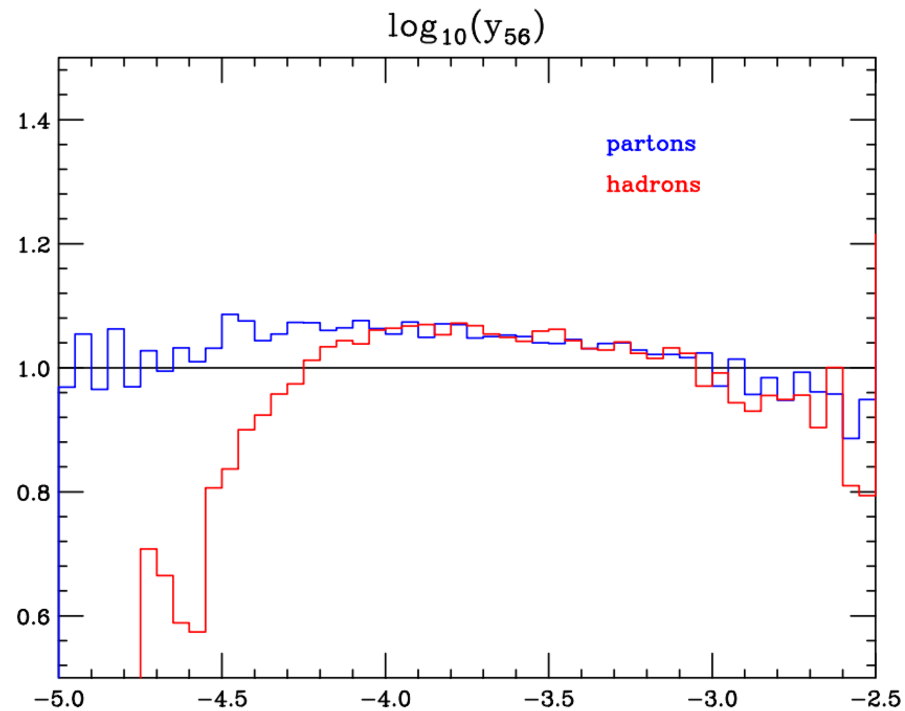

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- FC/LC

Outlook

- The reformulation of tree-level results in terms of colour flows suggests strategies to implement subleading-N effects in MCs
- Closed flows define generalised radiation regions and intensities, that could be implemented in an MC by means of a Sudakov with contributions of alternating signs to the exponent
- Dipole graphs offer an alternative way of proceeding whose showers are fully analogous to those of the leading-N case
- To be (better) understood: reconnections among showers, role of $U(1)$/self-connected gluons, hadronization models

We are starting to implement different strategies, using simple processes as templates